Method of Characteristics

Constant Coefficient Case: Consider the constant coefficient first-order partial differential equation (PDE)

$$au_x + bu_y = 0. \tag{1}$$

This can be solved by the *Method of Characteristics*.

This PDE is equivalent to requiring that the derivatives of the function u at $\mathbf{x} = (x, y)$ in the direction $\mathbf{v} = [a, b]^T$ be zero,

$$\lim_{\epsilon \to 0} \frac{u(\mathbf{x} + \epsilon \mathbf{v}) - u(\mathbf{x})}{\epsilon} = \frac{\partial}{\partial \epsilon} u(\mathbf{x} + \epsilon \mathbf{v})\bigg|_{\epsilon=0} = \nabla u \cdot \mathbf{v} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = au_x + bu_y = 0.$$

We will now treat the case when $a \neq 0$ which allows us to simplify the problem to

$$u_x + cu_y = 0.$$

This is obtained by dividing both sides by a to obtain $c = b/a$. We remark that any case with not both coefficients zero can be reduced to this by doing a change of variable $\tilde{x} = y$, $\tilde{y} = x$.

This requires the function is constant in the direction $\mathbf{v} = [1, c]^T$, so that $u(\mathbf{x} + \alpha \mathbf{v}) = u(\mathbf{x})$ for any $\alpha \in \mathbb{R}$. This means the value is the same at any point $\mathbf{x}_1$ on the same line as $\mathbf{x}$, provided that $\mathbf{x}_1 = \mathbf{x} + \alpha \mathbf{v}$ for some α. We can denote these lines by $\boldsymbol{\gamma}(s; c_0) = \mathbf{x}_0(c_0) + s\mathbf{v}$. These lines $\boldsymbol{\gamma}$ are the *Characteristic Curves* for this PDE. We parameterized here the family of curves $\boldsymbol{\gamma}(\cdot; c_0)$ using c_0 for some chosen reference point $\mathbf{x}_0 = \mathbf{x}_0(c_0)$ for each curve.

Since all of these lines intersect the y-axis, we use reference points $\mathbf{x}_0 = (0, y_0)$. For any point $\mathbf{x}$ there is always a way to choose α so that $\mathbf{x} + \alpha \mathbf{v}$ gives the y-intersection. This requires $\mathbf{x} + \alpha \mathbf{v} = \mathbf{x}_0 = (0, y_0)$, which implies $x + \alpha 1 = 0$ and $y + \alpha c = y_0$. This allows us to solve for $y_0 = y - xc$. This gives the *general solution* to the PDE

$$u(x, y) = f(y - xc),$$

where $f(s) = u(0, s)$.

Now we also alternatively could have constructed an equivalent solution by choosing as our reference to parameterize the lines by the location where they intersect the x-axis. In that case, we would have instead the reference points $\mathbf{x}_0 = (x_0, 0)$. Now for any point $\mathbf{x}$ we choose α so that $\mathbf{x} + \alpha \mathbf{v}$ gives the x-intersection. This requires $\mathbf{x} + \alpha \mathbf{v} = \mathbf{x}_0 = (x_0, 0)$, which implies $x + \alpha 1 = x_0$ and $y + \alpha c = 0$. This allows us to solve for $x_0 = x - \frac{y}{c}$. This gives an equivalent *general solution* to the PDE which can be expressed as

$$u(x, y) = g\left(x - \frac{y}{c}\right),$$

where $g(s) = u(s, 0)$.

In general, the characteristic curves can be parameterized in many different ways and how one does this in practice depends on the problem being considered. This often depends on where the data is specified. If the data is specified along the y-axis then the first parameterization above would be chosen. If the data is specified along the x-axis, we would choose the second approach. If the data were specified in some other way, such as along a curve intersecting all the lines, one might use that curve to parameterize the characteristics. The overall idea is to use that the solution of the PDE above is a function that remains constant on each of the characteristic curves. We then construct a solution to the PDE by making use of available information about what the constant value is on each of the characteristic curves. We now give a few examples to illustrate this approach.

