

# **An Introduction to Portfolio Theory**

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# Introduction

Portfolio theory deals with the problem of constructing for a given collection of assets an investment with desirable features. A variety of different asset characteristics can be taken into consideration, such as the amount of value, on average, an asset returns on over a period of time and the riskiness of reaping returns comparable to the average. The financial objectives of the investor and tolerance of risk determine what types of portfolios are to be considered desirable. In these notes we shall discuss a quantitative approach to constructing portfolios. In particular, we shall use the methods of constrained optimization to construct portfolios for a given collection of assets with desirable features as quantitated by an appropriate utility function and constraints. The materials presented here are taken from the following sources: *Theory of Finance - Mean Variance Analysis* by Simon Hubbert, and *Investments* by Bodie, Kane, and Marcus.

## Characterizing the Rates of Return of Assets and Portfolios

We shall concern ourselves with primarily two basic features of an asset. The first is the average return of an asset over a period of time. The second characteristic is how risky it is to obtain similar returns comparable to the average over the investment period.

For an asset with value  $S(0)$  at time 0 and value  $S(T)$  at time  $T$ , the rate of return  $\rho$  is defined by:

$$S(T) = (1 + \rho)S(0). \quad (1)$$

The rate of return can be thought of as an “effective interest rate” which would be required for a deposit of  $S(0)$  into a savings account at a bank to obtain the same change in value as the asset over the period  $[0, T]$ . For example, if  $S(0) = \$4$  and after one year  $S(1) = \$6$ , the rate of return of the asset is  $\rho = S(1) - S(0)/S(0) = 50\%$ . The rate of return of an asset is also sometimes referred to as the “yield” of the asset.

Since the outcome of an investment in an asset has some level of uncertainty, the value  $S(T)$  is unknown exactly at time 0. To model the uncertainty we shall consider the value of the asset at time  $T$  as a random variable. Correspondingly, the rate of return  $\rho$  defined by equation 1 is also a random variable. To characterize the asset we shall consider the average rate of return defined by:

$$\mu = E(\rho) \quad (2)$$

where  $E()$  denotes the expectation of a random variable. This is also sometimes referred to as the “expected rate of return”. While the expected rate of return is a useful way to characterize an asset and gives us some indication of how large the returns may be, it does not capture the uncertainty in obtaining a comparable return rate to the average.

To quantify how much the rate of return deviates from the expected return and in order to capture the riskiness of the asset, we shall use the variance defined by:

$$\sigma^2 = \text{Var}(\rho) = E(|\rho - \mu|^2). \quad (3)$$

For a given collection of  $n$  assets  $\{S_1, S_2, \dots, S_n\}$ , for the  $i^{th}$  asset we denote the rate of return by  $\rho_i$  and the variance by  $\sigma_i^2$ . For a collection of assets we shall find it useful to consider, in addition, how the random rates of return are coupled.

For example, in choosing investments it is important to take into account not only the individual returns of the assets but also how the returns are coupled among the assets. A natural investment strategy to reduce risk in which an asset loses value should a given event occur, is to try to find another asset which increases in value should this event happen. This requires that the assets exhibit a coupling in which values move in opposite directions should the event occur. To quantify this for random rates of return we shall use the covariance for the returns defined by:

$$\sigma_{i,j} = E((\rho_i - \mu_i)(\rho_j - \mu_j)). \quad (4)$$

We remark that  $\sigma_{i,j} = \sigma_{j,i}$  and that when  $i = j$  we have  $\sigma_{i,i} = \sigma_i^2$ .

To describe the coupling of all  $n$  assets we define the covariance matrix by:

$$V = \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} & \cdots & \sigma_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{n,1} & \sigma_{n,2} & \cdots & \sigma_{n,n} \end{bmatrix}. \quad (5)$$

We remark that  $V$  is a symmetric matrix and can also be shown to be positive definite.

A portfolio is an investment made in  $n$  assets using some amount of wealth  $W$ . Let  $W_i$  denote the amount of money invested in the  $i^{\text{th}}$  asset. We shall allow negative values of  $W_i$ , which for example can be interpreted as *short selling* an asset. In other words, we assume a liability and must deliver the asset at a future time. Since the total wealth invested is  $W$  we have:

$$\sum_{i=1}^n W_i = W. \quad (6)$$

To avoid working with absolute magnitudes of the assets and portfolios, we shall find it convenient to instead describe the investments in terms of relative values such as the rates of return of the assets and the relative portion of wealth invested in a given asset. For the fraction of wealth invested in the  $i^{\text{th}}$  asset we make the definition:

$$w_i = \frac{W_i}{W}. \quad (7)$$

Equation 6 then implies:

$$\sum_{i=1}^n w_i = 1. \quad (8)$$

The value  $Q_p$  of the portfolio at time  $t$  can be expressed as:

$$Q_p(t) = \sum_{i=1}^n \frac{W_i}{S_i(0)} S_i(t). \quad (9)$$

where we use that the portfolio has the value  $Q_p(0) = W$  from equation 6.

