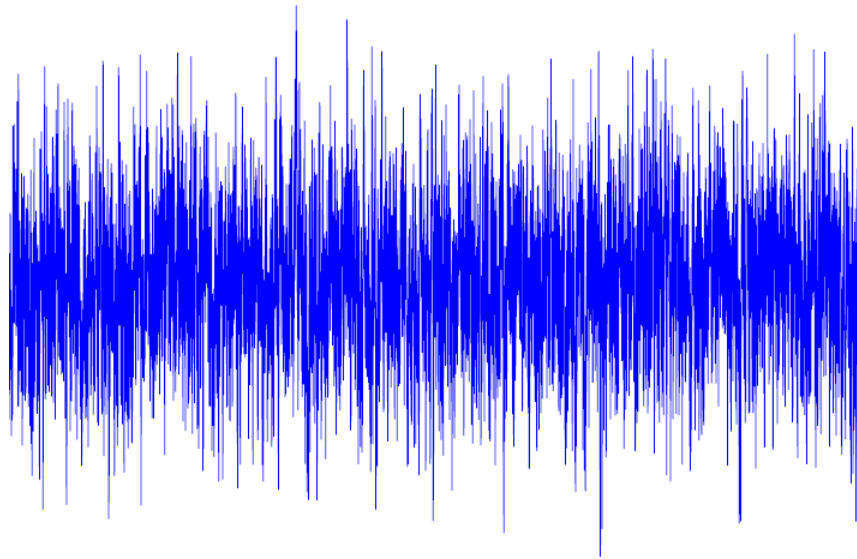


Stochastic Volatility (Draft I)

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Introduction

When using the Black-Scholes-Merton model to price derivative contracts the volatility σ of the underlying asset (stock) must be specified. Once this parameter is determined, the price of any contingent claim on the asset can in principle then be determined by applying the Black-Scholes-Merton formula for the given value of volatility σ . However, in practice, when considering the implied volatility σ in the Black-Scholes-Merton Model required to match contract prices realized in the marketplace, it is found that significantly different volatilities are required even when the contracts depend on the same underlying asset. This suggests that the lognormal asset price dynamics assumed in the Black-Scholes-Merton model is insufficient to fully capture the asset price dynamics occurring in the marketplace. In particular, it is found that for call options that the volatility changes for strike prices significantly larger or smaller than the current spot price of the asset, these features are referred to as the volatility smile and volatility skew.

The Black-Scholes-Merton theory can be extended to capture many features of the volatility smile and skew by introducing a more sophisticated dynamics for the underlying asset. One natural approach is to allow for the volatility of the underlying asset to evolve according to its own stochastic dynamics. This then presents a number of interesting issues, including how the model parameters should be calibrated to the marketplace and how contracts valued under the model should be hedged in practice. In these notes we discuss some basic stochastic models of the volatility. We then discuss the pricing of contingent claims whose payoffs depend not only on the underlying asset price, but also possibly on the realized variance or volatility of the asset. The material presented in these notes draws heavily on the lectures given in the Fall 2005 semester at the Courant Institute of Mathematical Sciences, New York University by Jim Gatheral of Merrill Lynch.

Stochastic Volatility Models

In this section we present a general class of stochastic volatility models for which a valuation formula can be derived. Let us consider as our asset price and stochastic volatility model the general class of stochastic processes satisfying the SDE's:

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sqrt{v_t} S_t dB_t^{(1)} \\ dv_t &= \alpha(S_t, v_t) dt + \eta \beta(S_t, v_t) \sqrt{v_t} B_t^{(2)} \end{aligned}$$

where the Brownian motions have correlation $\langle dB^{(1)} dB^{(2)} \rangle = \rho dt$.

