Separation of Variables

General Technique: Consider a linear pde on an interval $[0,\ell]$ of the form

$$\begin{cases} u_t & = & \mathcal{L}u, & t>0,\ 0<x<\ell \\ u(0,t) & = & u(\ell,t)=0, & t>0. \end{cases} \tag{1}$$

Let $\mathcal{L}$ be any linear operation that only depends on x. For example, a differential operator such as $\mathcal{L} = \frac{\partial^2}{\partial x^2}$. We have intentionally not included initial conditions, so we can focus first on the differential relations and boundary conditions. We attempt to construct a solution by using a functional form $u(x,t) = X(x)T(t)$. Substituting above yields

$$\begin{aligned} X(x)T'(t) &= \left(\mathcal{L}X(x)\right)T(t),\ t>0,\ 0<x<\ell \\ X(0)T(t) &= X(\ell)T(t) = 0. \end{aligned}$$

If we divide both sides by $X(x)T(t)$, we obtain

$$\frac{T'(t)}{T(t)} = \frac{\mathcal{L}X(x)}{X(x)} = -\lambda,$$

where λ is a constant. It follows that this expression is a constant, since the only way for two functions $f(t) = g(x)$ that have different independent variables (here t and x) is for the f,g to be trivial functions that have no dependence on t or x and hence are constant. In other words, the only way we can ensure equality for all choices of t and x is if $f(t) = C = g(x)$, where C is some constant. This allows us to reduce the problem to solving two differential equations of the form

$$\begin{aligned} T'(t) &= -\lambda T(t) \\ \mathcal{L}X(x) &= -\lambda X(x), \quad X(0) = X(\ell) = 0. \end{aligned}$$

The first has the solution $T(t) = C\exp(-t\lambda)$. The second equation has an interesting form. We can view this equation as an eigenvalue problem, where $-\lambda$ is an eigenvalue of the linear operation $\mathcal{L}$. The solution $X(x)$ would then be the *eigenfunction* of the operator $\mathcal{L}$ when the eigenvalue is $-\lambda$. Let's denote this solution as $X_\lambda(x)$. Then a solution of the pde above is given by

$$u_\lambda(x,t) = C\exp(-t\lambda)\,X_\lambda(x).$$

Since we did not specify initial conditions or other requirements, there could be many solutions of the pde in equation 1 by choosing different eigenvalues $-\lambda$. Since the equation is linear, any additive combination $u = a_1 u_{\lambda_1} + a_2 u_{\lambda_2}$ of solutions u_{λ_1} and u_{λ_2} would again be a solution. This provides one general way to obtain a rich collection of solutions making use of the properties of $\mathcal{L}$. As we will see, the eigenfunctions of differential operators often provide rich enough collections to represent general solutions of pdes in the form

$$u(x,t) = \sum_\lambda a_\lambda u_\lambda(x,t).$$

We remark this approach based on eigenfunctions also can serve other useful roles, such as in the analysis of the behaviors of the pde or in the development of numerical approximations.

We also consider the case when the time derivative is second-order

$$\begin{cases} u_{tt} = \mathcal{L}u, & t > 0,\ 0 < x < \ell \\ u(0,t) = u(\ell,t) = 0, & t > 0. \end{cases} \tag{2}$$

Let $\mathcal{L}$ be any linear operation that only depends on x. Much of the analysis will follow similarly, so we focus on where things differ in this case. By substituting $u(x,t) = X(x)T(t)$ above we obtain dividing both sides by $X(x)T(t)$ the equations

$$\frac{T''(t)}{T(t)} = \frac{\mathcal{L}X(x)}{X(x)} = -\lambda.$$

This reduces the problem to solving

$$\begin{aligned} T''(t) &= -\lambda T(t) \\ \mathcal{L}X(x) &= -\lambda X(x), \quad X(0) = X(\ell) = 0. \end{aligned}$$

The first has the solution $T(t) = C_1 \cos\left(t\sqrt{\lambda}\right) + C_2 \sin\left(t\sqrt{\lambda}\right)$. The second equation again can be viewed as an eigenvalue problem, where $-\lambda$ is an eigenvalue of the linear operation $\mathcal{L}$. In fact, it is the same eigenvalue problem as before, so the solutions $X_\lambda(x)$ also will be the same. This gives solutions of the form

$$u_\lambda(x,t) = \left(C_1 \cos\left(t\sqrt{\lambda}\right) + C_2 \sin\left(t\sqrt{\lambda}\right)\right) X_\lambda(x).$$

For many pdes, we can use combinations of these functions for representing solutions in the form

$$u(x,t) = \sum_\lambda u_\lambda(x,t) = \sum_k \left(a_k \cos\left(t\sqrt{\lambda}\right) + b_k \sin\left(t\sqrt{\lambda}\right)\right) X_{\lambda_k}(x).$$

Here, we parametrized the constants as the coefficients a_k, b_k over an index $k \in \mathbb{Z}$ assuming a discrete collection of eigenvalues. To obtain more specific solutions, we will next consider particular pdes using these approaches.