The rate of return of the portfolio  $\rho_p$  at time  $t$  is given by:

$$\begin{aligned} \rho_p(t) &= \frac{Q_p(t) - Q_p(0)}{Q_p(0)} \\ &= \frac{\sum_{i=1}^n \frac{W_i}{S_i(0)} S_i(t) - W}{W} \\ &= \sum_{i=1}^n \frac{W_i}{W} \frac{S_i(t)}{S_i(0)} - \sum_{i=1}^n \frac{W_i}{W} \\ &= \sum_{i=1}^n w_i \frac{(S_i(t) - S_i(0))}{S_i(0)} \\ &= \sum_{i=1}^n w_i \rho_i. \end{aligned} \quad (10)$$

In other words, the rate of return for a portfolio is the weighted average of the rates of return of the assets where the weights are determined by the fraction of wealth invested in each asset.

The expected rate of return  $\mu_p$  of the portfolio is given by:

$$\mu_p = E \left( \sum_{i=1}^n w_i \rho_i \right) \quad (11)$$

$$= \sum_{i=1}^n w_i \mu_i. \quad (12)$$

where the linearity property of expectations has been used.

The variance  $\sigma_p^2$  for the rate of return of the portfolio is given by:

$$\begin{aligned} \sigma_p^2 &= E (|\rho_p - \mu_p|^2) \quad (13) \\ &= E \left( \left| \sum_{i=1}^n w_i (\rho_i - \mu_i) \right|^2 \right) \\ &= E \left( \left( \sum_{i=1}^n w_i (\rho_i - \mu_i) \right) \left( \sum_{j=1}^n w_j (\rho_j - \mu_j) \right) \right) \\ &= \sum_{i,j=1}^n w_i w_j E ((\rho_i - \mu_i) (\rho_j - \mu_j)) \\ &= \sum_{i,j=1}^n w_i w_j \sigma_{i,j} \\ &= \mathbf{w}^T V \mathbf{w}. \end{aligned}$$

where  $\mathbf{w}^T = [w_1, \dots, w_N]$  and  $V$  is defined in equation 5.

For a portfolio  $a$  and a portfolio  $b$  we shall quantitatively describe the coupling between two portfolios by using the covariance of the random rates of returns of the two portfolios  $\rho_p^{(a)}$  and  $\rho_p^{(b)}$ . This is given by:

$$\begin{aligned} \sigma^{(a,b)} &= E \left( \left( \rho_p^{(a)} - \mu_p^{(a)} \right) \left( \rho_p^{(b)} - \mu_p^{(b)} \right) \right) \quad (14) \\ &= E \left( \left( \sum_{i=1}^n w_i^{(a)} (\rho_i - \mu_i) \right) \left( \sum_{j=1}^n w_j^{(b)} (\rho_j - \mu_j) \right) \right) \\ &= \sum_{i,j=1}^n w_i^{(a)} w_j^{(b)} E ((\rho_i - \mu_i) (\rho_j - \mu_j)) \\ &= \sum_{i,j=1}^n w_i^{(a)} w_j^{(b)} \sigma_{i,j} \\ &= \left( \mathbf{w}_p^{(a)} \right)^T V \mathbf{w}_p^{(b)}. \end{aligned}$$

To summarize, we shall characterize the average rate of return, riskiness in obtaining comparable returns to the average, and coupling among the returns for both portfolios and individual assets. This will be done quantitatively by using, respectively, the expected rate of return, variance of the return, and covariance of the returns.

## Desirable Portfolios (Markowitz Theory)

Determining what constitutes a desirable portfolio depends on many factors. The primary factors we shall consider are the financial objectives of the investor and his or her tolerance for risk in achieving these objectives.

For example, imagine two assets such as stock of a biotechnology start-up company and blue chip stock in a company such as General Electric. The biotechnology company is a rather unproven enterprise that could go bankrupt should some important technical obstacle arise on their way to producing a product, while General Electric is a proven enterprise with a solid track record. Given these circumstances a wise investor would require some form of compensation for the riskiness of choosing the biotechnology stock over the blue chip stock. The biotechnology stock while risky likely has the potential to skyrocket in value should the enterprise carry through on its ambitious objectives, while the General Electric stock is more likely to have a more modest rate of return somewhere around its stated performance objectives.

To cast these considerations in terms of our quantitative theory, the scenario above suggests that if an investor is presented with two assets, the investor would choose the one having the larger variance only if this also entails having a larger expected return. This larger expected return acts as an incentive for the investor, we shall refer to this as a “risk premium” which compensates the investor for taking the larger risk. Another way to think about these preferences in our theory, is that if the two assets had the same expected rate of return, in other words there was no “risk premium”, then an investor would choose the one with the smallest variance (least risk).

With this understanding about the preferences of the investor, we shall consider a portfolio to be desirable if for a given expected rate of return  $\mu_p$  the portfolio has the least variance  $\sigma_p^2$ . Finding such a portfolio is referred to as the Markowitz problem and can be stated mathematically as the constrained optimization problem:

$$\begin{aligned} \text{minimize } f(w_1, \dots, w_n) &= \frac{1}{2} \sum_{i,j=1}^n w_i w_j \sigma_{i,j} & (15) \\ \text{subject:} & \\ g_1(w_1, \dots, w_n) &= \sum_{i=1}^n w_i \mu_i - \mu_p = 0 \\ g_2(w_1, \dots, w_n) &= \sum_{i=1}^n w_i - 1 = 0. & (16) \end{aligned}$$

The objective function is the variance of the portfolio, as computed in equation 13. The first constraint specifies that the constructed portfolio is to have expected rate of return  $\mu_p$  while the second constraint arises from equation 8 defining the portfolio. To solve the constrained optimization problem 15 analytically, we shall use the Method of Lagrange Multipliers. Many numerical methods also exist for these types of optimization problems.