We can now proceed along lines similar to the hedging arguments used in deriving the Black-Scholes-Merton formula in order to form a risk-free portfolio. Let $V(S, v, t)$ denote the value of a contingent claim depending on the current spot price S and spot variance v at time t . Now since there are two sources of randomness in the model we must hedge with at least two financial assets not having completely correlated sensitivities to S and v . The first natural choice is to use the underlying stock S , while the second which we denote by V_1 is much less obvious and an important issue in practice. Now let us form the hedging portfolio

$$\Pi = V - \Delta \cdot S - \Delta_1 \cdot V_1.$$

An expression for the change in value of the portfolio Π over an interval in time dt can be obtained by using Ito's Lemma:

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \rho \eta v \beta S \frac{\partial^2 V}{\partial S \partial v} + \frac{1}{2} \eta^2 v \beta^2 \frac{\partial^2 V}{\partial v^2} \right) dt \\ &- \Delta_1 \left(\frac{\partial V_1}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho \eta v \beta S \frac{\partial^2 V_1}{\partial S \partial v} + \frac{1}{2} \eta^2 v \beta^2 \frac{\partial^2 V_1}{\partial v^2} \right) dt \\ &+ \left(\frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta \right) dS \\ &+ \left(\frac{\partial V}{\partial v} - \Delta_1 \frac{\partial V_1}{\partial v} \right) dv. \end{aligned}$$

The risk can be hedged away to leading order by setting the coefficients of dS and dv to zero. This can be obtained by setting Δ and Δ_1 to:

$$\begin{aligned}\Delta &= \frac{\partial V}{\partial S} - \frac{\partial V_1}{\partial S} \Delta_1 \\ \Delta_1 &= \frac{\frac{\partial V}{\partial v}}{\frac{\partial V_1}{\partial v}}.\end{aligned}$$

Now with this choice of Δ and Δ_1 the change in value of the portfolio $d\Pi$ is deterministic to leading order. Since investing in the portfolio gives a deterministic rate of return by the principles of no arbitrage it must have as its return the risk-free interest rate r . Expressing this in terms of the change in value of the portfolio gives:

$$\begin{aligned}d\Pi &= r\Pi dt \\ &= r(V - \Delta \cdot S - \Delta_1 \cdot V_1) dt.\end{aligned}$$

We can now deduce a valuation formula for the contingent claim by equating this with the expression for $d\Pi$ above. This gives the following differential equation for the option:

$$\begin{aligned}& \frac{\frac{\partial V}{\partial t} + \frac{1}{2}vS^2 \frac{\partial^2 V}{\partial S^2} + \rho\eta v\beta S \frac{\partial^2 V}{\partial S \partial v} + \frac{1}{2}\eta^2 v\beta^2 \frac{\partial^2 V}{\partial v^2} + rS \frac{\partial V}{\partial S} - rV}{\frac{\partial V}{\partial v}} \\ &= \frac{\frac{\partial V_1}{\partial t} + \frac{1}{2}vS^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\eta v\beta S \frac{\partial^2 V_1}{\partial S \partial v} + \frac{1}{2}\eta^2 v\beta^2 \frac{\partial^2 V_1}{\partial v^2} + rS \frac{\partial V_1}{\partial S} - rV_1}{\frac{\partial V_1}{\partial v}}\end{aligned}$$

Since V and V_1 are distinct functions this requires that the left and right hand sides must be equal to a function of only the underlying parameters S, v, t . This gives the valuation formula for V :

$$\frac{\partial V}{\partial t} + \frac{1}{2}vS^2 \frac{\partial^2 V}{\partial S^2} + \rho\eta v\beta S \frac{\partial^2 V}{\partial S \partial v} + \frac{1}{2}\eta^2 v\beta^2 \frac{\partial^2 V}{\partial v^2} + rS \frac{\partial V}{\partial S} - rV = g(s, v, t) \frac{\partial V}{\partial v}.$$

Conventionally the function on the RHS is expressed as $g(s, v, t) = \alpha - \psi\beta$, where ψ is interpreted as the market price of volatility risk.

