Finite Difference Methods and Von Neumann Analysis

Many partial differential equations can not be solved with closed form solutions. Obtaining analytic solutions becomes especially challenging when considering general functions and geometries. As an alternative we can seek discrete models which approximate the solutions of the partial differential equation while yielding algorithms amenable to efficient computational implementation. A widely used strategy for this purpose is to use *finite difference* approximations to derivatives.

Finite Difference Methods. In finite difference methods a function u is sampled on a lattice x_m to obtain the values $u_m = u(x_m)$. The derivatives of the function appearing in the PDE are approximated by using differences of these values on the lattice. For example, consider the advection equation

$$u_t + au_x = 0, \quad u(x,0) = \phi(x).$$

The function is approximated by representing its values on the lattice (x_m, t_n), where $x_m = m\delta x$ and $t_n = n\delta t$. We denote the function values by $u_m^n = u(x_m, t_n)$. There are many ways we could approximate the derivatives by difference quotients. One such choice is to use

$$\begin{aligned} u_t &= \frac{\partial u}{\partial t}(x_m, t_n) \approx \frac{u_m^{n+1} - u_m^n}{\delta t} \\ u_x &= \frac{\partial u}{\partial x}(x_m, t_n) \approx \frac{u_m^n - u_{m-1}^n}{\delta x}. \end{aligned}$$

Since the lattice spacing δt, δx is held fixed, instead of taking the limit to zero as one would do to obtain the derivatives in calculus, this is referred to as a *finite difference approximation.* By Taylor expansion we can see these each would converge to the derivative as $\delta t \to 0$ and $\delta x \to 0$. This yields the following discrete model v_m^n aiming to mimic the behavior of the PDE

$$\frac{v_m^{n+1} - v_m^n}{\delta t} = a\frac{v_m^n - v_{m-1}^n}{\delta x}.$$

This can be expressed as the recurrence

$$v_m^{n+1} = (1 - a\lambda)v_m^n - a\lambda v_{m-1}^n,$$

where $\lambda = \delta t/\delta x$. We notice something already distinct about this choice, where the value v_m^n is constructed only by using values from the left in space v_m^n, v_{m-1}^n. In other words, information from the initial conditions only propagates with this discretization from left to right. Thinking about the method of characteristics we have information propagate from left to right when $a > 0$ and from right to left when $a < 0$. As we will discuss in more detail shortly, this scheme is most promising when $a > 0$, and a bit suspect when $a < 0$. Luckily, there are also many other choices for discretizations and as we shall see some choices work better in practice than others.

Von Neumann Analysis. An important issue we need to address is how to determine if the discretization is likely to work well in practice or if it inherently has poor properties in attempting to approximate the PDE. As we already saw, it is often relatively easy to construct discretizations that are *consistent* with the derivatives. However, what will be crucial is that the numerical scheme is *stable* in the sense that

$$\|v^n\|_2 \leq C_T \|v^0\|_2.$$

We use here the lattice ℓ^2-norm given by

$$\|v^n\|_2^2 = \sum_m |v_m^n|^2.$$

The constant C_T only depends on a fixed time T. For n we require that $n\delta t = T$ for a fixed T. The stability condition requires that as $\delta t \to 0$, which has $n = T/\delta t \to \infty$, that there exists a constant C_T establishing the inequality above. This requires that we show that v^n maintains a magnitude that does not grow unbounded as $n \to \infty$.