Solution of Parabolic PDEs with Dirichlet Boundary Conditions:
Consider

$$\begin{cases} u_t = \kappa u_{xx}, & t > 0,\ 0 < x < \ell \\ u(0,t) = u(\ell,t) = 0, & t > 0 \\ u(x,0) = \phi(x), & t = 0. \end{cases} \tag{3}$$

Construct a solution by using the solution form

$$u(x,t) = X(t)T(t).$$

Substituting this into the pde, and ignoring for now the initial condition, we obtain

$$\begin{aligned} X(x)T'(t) &= \kappa X''(x)T(t),\ t > 0,\ 0 < x < \ell \\ X(0)T(t) &= X(\ell)T(t) = 0. \end{aligned}$$

We aim to obtain an expression that separates the two independent variables x and t. This can be accomplished above if we divide both sides by $\kappa X(x)T(t)$, which yields

$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

where λ is a constant. It follows that this expression is a constant, since the only way for two functions $f(t) = g(x)$ with different independent variables to be equal is if they have no no dependence on t or x. This allows us to obtain for the pde the two Ordinary Differential Equations (ODEs)

$$\begin{aligned} T'(t) &= -\kappa\lambda T(t) \\ X''(t) &= -\lambda X(x), \quad X(0) = X(\ell) = 0. \end{aligned}$$

We can solve these to obtain the solutions

$$\begin{aligned} T(t) &= \tilde{C}_1 \exp\left(-\kappa\lambda t\right) \\ X(x) &= \tilde{C}_2 \sin(\sqrt{\lambda}x) + \tilde{C}_3 \cos(\sqrt{\lambda}x). \end{aligned}$$

The boundary conditions $X(0) = X(\ell) = 0$ requires $\tilde{C}_3 = 0$ and $\sqrt{\lambda} = \frac{\pi}{\ell}k$ for some $k \in \mathbb{Z}$. This gives $\lambda = \lambda_k = \frac{k^2\pi^2}{\ell^2}$. We remark that $-\lambda$ is the eigenvalue of the differential operator $\mathcal{L} = \frac{\partial^2}{\partial x^2}$ on $[0, \ell]$ with Dirichlet boundary conditions and $\sin(\sqrt{\lambda_k}x) = \sin\left(\frac{k\pi}{\ell}x\right)$ are the eigenfunctions. Now using that $u(x,t) = X(x)T(t)$ we obtain solutions to the pde of the form

$$u(x,t) = A\exp\left(-t\kappa\lambda_k\right)\sin\left(\sqrt{\lambda_k}x\right),$$

where A is a constant. Since any linear combination of these solutions is again a solution, we obtain the general solution

$$u(x,t) = \sum_{k=1}^{\infty} a_k \exp\left(-t\kappa\lambda_k\right)\sin\left(\sqrt{\lambda_k}x\right) = \sum_{k=1}^{\infty} a_k \exp\left(-t\kappa k^2\pi^2/\ell^2\right)\sin\left(\frac{k\pi}{\ell}x\right). \quad (4)$$

The reason we do not need to sum over $k = -\infty$ to ∞ is that $\sin(\cdot)$ is an odd function. Since we have $\sin(\lambda_{-k}y) = -\sin(\lambda_k y)$, this allows us to combine into one contribution of a_k the terms associated with $\pm k$. The term with $k = 0$ is always zero in this case.