A particular model for which much analysis has been done is the Heston Stochastic Volatility Model:

$$\begin{aligned}\frac{dS_t}{S_t} &= rdt + \sqrt{v_t} dB_t^{(1)} \\ dv_t &= -\lambda(v_t - \bar{v}) dt + \eta\sqrt{v_t} dB_t^{(2)}.\end{aligned}$$

In this model the variance v_t (volatility $\sqrt{v_t}$) is modeled by a mean-reverting process. The parameter λ then gives the time scale $\frac{1}{\lambda}$ for the reversion of v_t to the asymptotic variance \bar{v} . The parameter η is then the "volatility of volatility" and the Black-Scholes-Merton model is recovered with volatility $\sqrt{\bar{v}}$ in the limit $\eta \rightarrow 0$ or $\lambda \rightarrow \infty$. For this model economic arguments can be made which indicate that the market price of volatility risk is proportional to the variance $\psi = \theta v$. The valuation equation is then:

$$\frac{\partial V}{\partial t} + \frac{1}{2}vS^2 \frac{\partial^2 V}{\partial S^2} + \rho\eta vS \frac{\partial^2 V}{\partial S \partial v} + \frac{1}{2}\eta^2 v \frac{\partial^2 V}{\partial v^2} + rS \frac{\partial V}{\partial S} - rV = \lambda'(v - \bar{v}) \frac{\partial V}{\partial v}$$

where $\lambda' = \lambda - \theta$ and $\lambda'\bar{v}' = \lambda\bar{v}$.

Now in principle options depending on the underlying asset S_T and possibly even the variance v_T can be priced by developing a numerical scheme for the PDE and working backward in time from the payoff at maturity $f(S_T, v_T, T)$. However, in practice this price is not as readily justified as in the Black-Scholes-Merton case since the variance v_t is not a tradable asset in the marketplace and must somehow be dynamically

hedged. The model also has many parameters, which include $\rho, \eta, \lambda, \bar{v}, \theta$, that must be calibrated so as to model the marketplace. The choice of these parameters may significantly influence the prices obtained for the options, especially those depending directly on v_t . The issue of how to determine appropriate parameters which capture features of the marketplace, in particular the observed Black-Scholes-Merton implied volatility surface, is an active area of research. We shall leave as a discussion of the important issue of how to calibrate the model to another time. In these notes we shall discuss primarily how the model can be used mathematically to price options. In some special cases we shall discuss how these prices can be justified (hedged) by using assets available in the marketplace.

Pricing Options in the Heston Model

Let $x = \log\left(\frac{F_t}{K}\right)$, where the forward price is $F_t = S_t e^{r(T-t)}$, and let $\tau = T - t$. Then the valuation equation becomes:

$$-\frac{\partial V}{\partial \tau} + \frac{1}{2}vV_{xx} - \frac{1}{2}vV_x + \frac{1}{2}\eta^2vV_{vv} + \rho\eta vV_{xv} - \lambda'(v - \bar{v}')V_v = 0$$

where $V_x = \frac{\partial V}{\partial x}$ and $V_v = \frac{\partial V}{\partial v}$. An important consequence of this change of variable is that the PDE now has coefficients which are constant with respect to x , that is they only depend on v . This allows for the Fourier Transform to be applied which converts derivatives in x to multiplication, leaving only derivatives in τ and v . The Fourier Transform is defined by:

$$\hat{V}(k, v, \tau) = \int_{-\infty}^{\infty} e^{-ikx} V(x, v, \tau) dx$$

with inverse:

$$V(x, v, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{V}(k, v, \tau) dk.$$

Applying the Fourier Transform to the valuation PDE gives:

$$-\frac{\partial \hat{V}}{\partial \tau} + \frac{1}{2}k^2v\hat{V} - \frac{1}{2}ikv\hat{V} + \frac{1}{2}\eta^2v\hat{V}_{vv} + \rho\eta ikvV_v - \lambda'(v - \bar{v}')V_v = 0.$$

By grouping common terms in v this becomes:

$$v \left(\alpha \hat{V} - \beta \hat{V}_v + \gamma \hat{V}_{vv} \right) + \lambda' \bar{v}' \hat{V}_v - \frac{\partial \hat{V}}{\partial \tau} = 0$$

where $\alpha = -\frac{k^2}{2} - \frac{ik}{2}$, $\beta = \lambda' - \rho\eta ik$, $\gamma = \frac{\eta^2}{2}$.

