SYSTEMATIC STOCHASTIC REDUCTION OF INERTIAL FLUID-STRUCTURE INTERACTIONS SUBJECT TO THERMAL FLUCTUATIONS

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Abstract. We present analysis for the reduction of an inertial description of fluid-structure interactions subject to thermal fluctuations. We show how the viscous coupling between the immersed structures and the fluid can be simplified in the regime where this coupling becomes increasingly strong. Many descriptions in fluid mechanics and in the formulation of computational methods account for fluid-structure interactions through viscous drag terms to tranfer momentum from the fluid to immersed structures. In the inertial regime, this coupling often introduces a prohibitively small time-scale into the temporal dynamics of the fluid-structure system. This is further exacerbated in the presence of thermal fluctuations. We discuss here a systematic reduction technique for the full inertial equations to obtain a simplified description where this coupling term is eliminated. This approach also accounts for the effective stochastic equations for the fluid-structure dynamics. The analysis is based on use of the infinitesmal generator of the SPDEs and a singular perturbation analysis of the backward kolomogorov PDEs. We also discuss the physical motivations and interpretation of the obtained reduced description of the fluid-structure system.

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1. Introduction. The investigation of fluid-structure interactions has a rich history. Problems motivating past work include the study of aerodynamic oscillations induced in airplane wings and propellers [20, 22], the study of animal locomotion including swimming and insect flight [35, 29, 41], and the study of physiological problems such as blood flow through heart valves [25, 39, 23]. Central to studying these problems in practice is the development of descriptions which capture essential features of the fluid structure interactions while introducing approximations which facilitate analysis and the development of tractable numerical methods [20, 36]. This area of research is still very active [25, 11, 36, 10]. While we have above mentioned macroscopic problems involving fluid-structure interaction, recent developments in the sciences and technology motivate the study of fluid-structure interactions in new physical regimes often involving very small length and time scales. This includes the dynamic responses of soft materials and complex fluids [44, 46, 13, 37]. At the molecular level advances in computational modeling at the level of coarse-grained descriptions with implicit solvent also motivate the need for the development of new descriptions of fluid-structure interactions to account through continuum fields the important roles played by momentum transfer of the neglected solvent fluid in dynamic responses. This includes the study of implicit solvent models for polymers, gels, and lipid bilayers (cite). For these problems thermal fluctuations often play an important role and pose additional challenges in the study of fluid-structure systems.

Significant past work has been done on the formulation of descriptions of fluid-structure

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interactions subject to thermal fluctuations. Many of these analytic and numerical approaches originate from the polymer physics community [17, 21, 42, 9]. To obtain descriptions tractable for analysis and numerical simulation, these approaches typically place an emphasis on approximations which retain only the structure degrees of freedom. This often results in significant simplifications in the descriptions and in significant computational savings. This eliminates the many degrees of freedom associated with the fluid and avoids having to resolve the potentially intricate and stiff stochastic dynamics of the fluid. These approaches have worked especially well for the study of bulk phenomena in free solution and the study of complex fluids and soft materials [42, 17, 28].

Recent applications arising in the sciences and in technological fields present situations in which resolving the dynamics of the fluid may be important and even advantageous both for modeling and computation. This includes modeling the spectroscopic responses of biological materials [47, 24, 34], studying transport in microfluidic and nanofluidic devices [44, 38], and investigating dynamics in biological systems [2, 15]. There are also other motivations for representing the fluid explicitly and resolving its stochastic dynamics. This includes the development of hybrid fluid-particle models in which thermal fluctuations mediate important effects when coupling continuum and particle descriptions [16, 18], the study of hydrodynamic coupling and diffusion in the vicinity of surfaces having complicated geometries [44], and the study of systems in which there are many interacting mechanical structures [6, 40, 39, 3, 8, 10, 45, 36, 48, 26, 31, 5, 12, 7, 19, 6]. To help obtain less stiff equations for amenable to practical numerical approximation, some prior work has been done on the above mentioned fluid-structure interaction approaches. For example, for the Stochastic Immersed Boundary Method a reduction procedure was developed to eliminate the fluid degrees of freedom and obtain a purely particle based description [32].