## The Method of Lagrange Multipliers

In the Method of Lagrange Multipliers the constraints are taken into account by solving an unconstrained optimization problem for the following utility function, referred to as the Lagrangian:

$$L(w_1, \dots, w_n | \lambda_1, \lambda_2) = f(w_1, \dots, w_n) - \lambda_1 g_1(w_1, \dots, w_n) - \lambda_2 g_2(w_1, \dots, w_n). \quad (17)$$

By design a critical point of the Lagrangian is also a critical point of  $f$  and satisfies the constraints

$g_1 = 0 = g_2$ . This follows since the condition that  $(\mathbf{w}, \lambda_1, \lambda_2)$  be a critical point of  $L$  is given by:

$$\nabla_{\mathbf{w}} L = \begin{bmatrix} \frac{\partial L}{\partial w_1} \\ \frac{\partial L}{\partial w_2} \\ \vdots \\ \frac{\partial L}{\partial w_n} \end{bmatrix} = 0 \quad (18)$$

and

$$\frac{\partial L}{\partial \lambda_1} = 0, \quad \frac{\partial L}{\partial \lambda_2} = 0. \quad (19)$$

In particular, we have that

$$\frac{\partial L}{\partial \lambda_1} = g_1 = 0 \quad (20)$$

$$\frac{\partial L}{\partial \lambda_2} = g_2 = 0. \quad (21)$$

From this and equation 17, 18 we have  $\mathbf{w}$  is a critical point of  $f$ .

Geometrically, the condition is equivalent to the level surfaces determined by constant values of the utility function  $f$  meeting the level surfaces of the constraints  $g_1 = 0$  and  $g_2 = 0$  so that the normals of the surface align. In other words, moving in the permissible directions determined by the intersection of the tangent planes of  $g_1 = 0$  and  $g_2 = 0$  the value of  $f$  to first order remains constant. We shall demonstrate more concretely how this method can be used in practice in the sections below.

## Optimal Portfolios of $n$ Risky Assets

We now discuss the portfolios which are optimal in the sense that for a specified expected return the portfolio minimizes the variance of the return. To ensure a well-defined solution exists, we shall make two assumptions about the collection of assets: (i) the random returns are linearly independent, in the sense that any one return can not be expressed as a linear combination of the others, (ii) the expected return rates  $\mu_i$  of the assets are not all equal. These assumptions ensure there is a unique solution of the Markowitz problem given in equation 15.

We shall now use the Method of Lagrange Multipliers to solve the Markowitz problem analytically. The Lagrangian is given by:

$$L(\mathbf{w}, \lambda_1, \lambda_2) = \frac{1}{2} \mathbf{w}^T V \mathbf{w} + \lambda_1 (\mu_p - \mathbf{w}^T \boldsymbol{\mu}) + \lambda_2 (1 - \mathbf{w}^T \mathbf{1}). \quad (22)$$

To find the critical points of the Lagrangian, and hence the optimal portfolio, we must solve the first order equations:

$$\nabla_{\mathbf{w}} L = V \mathbf{w}_p - \lambda_1 \boldsymbol{\mu} - \lambda_2 \mathbf{1} = \mathbf{0} \quad (23)$$

and

$$\frac{\partial L}{\partial \lambda_1} = \mu_p - \mathbf{w}_p^T \boldsymbol{\mu} = 0 \quad (24)$$

$$\frac{\partial L}{\partial \lambda_2} = 1 - \mathbf{w}_p^T \mathbf{1} = 0. \quad (25)$$

From 23 we have:

$$\mathbf{w}_p = \lambda_1 (V^{-1} \boldsymbol{\mu}) + \lambda_2 (V^{-1} \mathbf{1}) \quad (26)$$

and from 24 and 25 we have:

$$(\boldsymbol{\mu}^T V^{-1} \boldsymbol{\mu}) \lambda_1 + (\boldsymbol{\mu}^T V^{-1} \mathbf{1}) \lambda_2 = \mu_p \quad (27)$$

$$(\mathbf{1}^T V^{-1} \boldsymbol{\mu}) \lambda_1 + (\mathbf{1}^T V^{-1} \mathbf{1}) \lambda_2 = 1. \quad (28)$$

Using that

$$(\boldsymbol{\mu}^T V^{-1} \mathbf{1}) = (\boldsymbol{\mu}^T V^{-1} \mathbf{1})^T = \mathbf{1}^T (V^{-1})^T \boldsymbol{\mu} = \mathbf{1}^T V^{-1} \boldsymbol{\mu} \quad (29)$$

we can express 27 and 28 as:

$$\begin{bmatrix} B & A \\ A & C \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mu_p \\ 1 \end{bmatrix} \quad (30)$$

where

$$\begin{bmatrix} B & A \\ A & C \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}^T V^{-1} \boldsymbol{\mu} & \boldsymbol{\mu}^T V^{-1} \mathbf{1} \\ \mathbf{1}^T V^{-1} \boldsymbol{\mu} & \mathbf{1}^T V^{-1} \mathbf{1} \end{bmatrix}. \quad (31)$$

To ensure there is a solute of 31 requires that the determinant be non-zero:

$$D = BC - A^2 \neq 0. \quad (32)$$

We shall now show this is indeed the case, under our assumptions. A matrix  $V$  is called *positive definite* if:

$$\mathbf{w}^T V \mathbf{w} > 0 \text{ for any } \mathbf{w} \neq \mathbf{0}. \quad (33)$$

The inverse  $V^{-1}$  of a positive definite matrix is also positive definite. We shall show  $D \neq 0$  as a consequence of the positive definiteness of  $V$ .