To find a solution to this equation consider let us consider the following form for the solution:

$$\tilde{V}(k, v, \tau) = 2\pi \exp(C(k, \tau)\bar{v}' + D(k, \tau)v) \hat{V}(k, v, 0).$$

and make this substitution for \hat{V} above. This gives that

$$\begin{aligned} \frac{\partial \tilde{V}}{\partial \tau} &= \left(\bar{v}' \frac{\partial C}{\partial \tau} + v \frac{\partial D}{\partial \tau} \right) \tilde{V} \\ \tilde{V}_v &= D \tilde{V} \\ \tilde{V}_{vv} &= D^2 \tilde{V}. \end{aligned}$$

This substitution in effect converts differentiation in v to multiplication and reduces the entire system to a system of ODE's:

$$\begin{aligned} \frac{\partial C}{\partial \tau} &= \lambda' D \\ \frac{\partial D}{\partial \tau} &= \alpha - \beta D + \gamma D^2 = \gamma(D - r_+)(D - r_-) \end{aligned}$$

where

$$r_{\pm} = \frac{\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\gamma} = \frac{\beta \pm \delta}{\eta^2}.$$

The equations can be integrated with $(C(k,0) = 0, D(k,0) = 0)$ to obtain:

$$\begin{aligned} D(k, \tau) &= r_- \frac{1 - e^{-\delta\tau}}{1 - ge^{-\delta\tau}} \\ C(k, \tau) &= \lambda' \left(r_- r_+ - \frac{2}{\eta^2} \log \left(\frac{1 - ge^{-\delta\tau}}{1 - g} \right) \right) \end{aligned}$$

where $g = \frac{r_-}{r_+}$.

From equation 1 and using the determined C and D from above we obtain using the Inverse Fourier Transform the "risk-neutral" valuation formula for $V(x, v, \tau)$:

$$V(x, v, \tau) = \int_{-\infty}^{\infty} \phi_T(k, v, \tau) \hat{V}(k, v, 0) e^{ikx} dk$$

where

$$\phi_T(k, v, \tau) = \exp(C(k, \tau)\bar{v}' + D(k, \tau)v).$$

From this expression a numerical approach for estimating the option value can be obtained by approximating the payoff function by evaluation at a finite number of lattice sites and using the Fast Discrete Fourier Transform, see (1).

Furthermore, since this holds for any payoff function $V(x, v, 0)$ we have that the characteristic function of the "risk-neutral" probability in x is $\phi_T(k, v, \tau)$ (for a fixed v and τ). To see this let $V(x, v, 0) = \theta(x - x_0)$ where θ is the Heaviside function defined by $\theta(y) = 0$ for $y \leq 0$ and $\theta(y) = 1$ for $y > 0$. The Fourier Transform is $\hat{V}(k, v, 0) = \hat{\theta}(k) = " - \frac{1}{ik} "$. This formally gives:

$$\Pr\{x_T > x_0\} = V(x, v, \tau) = \int_{-\infty}^{\infty} \exp(C(k, \tau)\bar{v}' + D(k, \tau)v + ikx_0) \frac{1}{-ik} dk.$$

The "risk-neutral" probability density is $\rho(x_0, v, \tau) = -\frac{\partial \Pr\{x_T > x_0\}}{\partial x_0}$. Differentiating the expression above gives:

$$\rho(x, v, \tau) = \int_{-\infty}^{\infty} \exp(C(k, \tau)\bar{v}' + D(k, \tau)v) e^{ikx} dk$$

showing formally that $\hat{\rho}(k, v, \tau) = \phi_T(k, v, \tau)$ by the invertibility of the Fourier Transform.