For this purpose, we will use fourier methods and Parseval's Lemma. We will leverage the continuous fourier transforms we previously developed but reverse the roles of the frequency and spatial components. We will use the fourier transforms

$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} v_m \exp(i2\pi\xi m), \quad \xi \in [-\pi, \pi],$$

and

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{v}(\xi) \exp(-i2\pi\xi m) d\xi.$$

The above assumes that the lattice spacing is one. In the case when $x_m = m\delta x = mh$, with $h = \delta x$, we have

$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} v_m \exp(i2\pi h\xi m) h, \quad \xi \in [-\pi/h, \pi/h],$$

and

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} \hat{v}(\xi) \exp(-i2\pi h\xi m) d\xi.$$

The Parseval's Lemma states that

$$\|\hat{v}\|_2 = \|v\|_2.$$

We use for general h the norms

$$\|\hat{v}\|_2^2 = \int_{-\pi/h}^{\pi/h} |\hat{v}(\xi)|^2 d\xi, \quad \|v\|_2^2 = \sum_m |v_m|^2 h.$$

We can now use these results to analyze the stability of our finite difference scheme above,

$$v_m^{n+1} = (1 - a\lambda) v_m^n - a\lambda v_{m-1}^n.$$

We can express each term as

$$\begin{aligned} v_m^{n+1} &= \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} \hat{v}^{n+1}(\xi) \exp(-i2\pi h\xi m) d\xi \\ v_m^n &= \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} \hat{v}^{n}(\xi) \exp(-i2\pi h\xi m) d\xi \\ v_{m-1}^n &= \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} \hat{v}^{n}(\xi) \exp(-i2\pi h\xi) \exp(-i2\pi h\xi m) d\xi. \end{aligned}$$

Substituting these expressions into the recurrence yields

$$v_m^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} \left[(1 - a\lambda) - a\lambda \exp(-i2\pi h\xi)\right] \hat{v}^{n}(\xi) \exp(-i2\pi h\xi m) d\xi.$$

This requires

$$\hat{v}^{n+1}(\xi) = \left[(1 - a\lambda) - a\lambda \exp(-i2\pi h\xi)\right] \hat{v}^{n}(\xi) = g(\xi h)\hat{v}^n(\xi),$$

where $g(\xi h) = (1 - a\lambda) - a\lambda \exp(-i2\pi h\xi)$. We see that

$$\hat{v}^n(\xi) = (g(\xi h))^n \; \hat{v}^0(\xi).$$

By Parseval's Lemma we have

$$\|v^n\|_2 = |g(\xi h)|^n \; \|\hat{v}^0\|_2.$$

We see this finite difference scheme is stable only if $|g(\xi h)|^n < C < \infty$ for some constant C as $n \to \infty$. This requires that $|g(\xi h)| \leq 1$. In our example above, this requires

$$\begin{aligned} |g(\xi h)|^2 &= ((1 - a\lambda) - a\lambda\cos(-2\pi h\xi))^2 + (a\lambda \sin(-2\pi h\xi))^2 \\ &= 1 - 4a\lambda(1 - a\lambda)\sin^2\left(\frac{1}{2}h\xi\right). \end{aligned}$$

Now since $0 \leq |g|^2 \leq 1$, we can express this as $-1 \leq -4a\lambda(1 - a\lambda)\sin^2\left(\frac{1}{2}h\xi\right) \leq 0$. Since $\sin^2\left(\frac{1}{2}h\xi\right) \leq 1$, we see that $|g(\xi h)| \leq 1$ only if $0 \leq a\lambda \leq 1$. This first requires that $a > 0$ and further that $\lambda \leq 1/a$. When this holds the finite difference method is stable. We see that if $a < 0$ or if λ is too large the finite difference method is unstable. Provided the method is stable and consistent, it turns out for well-posed PDEs, this is enough to ensure convergence. This is the content of the Lax-Richtmyer Equivalence Theorem.

The above analysis can be performed more succinctly by noting that our substitution of the fourier transforms is equivalent to substituting into our finite difference scheme the function $v_m^n = g(h\xi)^n \exp(i2\pi h\xi m)$, which can be simplified to $v_m^n = g(\theta)^n \exp(im\theta)$ where $\theta = 2\pi h\xi$. For the example above, we see this would yield

$$\begin{aligned} v_m^{n+1} &= (1 - a\lambda)v_m^n - a\lambda v_{m-1}^n \\ g(\theta)^{n+1}\exp(im\theta) &= (1 - a\lambda)g(\theta)^n \exp(im\theta) - a\lambda g(\theta)^n \exp(i(m-1)\theta) \\ &= (1 - a\lambda)g(\theta)^n \exp(im\theta) - a\lambda g(\theta)^n \exp(-i\theta)\exp(im\theta) \\ g &= (1 - a\lambda) - a\lambda \exp(-i\theta). \end{aligned}$$