We can solve the initial value problem provided we can find a choice of a_k to match $\phi(x)$. This requires

$$\phi(x) = u(x,0) = \sum_{k=1}^{\infty} a_k \sin\left(\sqrt{\lambda_k}x\right) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi}{\ell}x\right).$$

This can be done by using the sum-angle identity

$$\sin(k_1x)\sin(k_2x) = \frac{1}{2}\cos\left((k_2 - k_1)x\right) - \frac{1}{2}\cos\left((k_2 + k_1)x\right).$$

This yields the integral with $k_1, k_2 \in \mathbb{Z}$

$$\int_0^\pi \sin(k_1 x)\sin(k_2 x)\,dx = \begin{cases} \pi/2, & \text{if } k_1 = k_2 \\ 0, & \text{if } k_1 \neq k_2 \\ 0, & \text{if } k_1 = 0 \text{ or } k_2 = 0. \end{cases}$$

We can obtain the coefficients a_{k^*} by using

$$\begin{aligned} \int_0^\ell \phi(x)\sin\left(\frac{k^*\pi}{\ell}x\right)dx &= \sum_{k=1}^\infty a_k \int_0^\ell \sin\left(\frac{k\pi}{\ell}x\right)\sin\left(\frac{k^*\pi}{\ell}x\right)dx \\ &= \sum_{k=1}^\infty a_k \left(\frac{\ell}{\pi}\right)\int_0^\pi \sin(k\tilde{x})\sin(k^*\tilde{x})\,d\tilde{x} \\ &= \frac{\ell}{2}a_k^*. \end{aligned}$$

This also shows the integral with $k_1, k_2 \in \mathbb{Z}$ evaluates to

$$\int_0^\ell \sin\left(\frac{k_1\pi}{\ell}x\right)\sin\left(\frac{k_2\pi}{\ell}x\right)dx = \begin{cases} \frac{\ell}{2} & \text{if } k_1 = k_2 \\ 0, & \text{if } k_1 \neq k_2 \\ 0, & \text{if } k_1 = 0 \text{ or } k_2 = 0. \end{cases}$$

This gives the coefficients for matching the initial conditions

$$a_k = \frac{2}{\ell}\int_0^\ell \phi(x)\sin\left(\frac{k\pi}{\ell}x\right)dx.$$

This choice of coefficients along with equation 4 gives the solution to 3.

We remark that the above integration can be viewed as a type of operator $\mathcal{F}_k$ that acts on functions ϕ to obtain $a_k = \mathcal{F}_k[\phi]$. In particular, $\mathcal{F}_k[\phi] = \frac{2}{\ell}\int_0^\ell \phi(x)\sin\left(\frac{k\pi}{\ell}x\right)dx$. This is sometimes referred to as a *Sine Transform*. Related transforms and discretizations are used widely in numerical methods for pdes, signal processing, and in data compression of images (in JPEG format), among other applications. As we will discuss in more detail later, these operations also have close connections with the *Fourier Transform*.

Solution of Parabolic PDEs with Neumann Boundary Conditions:
Consider

$$\begin{cases} u_t &= \kappa u_{xx}, & t > 0,\ 0 < x < \ell \\ u_x(0,t) &= u_x(\ell,t) = 0, & t > 0 \\ u(x,0) &= \phi(x), & t = 0. \end{cases} \tag{5}$$

We proceed as before to construct solutions of the form $u(x,t) = X(t)T(t)$. In this case we have

$$\begin{aligned} X(x)T'(t) &= \kappa X''(x)T(t),\ t > 0,\ 0 < x < \ell \\ X'(0)T(t) &= X'(\ell)T(t) = 0. \end{aligned}$$

To separate the variables x and t, we divide both sides by $\kappa X(x)T(t)$, which yields

$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

where λ is a constant. We obtain the two differential equations

$$\begin{aligned} T'(t) &= -\kappa\lambda T(t) \\ X''(t) &= -\lambda X(x), \quad X'(0) = X('\ell) = 0. \end{aligned}$$

We can solve these to obtain the solutions

$$\begin{aligned} T(t) &= \tilde{C}_1 \exp\left(-\kappa\lambda t\right) \\ X(x) &= \tilde{C}_2 \sin(\sqrt{\lambda}x) + \tilde{C}_3 \cos(\sqrt{\lambda}x). \end{aligned}$$

The boundary conditions $X'(0) = X'(\ell) = 0$ requires $\tilde{C}_2 = 0$ and $\sqrt{\lambda} = \frac{\pi}{\ell}k$ for some $k \in \mathbb{Z}$. This gives $\lambda = \lambda_k = \frac{k^2\pi^2}{\ell^2}$. We remark that $-\lambda$ is the eigenvalue of the differential operator $\mathcal{L} = \frac{\partial^2}{\partial x^2}$ on $[0,\ell]$ with Neumann boundary conditions and $\cos(\sqrt{\lambda_k}x) = \cos\left(\frac{k\pi}{\ell}x\right)$ are the eigenfunctions.