Realized Variance and Realized Volatility

In the stochastic volatility model the variance processes is designed to capture features of the fluctuations of observed asset prices in the marketplace, but is not strictly speaking directly observable. One approach is to make the definition that the "realized variance" in the marketplace is:

$$\sigma_R^2(T) = \frac{1}{N} \sum_{j=1}^N \left[\log \left(\frac{S_{t_j + \Delta t}}{S_{t_j}} \right) \right]^2$$

with the corresponding "realized volatility":

$$\sigma_R(T) = \left(\frac{1}{N} \sum_{j=1}^N \left[\log \left(\frac{S_{t_j + \Delta t}}{S_{t_j}} \right) \right]^2 \right)^{1/2}$$

where time is discretized with $t_j = j\Delta t$.

If the assets are assumed to undergo lognormal stochastic dynamics (with a piecewise continuous stochastic volatility) then it can be shown that in the limit $\Delta t \rightarrow 0$:

$$\sigma_R^2(T) = \frac{1}{T} \int_0^T \sigma_t^2 dt$$

and

$$\sigma_R(T) = \left(\frac{1}{T} \int_0^T \sigma_t^2 dt \right)^{1/2}.$$

where σ_t is the continuous time limit of the volatility. For example, if the dynamics are assumed to be those of the Heston model then $\sigma_t^2 = v_t$.

Pricing a Variance Swap

A variance swap is a contingent claim which pays the owner the difference between the realized variance and some strike variance level:

$$\text{Payoff} = \mathcal{N} \cdot (\sigma_R^2(T) - K_v)$$

where \mathcal{N} is the numeraire (for example converting variance units to dollars). For simplicity, we shall always take $\mathcal{N} = \$1$ per unit variance. We now discuss one approach to determining the "fair price" of variance K_v , which makes the contract have zero value to each party participating in the swap.

In the case of a forward contract we found that the "fair price" could be determined by an arbitrage argument which gave a static position in the underlying asset and a bond. Replicating the payoff with a static position was made possible by the linearity of the payoff function. In principle we can try to proceed along similar lines for the variance swap, but, we can not buy as readily the realized variance $\sigma_R^2(T)$ since it is not a trading asset in the marketplace. However, while the variance is not directly traded one approach is to attempt to construct a portfolio of traded assets which closely tracks $\sigma_R^2(T)$. This requires that the portfolio have the feature that it is sensitive to fluctuations of the assets' values but does not give exposure to any particular value of the assets ($\Delta = 0, \Gamma \neq 0$). We shall now make this more clear.

Let us suppose that the asset prices undergo lognormal dynamics, with stochastic volatility σ_t :

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dB_t.$$

Then making the change of variable $X_t = \log(S_t/S_0)$ and applying Ito's Lemma we have:

$$d \log(S_t/S_0) = \left(\mu_t - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t dB_t.$$

Now subtracting we have:

$$\frac{dS_t}{S_t} - d \log(S_t/S_0) = \frac{1}{2} \sigma_t^2 dt.$$

This gives an expression for the realized variance:

$$\sigma_R^2(T) = \frac{2}{T} \left(\int_0^T \frac{dS_t}{S_t} - \log(S_T/S_0) \right).$$