To facilitate the development of tractable descriptions and efficient numerical methods for study of fluid-structure systems, we consider a rather general formalism which captures essential features of the coupled stochastic dynamics of the fluid and structures. To model the fluid-structure system, a mechanical description is utilized involving both Eulerian and Lagrangian reference frames. Such mixed descriptions arise rather naturally, since it is often convenient to describe the structure configurations in a Lagrangian reference frame while it is convenient to describe the fluid in an Eulerian reference frame. In practice, this presents a number of challenges for analysis and numerical studies. A central issue concerns how to couple the descriptions to represent accurately the fluid-structure interactions, while obtaining a coupled description which can be treated efficiently ultimately by numerical methods. An important issue often arising concerns the challenge of developing temporal integrators capable of handling a wide range of time-scales exhibited by the fluid-structure dynamics. Another important issue concerns how to account properly for thermal fluctuations in such numerical integrators and associated approximations. This all must be done carefully to be consistent with statistical mechanics.

To address these challenges and gain further insights into the fluid-structure equations, we develop a systematic reduction of the full stochastic description to a simplified description that removes the fastest temporal dynamics of the system and reduces the number of degrees of freedom. Our approach is based on the Infinitesmal Generator of the SELM SPDEs and a singular perturbation analysis of the Backward Kolomogorov PDEs following an approach recently introduced in [33, 32]. For the SELM approach, we consider the regime in which the fluid-structure coupling becomes strong. We then discuss our reduced descriptions of the fluid-structure interactions in a few different regimes and their physical interpretation.

2. Summary of the Stochastic Eulerian Lagrangian Method. We summarize here the Stochastic Eulerian Lagrangian Method, abbreviated as SELM. We present the general formalism and a number of alternative descriptions of the fluid-structure system. In many situations the stochastic differential equations for the full fluid-structure dynamics exhibits stiffness. To cope with this issue and to develop efficient numerical methods, simplified descriptions are discussed for various physical regimes. A more detailed discussion
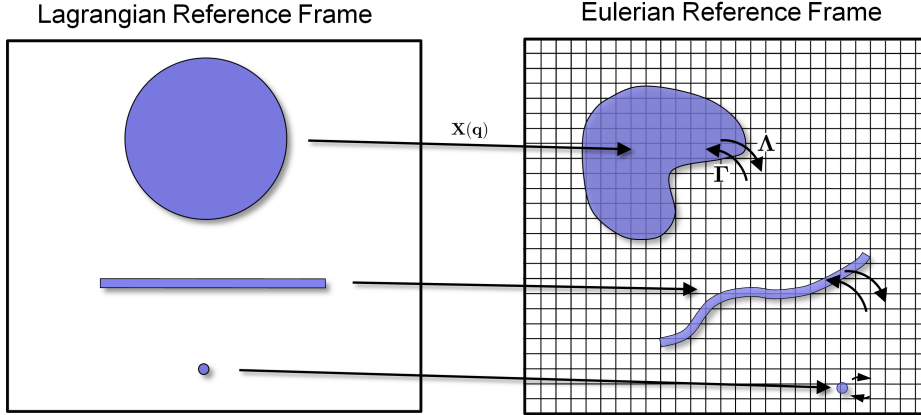
Fig. 2.1. *The description of the fluid-structure system utilizes both Eulerian and Lagrangian reference frames. The structure mechanics are often most naturally described using a Lagrangian reference frame. The fluid mechanics are often most naturally described using an Eulerian reference frame. The mapping $\mathbf{X}(\mathbf{q})$ relates the Lagrangian reference frame to the Eulerian reference frame. The operator Γ prescribes how structures are to be coupled to the fluid. The operator Λ prescribes how the fluid is to be coupled to the structures. A variety of fluid-structure interactions can be represented in this way. This includes rigid and deformable bodies, membrane structures, polymeric structures, or point particles.*

and derivation of SELM and the reduced equations in each of the physical regimes is given in Section 6.

To study the dynamics of fluid-structure interactions in the presence of thermal fluctuations, we utilize a mechanical description involving Eulerian and Lagrangian reference frames. Such mixed descriptions arise rather naturally, since it is often convenient to describe the structure configurations in a Lagrangian reference frame while it is convenient to describe the fluid in an Eulerian reference frame. In principle more general descriptions using other reference frames could also be considered. Descriptions for fluid-structure systems having these features can be described rather generally by the following dynamic equations

$$\rho \frac{d\mathbf{u}}{dt} = \mathcal{L}\mathbf{u} + \Lambda[\Upsilon(\mathbf{v} - \Gamma\mathbf{u})] + \lambda + \mathbf{f}_{\mathrm{thm}} \tag{2.1}$$