This was obtained by using that $\exp(i(m-1)\theta) = \exp(-i\theta)\exp(im\theta)$ and dividing both sides by $g(\theta)^n \exp(im\theta)$. This allowed us to solve for g. We then analyze the stability by determining if the conditions are met for $|g| \leq 1$ for all θ.

We remark that for some problems the function g can also depend on the time step $k = \delta t$, as $g(\theta, k)$. When this is the case we can loosen this bound slightly, since what we really need is that $|g(\theta,k)|^n \leq C$ as $k = \delta t \to 0$. In this case we only need to have that $|g(\theta,k)| \leq 1 + Kk$. This follows since $\lim_{n\to\infty}(1+\frac{r}{n})^n = e^r$, so we would have $|g(\theta,k)|^n \leq (1+Kk)^n = (1+\frac{KT}{n})^n \to e^{TK} < C$ since $k = \delta t = T/n$ and T, K are fixed.

This gives a brief introduction to *Von Neumann Analysis* for *finite difference methods*. As a brief summary, the steps are (i) to substitute into the scheme $v_m^n = g^n \exp(im\theta)$, (ii) solve for the function $g(\theta,k)$, (iii) perform analysis to determine under what conditions we have $|g(\theta,k)| \leq 1 + Kk$. When this inequality condition is met, the finite difference method is stable and for well-posed PDEs the method will produce approximations that converge to the solution of the PDE by the Lax-Richtmyer Equivalence Theorem.

First-Order Hyperbolic PDE: Transport Equation

Consider the transport equation

$$\begin{cases} u_t + au_x, & t > 0, x \in \mathbb{R} \\ u(x,0) = \phi(x), & t = 0, x \in \mathbb{R}. \end{cases}$$

We first seek a model of the PDE at time t_n by approximating the derivatives by the forward-difference and central differences

$$\begin{aligned} u_t(x_m, t_n) &\approx \frac{u_m^{n+1} - u_m^n}{\delta t}, \\ u_x(x_m, t_n) &\approx \frac{u_{m+1}^n - u_{m-1}^n}{2\delta x}. \end{aligned}$$

Substituting this in place of the derivatives yields

$$\frac{v_m^{n+1} - v_m^n}{\delta t} = -a\frac{v_{m+1}^n - v_{m-1}^n}{2\delta x}.$$

Let $\lambda = \delta t / 2\delta x$ then we can rewrite this as

$$v_m^{n+1} = v_m^n - \lambda a \left(v_{m+1}^n - v_{m-1}^n\right).$$

To perform the von Neumann Analysis we substitute $v_m^n = g^n \exp(im\theta)$ which yields

$$g^{n+1}e^{im\theta} = g^n e^{im\theta}\left(1 - \lambda a\left(e^{i\theta} - e^{-i\theta}\right)\right).$$

By dividing both sides by $g^n e^{im\theta}$ and using Euler identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we obtain

$$g(\theta) = 1 - i2\lambda a\left(\sin(\theta)\right).$$

For stability we need that $|g(\theta)| \leq 1$ for all θ. However, we have

$$|g(\theta)|^2 = 1^2 + 4\lambda^2 a^2 \sin^2(\theta) > 1, \quad \text{provided } \theta \neq \alpha\pi \ \ \alpha \in \mathbb{Z}.$$

This shows the finite difference method based on these central differences is **unstable.** This finite difference method does not yield useful results for approximating the solution of the transport PDE. While we had a stable method based on backward differences in space that worked for $a > 0$, how do we obtain a method that is stable for all choices of $a \in \mathbb{R}$?