Now using that $u(x,t) = X(x)T(t)$, we obtain solutions to the pde of the form

$$u(x,t) = A\exp\left(-t\kappa\lambda_k\right)\cos(\sqrt{\lambda_k}x),$$

where A is a constant. We can combine these solutions to obtain the general solution

$$u(x,t) = \sum_{k=0}^{\infty} \tilde{a}_k \exp\left(-t\kappa k^2\pi^2/\ell^2\right)\cos\left(\frac{k\pi}{\ell}x\right). \tag{6}$$

We can solve the initial value problem provided we can find coefficients that match the initial conditions. This requires

$$u(x,0) = \phi(x) = \sum_{k=0}^{\infty} \tilde{a}_k \cos\left(\frac{k\pi}{\ell}x\right). \tag{7}$$

This can be done by using the sum-angle identity

$$\cos(k_1x)\cos(k_2x) = \frac{1}{2}\cos\left((k_2-k_1)x\right) + \frac{1}{2}\cos\left((k_2+k_1)x\right).$$

This yields the integral with $k_1, k_2 \in \mathbb{Z}$

$$\int_0^{\pi} \cos\left(k_1x\right)\cos\left(k_2x\right)dx = \begin{cases} \pi/2, & \text{if } k_1 = k_2 \neq 0 \\ \pi, & \text{if } k_1 = k_2 = 0 \\ 0, & \text{if } k_1 \neq k_2. \end{cases}$$

We can obtain the coefficients $\tilde{a}_{k^*}$ by using

$$\begin{aligned}
\int_0^\ell \phi(x)\cos\left(\frac{k^*\pi}{\ell}x\right)dx &= \sum_{k=1}^\infty \tilde{a}_k \int_0^\ell \cos\left(\frac{k\pi}{\ell}x\right)\cos\left(\frac{k^*\pi}{\ell}x\right)dx \\
&= \sum_{k=1}^\infty \tilde{a}_k \left(\frac{\ell}{\pi}\right)\int_0^\pi \cos\left(k\tilde{x}\right)\cos\left(k^*\tilde{x}\right)d\tilde{x} \\
&= \begin{cases} \frac{\ell}{2}\tilde{a}_k^*, & \text{if } k^* \neq 0 \\ \ell\tilde{a}_k^*, & \text{if } k^* = 0. \end{cases}
\end{aligned}$$

This shows the integral with $k_1, k_2 \in \mathbb{Z}$ evaluates to

$$\int_0^\ell \cos\left(\frac{k_1\pi}{\ell}x\right)\cos\left(\frac{k_2\pi}{\ell}x\right)dx = \begin{cases} \frac{\ell}{2} & \text{if } k_1 = k_2 \neq 0 \\ \ell & \text{if } k_1 = k_2 = 0 \\ 0, & \text{if } k_1 \neq k_2. \end{cases}$$

This gives the coefficients for matching the initial conditions when $k \neq 0$

$$\tilde{a}_k = \frac{2}{\ell}\int_0^\ell \phi(x)\cos\left(\frac{k\pi}{\ell}x\right)dx.$$

and when $k = 0$

$$\tilde{a}_0 = \frac{1}{\ell}\int_0^\ell \phi(x)dx.$$

This choice of coefficients along with equation 6 gives the solution to 5.

In practice, it is often convenient to be able to compute coefficients using just one form for the integration. For this purpose, notation is typically used with $a_0 = 2\tilde{a}_0$ and $a_k = \tilde{a}_k$. This gives for the solution

$$u(x,t) = \frac{1}{2}a_0 + \sum_{k=1}^\infty a_k \exp\left(-t\kappa k^2\pi^2/\ell^2\right)\cos\left(\frac{k\pi}{\ell}x\right). \tag{8}$$

The coefficients can be obtained from the initial conditions using for $k \geq 0$

$$a_k = \frac{2}{\ell}\int_0^\ell \phi(x)\cos\left(\frac{k\pi}{\ell}x\right)dx.$$

This choice of coefficients using equation 8 gives another way to represent the solution of 5.