Let us consider the vector:

$$A\boldsymbol{\mu} - B\mathbf{1}. \quad (34)$$

The vector vanishes only if

$$0 = A\boldsymbol{\mu} - B\mathbf{1} = \boldsymbol{\mu}^T V^{-1} \mathbf{1} \boldsymbol{\mu} - \boldsymbol{\mu}^T V^{-1} \boldsymbol{\mu} \mathbf{1}. \quad (35)$$

This is ruled out, since the only solution of this equation is  $\boldsymbol{\mu} = \mathbf{1}$ , which is forbidden by the assumption (ii) that the  $\mu_i$  are not all equal. Thus under assumption (ii) we can assume that  $A\boldsymbol{\mu} - B\mathbf{1} \neq 0$ .

From the positive definiteness of  $V$  we have that  $V^{-1}$  is positive definite and that:

$$0 < (A\boldsymbol{\mu} - B\mathbf{1})^T V^{-1} (A\boldsymbol{\mu} - B\mathbf{1}) \quad (36)$$

$$= A^2 \boldsymbol{\mu}^T V^{-1} \boldsymbol{\mu} - AB \boldsymbol{\mu}^T V^{-1} \mathbf{1} - BA \mathbf{1}^T V^{-1} \boldsymbol{\mu} + B^2 \mathbf{1}^T V^{-1} \mathbf{1} \quad (37)$$

$$= B^2 C - A^2 B = B(BC - A^2). \quad (38)$$

From the fact that  $B = \boldsymbol{\mu}^T V^{-1} \boldsymbol{\mu} > 0$  by positive definiteness of  $V$  we have that  $D = BC - A^2 > 0$ .

In particular,  $D \neq 0$ , and we can invert the matrix in equation 31 to obtain:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{1}{D} \begin{bmatrix} A & -C \\ -C & B \end{bmatrix} \begin{bmatrix} \mu_p \\ 1 \end{bmatrix}. \quad (39)$$

More explicitly, this gives the solution for  $\lambda_1, \lambda_2$ :

$$\lambda_1 = \frac{C\mu_p - A}{D} \quad (40)$$

$$\lambda_2 = \frac{B - A\mu_p}{D}. \quad (41)$$

The weights of the portfolio can be found by substitution into equation 26:

$$\mathbf{w}_p = \left( \frac{C\mu_p - A}{D} \right) V^{-1}\boldsymbol{\mu} + \left( \frac{B - A\mu_p}{D} \right) V^{-1}\mathbf{1} \quad (42)$$

$$= \frac{1}{D} (BV^{-1}\mathbf{1} - AV^{-1}\boldsymbol{\mu}) + \frac{1}{D} (CV^{-1}\boldsymbol{\mu} - AV^{-1}\mathbf{1}) \mu_p. \quad (43)$$

For the definitions:

$$\mathbf{g} = \frac{1}{D} (BV^{-1}\mathbf{1} - AV^{-1}\boldsymbol{\mu}) \quad (44)$$

$$\mathbf{h} = \frac{1}{D} (CV^{-1}\boldsymbol{\mu} - AV^{-1}\mathbf{1}) \quad (45)$$

we can express this more succinctly as:

$$\mathbf{w}_p = \mathbf{g} + \mu_p \mathbf{h}. \quad (46)$$

We shall refer to the portfolio which minimizes the variance for a specified expected rate of return  $\mu_p$  as a *frontier portfolio*. We shall refer to the portfolio which has the minimum variance out of all frontier portfolios as the *minimum variance portfolio* and denote this by  $p^*$ .

The expression 46 for the solution shows explicitly how the weights of the portfolio with minimum variance depend on the desired expected rate of return  $\mu_p$ . In particular, it follows that all frontier portfolios can be attained by a linear combination of the two portfolios  $\mathbf{g}$  and  $\mathbf{h}$ . In other words, instead of buying the individual assets, in theory one could obtain an equivalent rate of return of any frontier portfolio by investing in only two mutual funds (frontier portfolios) of the market. This is known as the *two fund theorem*. In practice, however, things are more complicated since this presupposes that investors only care about the expected returns, variances, and covariances of the assets and that investors agree on the expected returns, variances, and covariances to associated with the assets.

Some interesting features of investing in portfolios, however, can be obtained from the theory. The covariance between the random rates of return for two frontier portfolios  $a$  and  $b$  can be expressed in terms of  $\mu_p^{(a)}$  and  $\mu_p^{(b)}$  as:

$$\begin{aligned} \text{cov}(\rho_p^{(a)}, \rho_p^{(b)}) &= (\mathbf{w}_p^{(a)})^T V \mathbf{w}_p^{(b)} \quad (47) \\ &= (\mathbf{w}_p^{(a)})^T V (\mathbf{g} + \mathbf{h}\mu_p^{(b)}) \\ &= (\mathbf{w}_p^{(a)})^T V \left( \frac{1}{D} (BV^{-1}\mathbf{1} - AV^{-1}\boldsymbol{\mu}) + \frac{1}{D} (CV^{-1}\boldsymbol{\mu} - AV^{-1}\mathbf{1}) \mu_p^{(b)} \right) \\ &= \frac{1}{D} (\mathbf{w}_p^{(a)})^T \left( (B\mathbf{1} - A\boldsymbol{\mu}) + (C\boldsymbol{\mu} - A\mathbf{1}) \mu_p^{(b)} \right) \\ &= \frac{C}{D} \left( \mu_p^{(a)} \mu_p^{(b)} - \frac{A}{C} (\mu_p^{(a)} + \mu_p^{(b)}) + \frac{B}{C} \right) \\ &= \frac{C}{D} \left( \left( \mu_p^{(a)} - \frac{A}{C} \right) \left( \mu_p^{(b)} - \frac{A}{C} \right) + \frac{B}{C} - \left( \frac{A}{C} \right)^2 \right) \\ &= \frac{C}{D} \left( \mu_p^{(a)} - \frac{A}{C} \right) \left( \mu_p^{(b)} - \frac{A}{C} \right) + \frac{1}{C}. \end{aligned}$$

When the frontier portfolios are identical  $a = b$  we obtain the variance:

$$\sigma_p^2 = \text{cov}(\rho_p, \rho_p) = \frac{C}{D} \left( \mu_p - \frac{A}{C} \right)^2 + \frac{1}{C}. \quad (48)$$

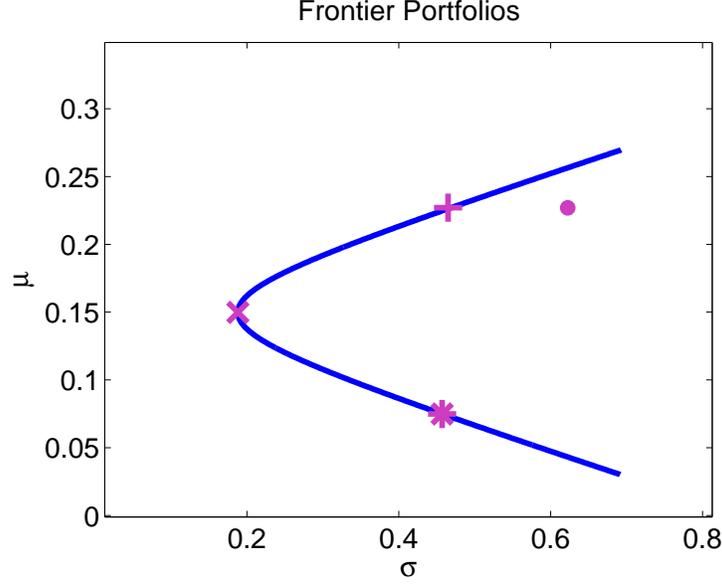


Figure 1: Frontier Portfolios for  $n$  Risky Assets. (curve) frontier portfolios, ( $\times$ ) denotes the minimum variance portfolio, ( $*$ ) denotes an inefficient frontier portfolio for given expected return  $\mu_p$ , ( $+$ ) denotes an efficient frontier portfolio for given expected return  $\mu_p$ , and ( $\cdot$ ) denotes a portfolio that does not minimize the variance for a given return.

This can equivalently be expressed as:

$$\frac{\sigma_p^2}{1/C} - \frac{(\mu_p - A/C)^2}{D/C^2} = 1. \quad (49)$$

From this, we notice that the relationship of the expected rate of return  $\mu_p$  and the variance of the frontier portfolio  $\sigma_p^2$  can be summarized as a plot in the  $(\sigma, \mu)$ -plane by a hyperbola with center  $(0, A/C)$ , asymptotes

$$\mu = \pm \sqrt{\frac{D}{C}} \sigma + \frac{A}{C}, \quad (50)$$

and vertex

$$v_0 = \left( \sqrt{\frac{1}{C}}, \frac{A}{C} \right). \quad (51)$$

From this we have that the minimum variance portfolio is given by:

$$\mathbf{w}_{p^*} = \mathbf{g} + \mathbf{h}\mu_{p^*} = \mathbf{g} + \frac{A}{C}\mathbf{h}. \quad (52)$$

The frontier portfolios and the minimum variance portfolio  $\times$  are plotted in Figure 1. From Figure 1 we see that it is possible for two portfolios with different rates of return to have the same variance  $\sigma_p^2$ . Since

both portfolios offer the same amount of risk, as quantitated by the variance, but one gives a better return, the portfolio with the greater return would be more desirable to an investor.

We shall refer to the portfolio with the lesser expected return

$$\mu_p < \mu_{p^*} = \frac{A}{C} \quad (53)$$

as *inefficient*. We shall refer to the portfolio with the greater expected return

$$\mu_p > \mu_{p^*} = \frac{A}{C} \quad (54)$$

as an *efficient* frontier portfolio. In Figure 1 we have for the variance  $\sigma_p^2 \approx 0.45$  two portfolios, the one denoted by (\*) is inefficient while the one denoted by (+) is efficient.

We now discuss a feature of the minimum variance portfolio. It can be shown that the covariance of the minimum variance portfolio  $p^*$  with any other frontier portfolio is the same. This follows from:

$$\text{cov}(\rho_{p^*}, \rho_p) = \frac{C}{D} \left( \mu_{p^*} - \frac{A}{C} \right) \left( \mu_p - \frac{A}{C} \right) + \frac{1}{C} = \frac{1}{C} = \sigma_{p^*}^2 \quad (55)$$

where we use that  $\mu_{p^*} = A/C$ . As a consequence, we see that it is not possible to find any portfolio which is completely independent of the minimum variance portfolio.