Lax-Friedrichs Method. Consider the finite difference method based at time t_n that uses

$$u_t(x_m, t_n) \approx \frac{u_m^{n+1} - \frac{1}{2}\left(u_{m+1}^n + u_{m-1}^n\right)}{\delta t}, \quad u_x(x_m, t_n) \approx \frac{u_{m+1}^n - u_{m-1}^n}{2\delta x}.$$

Using a similar analysis as above, we now obtain

$$g^{n+1}e^{im\theta} = g^n e^{im\theta}\left(\frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta} - \lambda a\left(e^{i\theta} - e^{-i\theta}\right)\right).$$

with $\lambda = \delta t/2\delta x$ and

$$g(\theta) = \cos(\theta) - i2\lambda a\sin(\theta).$$

Stability requires $|g(\theta)| \leq 1$ for all θ. We have

$$|g(\theta)|^2 = \cos^2(\theta) + 4\lambda^2 a^2 \sin^2(\theta) = 1 - \sin^2(\theta) + 4\lambda^2 a^2 \sin^2(\theta) \leq 1.$$

This can be expressed as $1-\sin^2(\theta)+4\lambda^2 a^2 \sin^2(\theta) \leq 1$, which becomes $0 \leq (1 - 4\lambda^2 a^2)\sin^2(\theta)$. Since $0 \leq \sin^2(\theta)$, we will have stability provided that $4\lambda^2 a^2 \leq 1$. This gives the **stability constraint** that $\lambda \leq 1/(2|a|)$. This requires the δt and δx satisfy the inequality $\delta t \leq \delta x/|a|$. We remark that intuitively the Lax-Friedrichs method achieves stability by introducing some dissipation through the averaging term $\frac{1}{2}(v_{m+1}^n + v_{m-1}^n)$. For any choice of $a \in \mathbb{R}$ the finite difference method will be stable provided we take an appropriate choice of δt and δx satisfying the inequality. The Lax-Friedrichs method and related variants can be used quite generally to obtain approximate solutions of transport PDEs and other hyperbolic systems.

Parabolic PDE: Diffusion Equation

Consider the diffusion equation

$$\begin{cases} u_t = \kappa\Delta u, & t > 0, x \in \mathbb{R} \\ u(x,0) = \phi(x), & t = 0, x \in \mathbb{R}. \end{cases}$$

We first seek a model of the PDE at time t_n by approximating the derivatives by the forward-difference and central differences

$$u_t(x_m, t_n) \approx \frac{u_m^{n+1} - u_m^n}{\delta t}, \quad u_{xx}(x_m, t_n) \approx \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\delta x^2}.$$

Substituting this in place of the derivatives yields

$$\frac{v_m^{n+1} - v_m^n}{\delta t} = \kappa\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{\delta x^2}.$$

Let $\lambda = \kappa\delta t/\delta x^2$ then we can rewrite this as

$$v_m^{n+1} = v_m^n + \lambda\left(v_{m+1}^n - 2v_m^n + v_{m-1}^n\right).$$

To perform the von Neumann Analysis we substitute $v_m^n = g^n \exp(im\theta)$ which yields

$$g^{n+1}e^{im\theta} = g^n e^{im\theta}\left(1 + \lambda\left(e^{i\theta} + e^{-i\theta} - 2\right)\right).$$

By dividing both sides by $g^n e^{im\theta}$ and using Euler identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we obtain

$$g(\theta) = 1 - 2\lambda\left(1 - \cos(\theta)\right) = 1 - 4\lambda\sin^2\left(\frac{\theta}{2}\right).$$