Example: *Transport PDE:*

$$2u_x + 6u_y = 0.$$

The characteristics are given by $\gamma(s) = \mathbf{x}_0 + s\mathbf{v}$ with $\mathbf{v} = [1, c]$ with $c = 6/2 = 3$. The general solution is

$$u(x, y) = f(y - 3x).$$

Example: *Transport PDE with Conditions on y-axis:*

$$\begin{cases} 2u_x + 6u_y & = & 0, \qquad x > 0 \\ u(0, y) & = & \phi(y). \end{cases}$$

The characteristics are given by $\gamma(s) = \mathbf{x}_0 + s\mathbf{v}$ with $\mathbf{v} = [1, c]$ with $c = 6/2 = 3$. Since the conditions give us information about the value of the function when the characteristics interset the y-axis, it is most natural to use of the form of the general solution

$$u(x, y) = f(y - 3x).$$

In this case we have $u(0, y) = f(y) = \phi(y)$. This gives the solution to the PDE

$$u(x, y) = \phi(y - 3x).$$

To illustrate how the general solutions we discussed are still equivalent, one could have also used the less natural expression for this problem for the general solution $u(x, y) = g\left(x - \frac{y}{c}\right)$. In that case, $u(0, y) = g(-\frac{y}{c}) = \phi(y)$ and we conclude $g(z) = \phi(-cz)$ which yields $u(0, y) = \phi\left(-c\left(x - \frac{y}{c}\right)\right) = \phi(y - cx)$. We end up with the same final solution, just through a more cumbersome calculation. As problems become more complicated, choosing a good parameterization for the characteristics can have a significant impact on the ease in solving the problem analytically.

Example: *Transport PDE with Conditions on x-axis:*

$$\begin{cases} u_x + 4u_y & = & 0, \qquad y > 0 \\ u(x, 0) & = & \phi(x). \end{cases}$$

The characteristics are given by $\gamma(s) = \mathbf{x}_0 + s\mathbf{v}$ with $\mathbf{v} = [1, c]$ with $c = 4/1 = 4$. Since the conditions give us information about the value of the function when the characteristics interset the y-axis, it is most natural to use of the form of the general solution

$$u(x, y) = g\left(x - \frac{y}{4}\right).$$

In this case we have $u(x,0) = g(x) = \phi(x)$. This gives the solution to the PDE

$$u(x,y) = \phi\left(x - \frac{y}{4}\right).$$

Summary: The form of the characteristic curves were determined by the vector field $\mathbf{v}$. Since $\mathbf{v}$ was constant these gave charactersitics that are straight lines. In general, the characteristics can be more complicated curves. In some cases they even can be curves in the whole plane that form closed loops, such as circles. In those cases, the domain of the PDE is typically more restricted. In this case, we will need methods to determine the characteristic curves as part of the construction of solutions of the PDE.

Variable Coefficient Case:

Consider the variable coefficient first-order partial differential equation (PDE)

$$a(x,y)u_x + b(x,y)u_y = 0.$$

We can again interpret this using directional derivatives as

$$\nabla u(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) = \begin{bmatrix} u_x \\ u_y \end{bmatrix} \cdot \begin{bmatrix} a(x,y) \\ b(x,y) \end{bmatrix} = a(x,y)u_x + b(x,y)u_y = 0.$$

Now the direction $\mathbf{v}$ for which the function has a zero derivative depends on $\mathbf{x}$. For smooth a, b we can view $\mathbf{v} = (a(x,y), b(x,y))$ as a smooth vector field. We can look for *integral curves* $\boldsymbol{\gamma}(s;c_0) = (\gamma_1(s;c_0), \gamma_2(s;c_0)$, which are defined as having their tangents align with $\mathbf{v}$ at each point,

$$\frac{\partial}{\partial s}\boldsymbol{\gamma}(s;c_0) = \mathbf{v}(\boldsymbol{\gamma}(s;c_0)).$$

This can also be expressed as

$$\frac{\partial}{\partial s}\gamma_1 = a(\gamma_1, \gamma_2) \tag{2}$$

$$\frac{\partial}{\partial s}\gamma_2 = b(\gamma_1, \gamma_2). \tag{3}$$

Since the curve given by $\boldsymbol{\gamma}$ has a tangent aligned with $\mathbf{v}$ we see that the function must be constant along these curves since

$$\frac{\partial}{\partial s}u(\boldsymbol{\gamma}(s;c_0)) = \nabla u \cdot \frac{\partial}{\partial s}\boldsymbol{\gamma} = \nabla u \cdot \mathbf{v} = 0.$$