However, for any other frontier portfolio  $a$  it can be shown that there exists a portfolio  $z(a)$  having zero covariance:

$$\text{cov}(\rho_p^{(a)}, \rho_p^{(z(a))}) = 0. \quad (56)$$

This is given by solving:

$$\text{cov}(\rho_p^{(a)}, \rho_p^{(z(a))}) = \frac{C}{D} \left( \mu_p^{(a)} - \frac{A}{C} \right) \left( \mu_p^{(z(a))} - \frac{A}{C} \right) + \frac{1}{C} = 0 \quad (57)$$

which has solution:

$$\mu_p^{(z(a))} = \frac{A}{C} - \frac{D/C^2}{\left( \mu_p^{(a)} - A/C \right)}. \quad (58)$$

## Optimal Portfolios for $n$ Risky Assets + a Risk-Free Asset

We shall now consider the case in which in addition to the the  $n$  risky assets there is one risk-free asset with return  $\rho_0$ . By risk-free we mean that the rate of return of the asset has zero variance. This will make our model somewhat more realistic since in actual markets an investor always has the opportunity to invest in an essentially risk-free treasury bond or put their money in a savings account. In this case the portion of wealth invested in the  $n$  risky assets no longer satisfies  $\sum_{i=1}^n w_i = 1$  since we can always invest the remaining fraction of wealth in the risk-free asset or borrow funds. In the case that

$$\sum_{i=1}^n w_i < 1 \quad (59)$$

we say that we are *under budget* and invest the remaining portion of wealth in the risk-free asset. In the case that

$$\sum_{i=1}^n w_i > 1 \quad (60)$$

we say that we are *over budget* and borrow at the risk-free rate the excessive portion of wealth invested in the  $n$  risky assets.

This leads to a reformulation of what is meant by an optimal portfolio. In this case, an investor is no longer constrained to invest all of his or her wealth in the  $n$  risky assets. The objective then becomes for a specified expected rate of return to find the portfolio with the minimum variance subject only to the constraint that the expected return is attained. This gives:

$$\text{minimize } f(w_1, \dots, w_n) = \frac{1}{2} \mathbf{w}^T V \mathbf{w} \quad (61)$$

$$\text{subject: } g(w_1, \dots, w_n) = \rho_0 + \mathbf{w}^T (\boldsymbol{\mu} - \rho_0 \mathbf{1}) - \mu_{p^+} = 0. \quad (62)$$

The Lagrangian in this case becomes:

$$L(\mathbf{w}|\lambda) = \frac{1}{2} \mathbf{w}^T V \mathbf{w} - \lambda (\rho_0 - \mu_{p^+} + \mathbf{w}^T (\boldsymbol{\mu} - \rho_0 \mathbf{1})). \quad (63)$$

The condition that  $(\mathbf{w}, \lambda)$  be a critical point becomes:

$$\nabla_{\mathbf{w}} L = V \mathbf{w}_{p^+} - \lambda (\boldsymbol{\mu} - \rho_0 \mathbf{1}) = \mathbf{0} \quad (64)$$

$$\frac{\partial L}{\partial \lambda} = \rho_0 - \mu_{p^+} + \mathbf{w}_{p^+}^T (\boldsymbol{\mu} - \rho_0 \mathbf{1}) = 0. \quad (65)$$

Using equation 64 we have:

$$\mathbf{w}_{p^+} = \lambda V^{-1} (\boldsymbol{\mu} - \rho_0 \mathbf{1}). \quad (66)$$

From equation 65 we have:

$$\lambda = \frac{\mu_{p^+} - \rho_0}{(\boldsymbol{\mu} - \rho_0 \mathbf{1})^T V^{-1} (\boldsymbol{\mu} - \rho_0 \mathbf{1})}. \quad (67)$$

Letting

$$H := (\boldsymbol{\mu} - \rho_0 \mathbf{1})^T V^{-1} (\boldsymbol{\mu} - \rho_0 \mathbf{1}) \quad (68)$$

we obtain:

$$\mathbf{w}_{p^+} = \left( \frac{\mu_{p^+} - \rho_0}{H} \right) V^{-1} (\boldsymbol{\mu} - \rho_0 \mathbf{1}). \quad (69)$$

Now it can be shown that in fact  $H > 0$ . Multiplying out the terms in equation 68 gives:

$$H = (\mathbf{1}^T V^{-1} \mathbf{1}) \rho_0^2 - 2 (\mathbf{1}^T V^{-1} \boldsymbol{\mu}) \rho_0 - 2 (\mathbf{1}^T V^{-1} \boldsymbol{\mu}). \quad (70)$$

Using the definitions of  $A, B, C$  made in equation 30 we obtain the expression:

$$H = C \rho_0^2 - 2A \rho_0 + B \quad (71)$$

which is a quadratic in  $\rho_0$ . The value of  $H$  has the same sign for all values of  $\rho_0$  if and only if there are no real roots. No real roots occur for the quadratic only if the discriminate is negative:

$$(2A)^2 - 4CB = 4(A^2 - BC) < 0. \quad (72)$$

This can be shown to hold by the positive definiteness of  $V^{-1}$  and follows immediately from equation 36.