We used the trigonometric identity $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\left(1 - \cos(\theta)\right)$. For stability we need that $|g(\theta)| \leq 1$ for all θ. This requires $-1 \leq 1 - 4\lambda\sin^2\left(\frac{\theta}{2}\right) \leq 1$. Using that $|\sin^2\left(\frac{\theta}{2}\right)| \leq 1$, we find this requires $0 \leq 4\lambda \leq 2$, $\Rightarrow 0 \leq \lambda \leq \frac{1}{2}$. This gives the **stability constraint**

$$\delta t \leq \frac{1}{2}\kappa^{-1}\delta x^2.$$

Provided this condition holds we have as $\delta x \to 0$ from the Lax-Richtmyer Theorem the finite difference method will produce an approximation that converges to the solution of the diffusion PDE. An important issue that arises in practice is that that as we refine the mesh with δx small we need to take $\delta t \sim \delta x^2$ very small to ensure stability. This can result in finite difference approximations that seem overly expensive in some cases, especially when the initial conditions are relatively smooth functions.

Crank-Nicolson Method for the Diffusion Equation. As an alternative way to approximate solutions of the PDE, we consider developing a finite difference method based at time $t_{n+1/2}$ for symmetry. We approximate the derivatives by the central differences

$$\begin{aligned} u_t(x_m, t_{n+1/2}) &\approx \frac{u_m^{n+1} - u_m^n}{\delta t}, \\ u_{xx}(x_m, t_{n+1/2}) &\approx \frac{1}{2}\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\delta x^2} + \frac{1}{2}\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\delta x^2}. \end{aligned}$$

Substituting this in place of the derivatives yields the *Crank-Nicolson Method*

$$\frac{v_m^{n+1} - v_m^n}{\delta t} = \frac{\kappa}{2}\left[\frac{v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}}{\delta x^2} + \frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{\delta x^2}\right].$$

Let $\lambda = \kappa\delta t/\delta x^2$ then we can rewrite this as

$$v_m^{n+1} - \frac{1}{2}\lambda\left(v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}\right) = v_m^n + \frac{1}{2}\lambda\left(v_{m+1}^n - 2v_m^n + v_{m-1}^n\right).$$

To perform the von Neumann Analysis we substitute $v_m^n = g^n \exp(im\theta)$ which yields

$$g^{n+1}e^{im\theta}\left(1 - \frac{1}{2}\lambda\left(e^{i\theta} + e^{-i\theta} - 2\right)\right) = g^n e^{im\theta}\left(1 + \frac{1}{2}\lambda\left(e^{i\theta} + e^{-i\theta} - 2\right)\right).$$

By dividing both sides by $g^n e^{im\theta}$ and using Euler identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we obtain

$$g(\theta) = \frac{1 - \lambda\left(1 - \cos(\theta)\right)}{1 + \lambda\left(1 - \cos(\theta)\right)}.$$

For stability, we need that $|g(\theta)| \leq 1$ for all θ. Let $\gamma = \lambda\,(1 - \cos(\theta))$, then we can express g as

$$g(\theta) = \frac{1-\gamma}{1+\gamma}.$$

Since $\gamma \geq 0$ for any $\lambda \geq 0$ and choice of θ, we have that $|g(\theta|) \leq 1$. This shows the Crank-Nicolson Method is **unconditionally stable**! In other words, for stability there is no restriction on the choice of discretization parameters δx and δt. Of course, we still do need to worry about the accuracy of the results produced by these methods. An advantage of unconditionally stable methods is that the accuracy becomes the main concern on how large the parameters can be taken.

This increased stability does come with a few extra computational steps. By letting $\mathbf{v}^n$ be a vector with components $[\mathbf{v}^n]_m = v_m^n$ and letting L be the finite difference operator $[L\mathbf{v}^n]_m = v_{m+1}^n - 2v_m^n + v_{m-1}^n$, we can express the collective updates as

$$\left(I - \frac{1}{2}\lambda L\right)\mathbf{v}^{n+1} = \left(I + \frac{1}{2}\lambda L\right)\mathbf{v}^n.$$

The approximation at time t_{n+1} is only given by these equations implicitly. At each time-step we have to solve the linear system to obtain

$$\mathbf{v}^{n+1} = \left(I - \frac{1}{2}\lambda L\right)^{-1}\left(I + \frac{1}{2}\lambda L\right)\mathbf{v}^n.$$

In practice, this often can be done either by using fourier transforms to obtain expressions similar to those above, or by developing iterative methods, both of which often yield efficient methods. The benefit of the Crank-Nicolson Method is that there are no stability restrictions on how we refine δx and δt and we are only restricted by considerations of accuracy.