We have

$$u(\boldsymbol{\gamma}(s_1;c_0)) = u(\boldsymbol{\gamma}(s_2;c_0)),$$

for any parameters s_1,s_2 corresponding to points on the curve $\boldsymbol{\gamma}(\cdot;c_0)$. These curves $\boldsymbol{\gamma}(\cdot;c_0)$ are called the *Characteristic Curves* for the PDE. This yields the general solutions for the PDE

$$u(x,y) = g(\boldsymbol{\gamma}(0;c_0)) = f(c_0),$$

where $g(\boldsymbol{\gamma}(0; c_0)) = f(c_0) = u(\boldsymbol{\gamma}(0; c_0))$. The $c_0 = c_0(x, y)$ is obtained from the requirement that $\mathbf{x} = (x, y)$ lies on the characteristic curve $\boldsymbol{\gamma}(\cdot; c_0)$. This requires that $\mathbf{x} = \boldsymbol{\gamma}(s; c_0))$ for some choice of s.

Example:

$$yu_x - xu_y = 0.$$

The characteristic equations are

$$\frac{\partial \gamma_1}{\partial s} = \gamma_2 \tag{4}$$

$$\frac{\partial \gamma_2}{\partial s} = -\gamma_1. \tag{5}$$

This has as a solution $\gamma_1(s; c_0) = c_0 \cos(s)$ and $\gamma_2(s; c_0) = c_0 \sin(s)$ with $c_0 \geq 0$. The PDE has $yu_x - xu_y = du(\boldsymbol{\gamma})/ds = 0$ so that for any $\mathbf{x}_1 = (x_1, y_1)$ and $\mathbf{x}_2 = (x_2, y_2)$ we have $u(x_1, y_1) = u(x_2, y_2)$ provided that $\mathbf{x}_1 = \boldsymbol{\gamma}(s_1; c_0)$ and $\mathbf{x}_2 = \boldsymbol{\gamma}(s_2; c_0)$ with the same constant c_0. Since these characteristics always intersect with the positive x-axis, we can use as reference points $\mathbf{x}_0 = (x_0, 0)$ with $x_0 \geq 0$.

For a given $\mathbf{x}_1$ point with $\mathbf{x}_1 = \boldsymbol{\gamma}(s_1) = c_0[\cos(s_1), \sin(s_1)]$, we must have $c_0 = \|\mathbf{x}_1\| = \sqrt{x_1^2 + y_1^2}$ (using convention $c_0 \geq 0$ to ensure a unique parameterization of the characteristics). We further have that $\|\mathbf{x}_0\| = \|[x_0, 0]\| = x_0 = c_0 = \|\mathbf{x}_1\|$ must hold. This gives the general solution

$$u(x, y) = f(\sqrt{x^2 + y^2}) = f(\|\mathbf{x}\|),$$

where $f(r) = u(r, 0)$.

Special Case with $a(\mathbf{x}) > 0$: In the case that $a(x, y) > 0$ we can formulate the PDE as

$$u_x + b(x, y)u_y = 0.$$

The characteristic equations then simplify to

$$\frac{\partial \gamma_1}{\partial s} = 1 \tag{6}$$

$$\frac{\partial \gamma_2}{\partial s} = b(\gamma_1, \gamma_2), \tag{7}$$

where $\boldsymbol{\gamma}(s; c_0) = (\gamma_1(s; c_0), \gamma_2(s; c_0))$. This would give $\gamma_1 = s$ up to a constant, so $x(s) = \gamma_1(s) = s$ and $y(s) = \gamma_2(s) = \gamma_2(x) \Rightarrow y = y(x)$. In this case, we can rewrite the characteristic equations in terms of x, y as

$$\frac{dy}{dx} = b(x, y(x)). \tag{8}$$

We only need to solve for $y(x)$ since x can be used to parameterize the curve (plays a role similar to s above). This gives a family of solutions $y = y(x; c_0)$. In this case, general solutions are given by

$$u(x, y) = f(c_0),$$

where $f(c_0) = u(0, y(0; c_0))$. The $c_0 = c_0(x, y)$ is obtained from the requirement that $\mathbf{x} = (x, y)$ lies on the characteristic curve $\boldsymbol{\gamma}(\cdot; c_0)$, here $\boldsymbol{\gamma}(x; c_0) = (x, y(x; c_0))$. This requires c_0 such that $\mathbf{x} = \boldsymbol{\gamma}(x; c_0) = (x, y(x; c_0))$.