Solution of Hyperbolic PDEs with Dirichlet Boundary Conditions:
Consider the wave equation on the interval $[0,\ell]$

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t > 0,\ 0 < x < \ell \\ u(0,t) = u(\ell,t) = 0, & t > 0 \\ u(x,0) = \phi(x),\ \ u_t(x,0) = \psi(x), & t = 0. \end{cases} \tag{9}$$

We construct solutions of the form $u(x,t) = X(t)T(t)$. Substituting this into the pde, we obtain

$$\begin{aligned} X(x)T''(t) &= c^2X''(x)T(t),\ t > 0,\ 0 < x < \ell \\ X(0)T(t) &= X(\ell)T(t) = 0. \end{aligned}$$

Dividing by $c^2X(x)T(t)$ yields

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

where λ is a constant. This reduces the problem to two differential equations

$$\begin{aligned} T''(t) &= -c^2\lambda T(t) \\ X''(t) &= -\lambda X(x), \quad X(0) = X(\ell) = 0. \end{aligned}$$

We can solve these to obtain the solutions

$$\begin{aligned} T(t) &= \tilde{C}_1 \cos\left(t\sqrt{c^2\lambda}\right) + \tilde{C}_2 \sin\left(t\sqrt{c^2\lambda}\right) \\ X(x) &= \tilde{C}_3 \sin(\sqrt{\lambda}x) + \tilde{C}_4 \cos(\sqrt{\lambda}x). \end{aligned}$$

The boundary conditions $X(0) = X(\ell) = 0$ requires $\tilde{C}_4 = 0$ and $\sqrt{\lambda} = \frac{\pi}{\ell}k$ for some $k \in \mathbb{Z}$. This gives $\lambda = \lambda_k = \frac{k^2\pi^2}{\ell^2}$. Now using that $u(x,t) = X(x)T(t)$ we obtain solutions to the pde of the form

$$u(x,t) = \left(A\cos\left(t\sqrt{c^2\lambda_k}\right) + B\sin\left(t\sqrt{c^2\lambda_k}\right)\right)\sin\left(\sqrt{\lambda_k}x\right),$$

where A, B are constants. Since linear combinations are solutions, we obtain the general solution

$$u(x,t) = \sum_{k=1}^{\infty}\left(a_k\cos\left(tkc\pi/\ell\right) + b_k\sin\left(tkc\pi/\ell\right)\right)\sin\left(\frac{k\pi}{\ell}x\right). \tag{10}$$

We can solve the initial value problem provided we can find a choice of a_k, b_k that match $\phi(x)$ and $\psi(x)$. This requires

$$\begin{aligned} \phi(x) &= u(x,0) = \sum_{k=1}^{\infty} a_k \sin\left(\sqrt{\lambda_k}x\right) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi}{\ell}x\right) \\ \psi(x) &= u_t(x,0) = \sum_{k=1}^{\infty} b_k\left(\sqrt{c^2\lambda_k}\right)\sin\left(\sqrt{\lambda_k}x\right) = \sum_{k=1}^{\infty} b_k\left(\frac{kc\pi}{\ell}\right)\sin\left(\frac{k\pi}{\ell}x\right). \end{aligned}$$

This can be done by using that the integral

$$\int_0^\ell \sin\left(\frac{k_1\pi}{\ell}x\right)\sin\left(\frac{k_2\pi}{\ell}x\right)dx = \begin{cases} \frac{\ell}{2} & \text{if } k_1 = k_2 \\ 0, & \text{if } k_1 \neq k_2 \\ 0, & \text{if } k_1 = 0 \text{ or } k_2 = 0. \end{cases}$$

This gives the coefficients for matching the initial conditions

$$\begin{aligned} a_k &= \frac{2}{\ell}\int_0^\ell \phi(x)\sin\left(\frac{k\pi}{\ell}x\right)dx \\ b_k &= \frac{2}{kc\pi}\int_0^\ell \psi(x)\sin\left(\frac{k\pi}{\ell}x\right)dx. \end{aligned}$$

By using this choice of coefficients in equation 10 we obtain the solution to equation 9.

Summary:
The *Separation of Variables* approach provides powerful methods for obtaining solutions of pdes. This typically results in solutions represented by infinite series expansions involving the eigenvalues and eigenfunctions of the spatial differential operator. These representations are used in analysis to gain insights into the behaviors of pdes and for developing numerical approximations. As we shall discuss in more detail, these methods have close connections with the *Fourier Transform*, methods in harmonic analysis, and related approaches for pdes.