Second-Order Hyperbolic PDE: Wave Equation

Consider the wave equation

$$\begin{cases} u_{tt} & = & c^2\Delta u, & t > 0, x \in \mathbb{R} \\ u(x,0) & = & \phi(x), & t = 0, x \in \mathbb{R}. \\ u_t(x,0) & = & \psi(x), & t = 0, x \in \mathbb{R}. \end{cases}$$

As an initial method we will first approximate the derivatives by the central differences

$$u_t(x_m,t_n) \approx \frac{u_m^{n+1} - 2u_m^n + u_m^{n-1}}{\delta t^2}, \quad u_{xx}(x_m,t_n) \approx \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\delta x^2}.$$

We remark there are also other alternative more sophisticated methods available for hyperbolic systems that often work better than this, but this gives one method to start. Substituting this in place of the derivatives yields

$$\frac{v_m^{n+1} - 2v_m^n + v_m^{n-1}}{\delta t^2} \quad = \quad c^2\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{\delta x^2}.$$

Let $\lambda = c^2\delta t^2/\delta x^2$ then we can rewrite this as

$$v_m^{n+1} = \lambda v_{m+1}^n + 2(1-\lambda)v_m^n + \lambda v_{m-1}^n - v_m^{n-1}.$$

To perform the von Neumann Analysis we substitute $v_m^n = g^n \exp(im\theta)$ which yields

$$g^{n+1}e^{im\theta} = g^n e^{im\theta}\lambda e^{i\theta} + g^n e^{im\theta}2(1-\lambda) + g^n e^{im\theta}\lambda e^{-i\theta} - g^{n-1}e^{im\theta}.$$

By dividing both sides by $g^{n-1}e^{im\theta}$ and using Euler identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we obtain that g must satisfy the following quadratic equation

$$g^2 = (2\lambda\cos(\theta) + 2(1-\lambda))\, g - 1 \quad \Rightarrow \quad g^2 - \beta g + 1 = 0,$$

where $\beta = 2 - 2\alpha$ with $\alpha = \lambda(1-\cos(\theta))$. This arose since we had a second-order recurrence equation with v^{n+1} depending on the two previous time-steps v^n and v^{n-1}. The solutions for g are given by the roots of the quadratic equation as

$$g_{\pm} = \frac{\beta \pm \sqrt{\beta^2 - 4}}{2} = 1 - \alpha \pm \sqrt{\alpha^2 - 2\alpha}.$$

We used here that $\beta^2 = 4 + 4\alpha^2 - 8\alpha$. For stability, we need that $|g_{\pm}(\theta)| \leq 1$ for all θ. The roots for g are real when $\alpha^2 - 2\alpha \geq 0 \;\Rightarrow\; \alpha \geq 2$. For $\alpha = 2$ we would have $g_{\pm} = -1$. In the case of $\alpha > 2$, we have $1 - \alpha < -1$ and $|g_-| > 1$ for the negative root. This requires the roots be complex and that $\alpha \leq 2$. To avoid $1 - \alpha$ becoming more than one, we also need to have $\alpha \geq 0$. Together these give the conditions $0 \leq \alpha \leq 2$ to ensure that $|g_{\pm}| \leq 1$. Since this must hold for all choices of θ, this provides the **stability condition** that $0 \leq 2\lambda \leq 2$ which requires $\lambda \leq 1$. Since $\lambda = c^2\delta t^2/\delta x^2$ this constrains the discretization to have $\delta t \leq \delta x/c$.