The portfolio with the smallest variance for the specified expected return  $\mu_{p^+}$  is then given by:

$$\begin{aligned}
\sigma_{p^+}^2 &= \mathbf{w}_{p^+}^T V \mathbf{w}_{p^+} \\
&= \left( \left( \frac{\mu_{p^+} - \rho_0}{H} \right) V^{-1} (\boldsymbol{\mu} - \rho_0 \mathbf{1}) \right)^T V \left( \frac{\mu_{p^+} - \rho_0}{H} \right) V^{-1} (\boldsymbol{\mu} - \rho_0 \mathbf{1}) \\
&= \left( \frac{\mu_{p^+} - \rho_0}{H} \right)^2 (\boldsymbol{\mu} - \rho_0 \mathbf{1})^T V^{-1} V V^{-1} (\boldsymbol{\mu} - \rho_0 \mathbf{1}) \\
&= (\boldsymbol{\mu} - \rho_0 \mathbf{1})^T \left( \frac{\mu_{p^+} - \rho_0}{H} \right)^2 (\boldsymbol{\mu} - \rho_0 \mathbf{1}) \\
&= \frac{(\mu_{p^+} - \rho_0)^2}{H}.
\end{aligned} \tag{73}$$

This can be expressed by:

$$\sigma_{p^+} = \frac{|\mu_{p^+} - \rho_0|}{\sqrt{H}}. \tag{74}$$

The relationship between the variance and expected returns of the portfolio can be summarized in the  $(\sigma, \mu)$ -plane as two half-lines with slopes  $\pm\sqrt{H}$  intersecting the point  $(0, \rho_0)$ .

For any frontier portfolio  $a$ , other than the one with  $\mu_{p^+} = A/C$ , a portfolio  $z(a)$  having zero covariance with  $a$  can be found. The expected return of this portfolio is:

$$\mu_{p^+}^{z(a)} = \frac{A}{C} - \frac{D/C^2}{\mu_{p^+}^{(a)} - A/C} \tag{75}$$

and has investment weights:

$$\mathbf{w}_{p^+}^{z(a)} = \mathbf{g} + \mathbf{h} \mu_{p^+}^{z(a)}. \tag{76}$$

Geometrically, the return  $\mu_{p^+}^{z(a)}$  of the zero-covariance portfolio  $z(a)$  corresponds to the  $\mu$ -intercept of the tangent line at the point  $(\mu_{p^+}^{(a)}, \sigma_{p^+}^{(a)})$  of the hyperbola. In other words, to find the expected return of the zero-covariance portfolio of  $a$ , we draw, starting from the point corresponding to the portfolio  $a$ , the tangent line and determine where this line crosses the  $\mu$  axis. The corresponding variance  $\sigma_{p^+}^{z(a)}$  to the portfolio  $z(a)$  is then found by drawing a horizontal line in the  $\sigma$  direction and determining the intersection with the frontier hyperbola, see Figure 2.

We now show that there is an interesting geometric relationship between the frontier portfolios consisting entirely of the  $n$  risky assets and the frontier portfolios which include the risk-free asset. In particular, when the risk-free rate is smaller than the expected return of the minimum variance portfolio  $p^*$ ,  $\rho_0 < A/C$ , the half-line of efficient frontier portfolios defined by equation 8 intersects as a tangent line the hyperbolic curve of frontier portfolios. This has as an important financial consequence that all efficient frontier portfolios made of  $n$  assets and a risk-free asset can be attained by a linear combination of the risk-free asset alone and some frontier portfolio consisting only of the  $n$  risky assets. This is referred to as the *one fund theorem*. We remark that the condition  $\rho_0 < A/C$  can be interpreted financially as asserting that the rate of return of the risk-free asset be less than the least risky portfolio consisting purely of the  $n$  risky assets. This agrees with our intuition that an investor should be compensated for taking risks. We shall now show that the statements above indeed hold for the frontier portfolios constructed from the  $n$  risky assets and the risk-free asset.

To find the frontier portfolio  $a$  which gives the point of intersection with the hyperbolic frontier curve, we shall consider the zero-covariance portfolio  $z(a)$  which has expected return  $\mu_a = \rho_0$ . This requires:

$$\mu_{p^+}^{z(a)} = \frac{A}{C} - \frac{D/C^2}{\left( \mu_{p^+}^{(a)} - A/C \right)} = \rho_0. \tag{77}$$

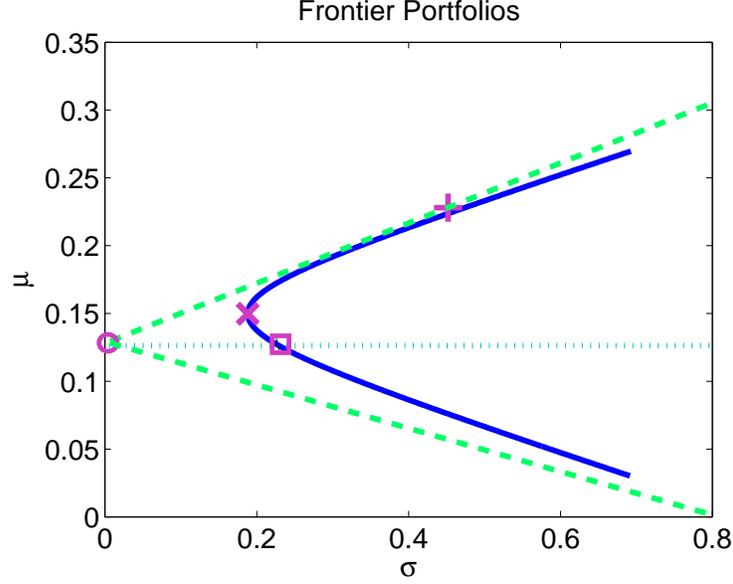


Figure 2: Frontier Portfolios for  $n$  Risky Assets + Risk-Free Asset. (curve) frontier portfolios consisting of only risky assets, ( $\times$ ) denotes the minimum variance portfolio, ( $+$ ) denotes an efficient frontier portfolio for given expected return  $\mu_p^{(a)}$ , ( $o$ ) denotes the  $\mu$ -intercept of the tangent line through  $(\sigma_p^{(a)}, \mu_p^{(a)})$ , and ( $\square$ ) denotes the portfolio  $z(a)$  with zero-covariance with  $a$ .