Example:

$$u_x + yu_y = 0.$$

The characteristic equations are

$$dy/dx = y \quad \Rightarrow \quad y(x; c_0) = c_0 \exp(x).$$

The PDE has $u_x + yu_y = du(x, y(x))/dx = 0$ so that for any $\mathbf{x}_1 = (x_1, y_1)$ and $\mathbf{x}_2 = (x_2, y_2)$ we have $u(x_1, y_1) = u(x_2, y_2)$ provided that $y_1 = c_0 \exp(x_1)$ and $y_2 = c_0 \exp(x_2)$ with the same constant c_0. Since these characteristics always intersect with the y-axis, we can use as reference points $\mathbf{x}_0 = (0, y_0)$. For a given $\mathbf{x}_1$ point with $y_1 = c_0 \exp(x_1)$ we must have $c_0 = y_1/\exp(x_1)$. This requires $y_0 = c_0 \exp(0) = c_0 = y_1/\exp(x_1)$. This gives the general solution

$$u(x, y) = f(y/\exp(x)),$$

where for this example $f(s) = u(0, s)$.

Summary for Homogeneous Case:

1. Formulate the characteristic curve equations for the PDE.
2. Solve the equations to get a family of characteristic curves $\gamma(\cdot; c_0)$.
3. General solutions are obtained of the form $u(x, y) = f(c_0)$, where $f(c_0) = u(\boldsymbol{\gamma}(0; c_0))$. The $c_0 = c_0(x, y)$ is obtained from the requirement that $\gamma(s; c_0) = (x, y)$ hold for some s.

As discussed above, this can be carried out readily in the case of constant coefficients or when $a(x, y) > 0$. In the more general setting, a system of ODEs has to be solved to obtain the characteristics either analytically or numerically. The *method of characteristics* can be utilized quite generally to solve PDEs.

Inhomogeneous Case:
Consider the variable coefficient first-order inhomogeneous partial differential equation (PDE)

$$a(x, y)u_x + b(x, y)u_y = f(x, y; u).$$

We can again interpret this using directional derivatives as

$$\nabla u(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) = \begin{bmatrix} u_x \\ u_y \end{bmatrix} \cdot \begin{bmatrix} a(x, y) \\ b(x, y) \end{bmatrix} = a(x, y)u_x + b(x, y)u_y = f(x, y; u).$$

We look for *characteristic curves* $\boldsymbol{\gamma}(s; c_0) = (\gamma_1(s; c_0), \gamma_2(s; c_0))$, which are defined as having their tangents align with $\mathbf{v}$ at each point,

$$\frac{\partial}{\partial s}\boldsymbol{\gamma}(s; c_0) = \mathbf{v}(\boldsymbol{\gamma}(s; c_0)).$$

This can also be expressed as

$$\frac{\partial}{\partial s}\gamma_1 = a(\gamma_1, \gamma_2), \qquad \frac{\partial}{\partial s}\gamma_2 = b(\gamma_1, \gamma_2). \tag{9}$$

Since the curve given by $\boldsymbol{\gamma}$ has a tangent aligned with $\mathbf{v}$ we see that u must satisfy in s the following ODE

$$\frac{\partial}{\partial s}u(\boldsymbol{\gamma}(s;c_0)) = \nabla u \cdot \frac{\partial}{\partial s}\boldsymbol{\gamma} = \nabla u \cdot \mathbf{v} = f\left(\boldsymbol{\gamma}(s;c_0); u(\boldsymbol{\gamma}(s;c_0))\right).$$

To simplify this notation we introduce $w(s) = w(s;c_0) = u(\boldsymbol{\gamma}(s;c_0))$ then the ODE becomes

$$\frac{dw}{ds} = f(s;w).$$

In this notation, we have $f(s;w) := f\left(\boldsymbol{\gamma}(s;c_0); u(\boldsymbol{\gamma}(s;c_0))\right) = f\left(\boldsymbol{\gamma}(s;c_0); w(s)\right)$. By solving this ODE for w, we have at each point $\mathbf{x} = \boldsymbol{\gamma}(s;c_0)$ along the characteristic curve that

$$u(\mathbf{x}) = u(\boldsymbol{\gamma}(s;c_0)) = w(s).$$

In other words to obtain u, this reduces the problem to finding for any given $\mathbf{x}$ the corresponding c_0 for the characteristic curve and the value for s so $\mathbf{x} = \boldsymbol{\gamma}(s;c_0)$. This yields the general solution for the PDE