We remark that this condition arises often, especially for explicit methods, and is referred to as the Courant-Friedrichs-Lewy (CFL) condition. It can intuitively be interpreted as requiring that we not propagate information of the solution on the mesh with spacing δx faster than would occur in the wave equation, which has the wave speed c. The time-scale τ for propagation of a wave with speed c over the distance δx is given by $\tau = \delta x/c$. The stability requires that our time-step satisfy $\delta t \leq \tau$.

While this finite difference method is stable, it will tend to accumulate many discretization artifacts and not handle well the case when there are sharp jumps in the solution. There are many alternative methods that have been designed that are better suited to the behaviors of the wave equation and more general applications of hyperbolic PDEs. We discuss one such approach based on a coupled system of first-order hyperbolic PDEs.

Lax-Wendroff Method. We formulate the wave equation as a coupled system of first-order hyperbolic PDEs. This is done by expressing the wave equation in "conservation form" by defining

$$\frac{\partial \mathbf{w}}{\partial t} = -\frac{\partial}{\partial x}\mathbf{F}, \quad \mathbf{w} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} -cv \\ -cu \end{bmatrix}.$$

The system above can be expressed completely in terms of $\mathbf{w} = [w^{(1)}, w^{(2)}]^T$ with $w^{(1)} = u,\ w^{(2)} = v$. We can express the flux term as

$$\mathbf{F}(\mathbf{w}) = A\mathbf{w}, \quad A = \begin{bmatrix} 0 & -c \\ -c & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} c^2 & 0 \\ 0 & c^2 \end{bmatrix}.$$

There are also other formulations for the wave equation. We can now discretize using the Lax-Wendroff Method, which approximates the derivatives as

$$\begin{aligned}\mathbf{w}_t(x_m, t_n) &\approx \frac{\mathbf{w}_m^{n+1} - \mathbf{w}_m^n}{\delta t}, \\ -A\mathbf{w}_x(x_m, t_n) &\approx -A\frac{\mathbf{w}_{m+1}^n - \mathbf{w}_{m-1}^n}{2\delta x} + \frac{1}{2}\delta t A^2 \frac{\mathbf{w}_{m+1}^n - 2\mathbf{w}_m^n + \mathbf{w}_{m-1}^n}{\delta x^2}.\end{aligned}$$

We use here that A^2 is positive definite. Part of the intuition for this choice of discretization is that the second term in $\mathbf{w}_x$ approximation provides some dissipation that helps stabilize the numerical method $-A\mathbf{w}_x \approx -A\mathbf{w}_x + \frac{1}{2}\delta t\, A^2\mathbf{w}_{xx}$. The extra term weakens with a diminishing contribution to the solution as we take $\delta t \to 0$. Let $\lambda = \delta t/\delta x$ be the constant that arose in advection problems and $\gamma = \delta t/\delta x^2$ be the constant that arose in diffusion problems. This gives the Lax-Wendroff Method

$$\mathbf{w}_m^{n+1} = \mathbf{w}_m^n - \frac{1}{2}\lambda A\left(\mathbf{w}_{m+1}^n - \mathbf{w}_{m-1}^n\right) + \frac{1}{2}\delta t\gamma A^2\left(\mathbf{w}_{m+1}^n - 2\mathbf{w}_m^n + \mathbf{w}_{m-1}^n\right).$$

To analyze stability, we can reduce this to a scalar problem by performing a change of variable to the eigenbasis, with $\mathbf{q} = P\mathbf{w}$ and P having columns that are the eigenvectors. The eigenvalues $r_\pm$ of A are readily seen to be $r_\pm = \pm c$. This gives that $PAP^{-1} = D$ where

$$D = \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix}.$$

By substituting in $\mathbf{w} = P^{-1}\mathbf{q}$ we obtain

$$P^{-1}\mathbf{q}_m^{n+1} = P^{-1}\mathbf{q}_m^n - \frac{1}{2}\lambda AP^{-1}\left(\mathbf{q}_{m+1}^n - P^{-1}\mathbf{q}_{m-1}^n\right) + \frac{1}{2}\delta t\gamma A^2P^{-1}\left(\mathbf{q}_{m+1}^n - 2\mathbf{q}_m^n + \mathbf{q}_{m-1}^n\right).$$