Solving for  $\mu_{p^+}^{(a)}$  gives:

$$\mu_{p^+}^{(a)} = \frac{A}{C} - \frac{D/C^2}{\rho_0 - A/C}. \quad (78)$$

The variance corresponding to this portfolio is then given by equation 73, which gives:

$$\begin{aligned} \sigma_{p^+}^2 &= \frac{D}{C^2} \left( \frac{1}{C(\rho_0 - A/C)^2} \right) + \frac{C}{D} \\ &= \frac{D}{C^2} \left( \frac{D + C^2(\rho_0 - A/C)^2}{CD(\rho_0 - A/C)^2} \right) \\ &= \frac{1}{C^2} \left( \frac{C^2 \left( \rho_0^2 - 2\frac{A}{C}\rho_0 + \frac{A^2}{C^2} \right) + D}{C \left( \rho_0 - \frac{A}{C} \right)^2} \right) \\ &= \frac{1}{C^2} \left( \frac{C\rho_0^2 - 2A\rho_0 + \left( \frac{A^2}{C} + \frac{D}{C} \right)}{\left( \rho_0 - \frac{A}{C} \right)^2} \right). \end{aligned} \quad (79)$$

From equation 32 we have that  $D = BC - A^2 \neq 0$ . Dividing the numerator by one of the factors of  $C$  we

obtain

$$C\rho_0^2 - 2A\rho_0 + B = H \quad (80)$$

where  $H$  was defined in equation 68.

Since  $\rho_0 < A/C$  we have that

$$\sigma_{p^+}^{(a)} = \pm \frac{\sqrt{H}}{C|\rho_0 - \frac{A}{C}|}. \quad (81)$$

We have now determined a point on the hyperbolic frontier curve which has a tangent line intersecting  $(0, \rho_0)$ . To validate the claims made above, we must show that this tangent line in fact coincides with the half-line of frontier portfolios which include the risk-free asset. In particular, we must check that the tangent line has slope  $\sqrt{H}$ .

The slope of the line passing through  $(0, \rho_0)$  and  $(\sigma_{p^+}^{(a)}, \mu_{p^+}^{(a)})$  is given by:

$$\frac{\mu_{p^+}^{(a)} - \rho_0}{\sigma_{p^+}^{(a)}}. \quad (82)$$

We shall first compute:

$$\begin{aligned} \mu_{p^+}^{(a)} - \rho_0 &= \frac{A}{C} - \frac{D/C^2}{(\rho_0 - \frac{A}{C})} - \rho_0 \\ &= \frac{1}{C(\rho_0 - \frac{A}{C})} \left( A \left( \rho_0 - \frac{A}{C} \right) - \frac{D}{C} - C\rho_0 \left( \rho_0 - \frac{A}{C} \right) \right) \\ &= \frac{-1}{C(\rho_0 - \frac{A}{C})} \left( C\rho_0^2 - 2A\rho_0 + \left( \frac{A^2 + D}{C} \right) \right) \\ &= \frac{-1}{C(\rho_0 - \frac{A}{C})} (C\rho_0^2 - 2A\rho_0 + B) \\ &= \frac{-H}{C(\rho_0 - \frac{A}{C})} \end{aligned} \quad (83)$$

where we have used that:

$$B = \frac{A^2 + D}{C} \quad (84)$$

and

$$H = C\rho_0^2 - 2A\rho_0 + B. \quad (85)$$

We can now compute the slope using this and equation 79 to obtain:

$$\frac{\mu_{p^+}^{(a)} - \rho_0}{\sigma_{p^+}^{(a)}} = \frac{\frac{-H}{C(\rho_0 - \frac{A}{C})}}{\frac{-\sqrt{H}}{C(\rho_0 - \frac{A}{C})}} = \sqrt{H}. \quad (86)$$

This shows that the tangent line is the frontier curve for the portfolios constructed from the  $n$  risky assets and risk-free asset. Thus the collection of all frontier portfolios consisting of  $n$  risky assets and a risk-free asset can be obtained by investing in the risk-free asset along and the portfolio  $a$ , confirming the *one fund theorem*.

## Conclusions

In these notes we have shown how a quantitative theory for portfolio management can be developed using the expected return to model an investors financial objectives and variance to quantify risk. To use this theory in practice requires a significant amount of information about the assets. For example, one must somehow decide what expected return to assign to a given asset and still more challenging how to estimate the variances and covariances of the assets. A natural approach would be to use the past history of an asset, but changing economic conditions may make this a poor indicator of future returns for many types of assets. For further discussion of these issues and how portfolio theory may be applied in practice see the reference *Investments* by Bodie, Kane, and Marcus and the other references below.

## References

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