$$u(x,y) = g(\boldsymbol{\gamma}(0;c_0)) + w(s^*) = \phi(c_0) + w(s^*),$$

where $g(\boldsymbol{\gamma}(0;c_0)) = u(\boldsymbol{\gamma}(0;c_0)) = \phi(c_0)$, s^* is such that $\boldsymbol{\gamma}(s^*;c_0) = \mathbf{x} = (x,y)$, and w solves the ODE. The $c_0 = c_0(x,y)$ is obtained from the requirement that $\mathbf{x} = (x,y)$ lies on the characteristic curve $\boldsymbol{\gamma}(\cdot;c_0)$. This requires that $\mathbf{x} = \boldsymbol{\gamma}(s^*;c_0))$ for some choice of s^*.

The key difference with the homogeneous case is that once we have found the characteristic curves, we need to solve an additional ODE. This is required, since the value of u is no longer constant along characteristic curves. The ODE we solved for w captures how the initial values of u and the source contributions f propagate as we move along a characteristic curve.

Example:

$$\begin{cases} u_x + y u_y &= -1, \quad x > 0, y > 0 \\ u(0,y) &= \phi(y), \quad y > 0. \end{cases}$$

The characteristic equations are

$$\frac{\partial \gamma_1}{\partial s} = 1, \qquad \frac{\partial \gamma_2}{\partial s} = \gamma_2. \tag{10}$$

This has as a solution $\gamma_1(s;c_0) = s$ and $\gamma_2(s;c_0) = c_0 \exp(s)$. The PDE has $u_x + y u_y = du(\boldsymbol{\gamma})/ds = dw/ds = -1$. For any point $\mathbf{x}_1 = (x_1, y_1)$ we need to find the characteristic curve that has $\mathbf{x}_1 = \boldsymbol{\gamma}(s_1;c_0)$ and intersects the positive y-axis. We use for the reference points $\mathbf{x}_0 = (0, y_0)$ with $y_0 \geq 0$.

For a given $\mathbf{x}_1$ point, we need $\mathbf{x}_1 = (x_1, y_1) = \boldsymbol{\gamma}(s_1) = (s_1, c_0 \exp(s_1))$, which requires $s_1 = x_1$ and $y_1 = c_0 \exp(s_1)$. This gives $c_0 = y_1 \exp(-x_1)$ for the characteristic curve that passes through $\mathbf{x}_1 = (x_1, y_1)$. For this characteristic curve, we intersect the y-axis at the point $\mathbf{x}_0 = (0, y_0) = \boldsymbol{\gamma}(0; c_0) = (0, y_1 \exp(-x_1))$, so $y_0 = y_1 \exp(-x_1)$.

Now we need to solve the ODE $dw/ds = -1$. This has the solution $w(s) = w(0) - s$. We have $w(0) = u(\boldsymbol{\gamma}(0; c_0)) = u(0, y_0) = \phi(y_0) = \phi(y_1 \exp(-x_1))$. This gives the solution at the point $\mathbf{x}_1 = (x_1, y_1)$

$$u(x_1, y_1) = \phi(y_1 \exp(-x_1)) - x_1.$$

This follows since $u(x_1, y_1) = w(s_1)$ and $s_1 = x_1$. Since the construction above can be performed for any point $\mathbf{x} = (x, y)$ with $x > 0, y > 0$, the solution to the PDE is given by

$$u(x, y) = \phi(y \exp(-x)) - x.$$

Summary for Inhomogeneous Case:

1. Formulate the characteristic curve equations for the PDE.

2. Solve the equations to get a family of characteristic curves $\gamma(\cdot; c_0)$.

3. Solve the ODE for how the initial conditions ϕ and source contributions f propagate along the characteristic curves $dw/ds = f(s; w)$.

4. The solution is of the form $u(x, y) = \phi(c_0) + w(s^*)$, where $\phi(c_0) = u(\boldsymbol{\gamma}(0; c_0))$ and s^* satisfies $\gamma(s^*; c_0) = (x, y)$. The $c_0 = c_0(x, y)$ is obtained from the requirement that $\gamma(s; c_0) = (x, y)$ for some s.

The *method of characteristics* can be utilized quite generally to solve PDEs. The overall approach to constructing solutions also can be used to develop approximations and numerical methods. For instance by solving the systems of ODEs that arise either using analytic approximations or numerically. In practice, the method of characteristics are combined with other techniques to obtain insights into the behaviors of pdes and as part of constructing solutions.