By multiply each side by P we obtain the equivalent recurrence

$$\mathbf{q}_m^{n+1} = \mathbf{q}_m^n - \frac{1}{2}\lambda D\left(\mathbf{q}_{m+1}^n - \mathbf{q}_{m-1}^n\right) + \frac{1}{2}\delta t\gamma D^2\left(\mathbf{q}_{m+1}^n - 2\mathbf{q}_m^n + \mathbf{q}_{m-1}^n\right).$$

Clearly, if $\mathbf{q}_m^n$ is stable then so is $\mathbf{w}_m^n$, since they only differ by a non-singular linear change of variable. Since D is diagonal, the system decouples and we need only analyze the two scalar problems

$$q_m^{n+1} = q_m^n \pm \frac{1}{2}c\lambda\left(q_{m+1}^n - q_{m-1}^n\right) + \frac{1}{2}\delta t\gamma c^2\left(q_{m+1}^n - 2q_m^n + q_{m-1}^n\right).$$

We perform the von Neumann Analysis by substitution $q_m^n = g^n\exp(im\theta)$ which gives

$$g^{n+1}e^{im\theta} = g^ne^{im\theta} \pm ic\lambda g^ne^{im\theta}\sin(\theta) - \lambda^2c^2g^ne^{im\theta}\left(1 - \cos(\theta)\right).$$

We used that $\lambda^2 = \delta t\gamma$. Let $\mu = \lambda c$, then this gives

$$\begin{aligned}g(\theta) &= 1 \pm i\mu\sin(\theta) - \mu^2\left(1 - \cos(\theta)\right) \\ &= 1 \pm i\mu\sin(\theta) - 2\mu^2\sin^2\left(\frac{\theta}{2}\right).\end{aligned}$$

This has modulus

$$
\begin{aligned}
|g|^2 &= \mu^2 \sin^2(\theta) + \left(1 - 2\mu^2 \sin^2\left(\frac{\theta}{2}\right)\right)^2 \\
&= \mu^2 \sin^2(\theta) + 1 + 4\mu^4 \sin^4\left(\frac{\theta}{2}\right) - 4\mu^2 \sin^2\left(\frac{\theta}{2}\right) \\
&= 1 - 4\mu^2(1-\mu^2)\sin^2\left(\frac{\theta}{2}\right) \leq 1 \\
&\Rightarrow -\mu^2(1-\mu^2) \leq 0.
\end{aligned}
$$

We used here the trigonometric identity

$$
\sin^2(\theta) = 4\cos^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\theta}{2}\right) = 4\left(1 - \sin^2\left(\frac{\theta}{2}\right)\right)\sin^2\left(\frac{\theta}{2}\right).
$$

We have $|g| \leq 1$ provided that $\mu^2 \leq 1$ which requires $\lambda < 1/c$. Since $\lambda = \delta t/\delta x$ this requires $\delta t \leq \delta x/c$. With these choices for δt and δx the method will be stable and by the Lax-Richtmyer Theorem the finite difference method provides approximations that will converge to the solution of the wave equation. The Lax-Wendroff Method can also be applied to other hyperbolic PDEs and first-order hyperbolic systems of equations by putting them into the conservation form above. This and related approaches are widely used to obtain approximate solutions of hyperbolic PDEs for applications.

Summary. These approaches to analyzing finite difference methods is referred to as *von Neumann Analysis*. There are many variants of this approach to deal with boundary conditions, higher dimensions, and roles of geometry. The overall idea is to make use of fourier representations and Parseval's Lemma to determine how the amplitudes of the fourier modes behave and what constraints we need on δt and δx as we refine the discretizations with $\delta t, \delta x \to 0$. There are also other related eigenvalue methods that can be used to analyze stability. The fourier methods and eigenvalue approaches presented here can be used to gain insights helpful in the selection and design of finite difference methods to approximate the solutions of PDEs.