The first term can be interpreted financially as a dynamic position in which at each instant in time \$1 is invested in the underlying asset. The second term can be interpreted financially as selling short a contingent

claim with maturity T which has a logarithm payoff of the underlying asset. Since contracts paying the logarithm of an asset may not be readily available in the marketplace we seek a portfolio in terms of the more standard call and put options. Any piecewise linear payoff function can be realized exactly by a finite number of call and put options by an appropriate choice of strikes K and by making investments with the appropriate weights in each contract. Therefore, any continuous payoff function can be approximated arbitrarily well by a finite (possibly large) number of call and put options. In the limit of using an infinite number of options the payoff can be replicated exactly. For any twice continuously differentiable payoff function we have the following representation formula:

$$f(S) = f(S^*) + f'(S^*)(S - S^*) + \int_0^{S^*} f''(K)(K - S)^+ dK + \int_{S^*}^{\infty} f''(K)(S - K)^+ dK$$

where $S^* \in [0, \infty)$ is an arbitrary value. This gives for the logarithm contingent claim at maturity:

$$-\log(S_T/S_0) = -\log(S^*/S_0) - \frac{S_T - S^*}{S^*} + \int_0^{S^*} \frac{1}{K^2}(K - S_T)^+ dK + \int_{S^*}^{\infty} \frac{1}{K^2}(S_T - K)^+ dK.$$

The realized variance can then be replicated by the portfolio having value at maturity T :

$$\sigma_R^2(T) = \frac{2}{T} \left(\int_0^T \frac{dS_t}{S_t} - \log\left(\frac{S^*}{S_0}\right) - \left(\frac{S_T}{S^*} - 1\right) + \int_0^{S^*} \frac{1}{K^2}(K - S_T)^+ dK + \int_{S^*}^{\infty} \frac{1}{K^2}(S_T - K)^+ dK \right).$$

We remark that each term now corresponds either to a position in a bond, forward contract, or call or put option. The fair value of the variance swap K_v at time 0 is then given by the forward value of the portfolio above at time 0 with maturity at time T :

$$K_v = \frac{2}{T} \left(rT - \log\left(\frac{S^*}{S_0}\right) - \left(\frac{S_0}{S^*}e^{rT} - 1\right) + e^{rT} \int_0^{S^*} \frac{1}{K^2}P(K, T)dK + e^{rT} \int_{S^*}^{\infty} \frac{1}{K^2}C(K, T)dK \right)$$

where $P(K, T)$, $C(K, T)$ are short-hand for put and call options at time 0 with strike K having maturity T .

In practice an important issue is that in the marketplace call and put options will only be available for a finite number of strike prices K and maturities T . This means that prices (implied volatilities) for the calls and puts not directly available in the marketplace must somehow be estimated to obtain K_v . One approach is to calibrate the Heston model to the market for the available call and put options and to use the call and put prices (implied volatilities) under the model to "interpolate" the known values. These "interpolated" values under the model could then in principle be used to form positions which replicate the options for the strikes needed in the representation formula above. However, in practice this must be carefully checked to make sure sensible prices (implied volatilities) are obtained, especially for strikes that give options significantly out-of-the-money. For a further discussion of these issues and the hedging of variance swaps, see (2; 3; 6).

Pricing Volatility Swaps

A volatility swap is a contract which has a payoff based on the difference between the realized volatility and some strike volatility level:

$$\text{Payoff} = \mathcal{N} \cdot (\sigma_R(T) - K_{\sqrt{v}})$$

where $K_{\sqrt{v}}$ is the strike volatility and \mathcal{N} is the numeraire (for example converting volatility units to dollars such as $\mathcal{N} = \$1$ per unit volatility).

Pricing of a volatility swap presents significantly more difficulty than the variance swap. The "realized volatility" is the square root (a non-linear function) of the "realized variance". So even though the "realized variance" can be replicated in principle by a portfolio of the underlying asset, bond, and options to make use

of this portfolio for the "realized volatility" would require a dynamically adjusted position in the portfolio of the "realized variance" which would be difficult to carry out in practice. A price can be obtained in principle from a stochastic volatility model, such as the Heston model calibrated to the marketplace, but such models make pretty strict assumptions about how the volatility will evolve in time. In order to trade such options, practitioners would ideally like to have a robust hedging strategy to replicate the payoff of the contracts. How to perform such hedging of volatility options is at present an active area of research, see (3; 5–12).

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