$$m \frac{d\mathbf{v}}{dt} = -\Upsilon(\mathbf{v} - \Gamma\mathbf{u}) - \nabla_{\mathbf{X}}\Phi[\mathbf{X}] + \zeta + \mathbf{F}_{\mathrm{thm}} \tag{2.2}$$

$$\frac{d\mathbf{X}}{dt} = \mathbf{v}. \tag{2.3}$$

The $\mathbf{u}$ denotes the velocity of the fluid, ρ the uniform fluid density. The $\mathbf{X}$ denotes the configuration of the structure and $\mathbf{v}$ the velocity of the structure. The mass of the structure is denoted by m. To simplify the presentation we treat here only the case when ρ and m are constant, but with some modifications these could also be treated as variable. The λ, ζ are Lagrange multipliers for imposed constraints, such as incompressibility of the fluid or a rigid body constraint of a structure. The operator $\mathcal{L}$ is used to account for dissipation in the fluid, such as associated with Newtonian fluid stresses [1]. To account for how the fluid and structures are coupled, a few general operators are introduced, $\Gamma, \Upsilon, \Lambda$.

The linear operators $\Gamma, \Lambda, \Upsilon$ are used to model the fluid-structure coupling. The Γ operator describes how a structure depends on the fluid flow while $-\Upsilon$ is a negative definite dissipative operator describing the viscous interactions coupling the structure to the fluid. We assume throughout that this dissipative operator is symmetric, $\Upsilon = \Upsilon^T$. The linear operator Λ is used to attribute a spatial location for the viscous interactions between the structure and fluid. The linear operators are assumed to have dependence only on the

configuration degrees of freedom $\Gamma = \Gamma[\mathbf{X}]$, $\Lambda = \Lambda[\mathbf{X}]$. We assume further that Υ does not have any dependence on $\mathbf{X}$.

To account for the mechanics of structures, $\Phi[\mathbf{X}]$ denotes the potential energy of the configuration $\mathbf{X}$. The total energy associated with this fluid-structure system is given by

$$E[\mathbf{u}, \mathbf{v}, \mathbf{X}] = \int_\Omega \frac{1}{2}\rho|\mathbf{u}(\mathbf{y})|^2 d\mathbf{y} + \frac{1}{2}m\mathbf{v}^2 + \Phi[\mathbf{X}]. \tag{2.4}$$

The first two terms give the kinetic energy of the fluid and structures. The last term gives the potential energy of the structures.

As we shall discuss, it is natural to consider coupling operators Λ and Γ which are adjoint in the sense

$$\int_\mathcal{S} (\Gamma\mathbf{u})(\mathbf{q}) \cdot \mathbf{v}(\mathbf{q}) d\mathbf{q} = \int_\Omega \mathbf{u}(\mathbf{x}) \cdot (\Lambda\mathbf{v})(\mathbf{x}) d\mathbf{x} \tag{2.5}$$

for any $\mathbf{u}$ and $\mathbf{v}$. The $\mathcal{S}$ and Ω denote the spaces used to parameterize respectively the structures and the fluid. We denote such an adjoint by $\Lambda = \Gamma^\dagger$ or $\Gamma = \Lambda^\dagger$. This adjoint condition can be shown to have the important consequence that the fluid-structure coupling conserves energy when $\Upsilon \to \infty$ in the inviscid and zero temperature limit.

In practice, the conditions discussed above can be relaxed somewhat. For our present purposes these conditions help simplify the presentation. Each of these operators will be discussed in more detail below.

To account for thermal fluctuations, a random force density $\mathbf{f}_{\mathrm{thm}}$ is introduced in the fluid equations and $\mathbf{F}_{\mathrm{thm}}$ in the structure equations. These account for spontaneous changes in the system momentum which occurs as a result of the influence of unresolved microscopic degrees of freedom and unresolved events occurring in the fluid and in the fluid-structure interactions.

The thermal fluctuations consistent with the form of the total energy and relaxation dynamics of the system are taken into account by the introduction of stochastic driving fields in the momentum equations of the fluid and structures. The stochastic driving fields are taken to be Gaussian processes with mean zero and with δ-correlation in time [43]. By the fluctuation-dissipation principle [43] these have covariances given by

$$\langle \mathbf{f}_{\mathrm{thm}}(s)\mathbf{f}_{\mathrm{thm}}^T(t)\rangle = -\left(2k_BT\right)\left(\mathcal{L} - \Lambda\Upsilon\Gamma\right)\delta(t-s) \tag{2.6}$$

$$\langle \mathbf{F}_{\mathrm{thm}}(s)\mathbf{F}_{\mathrm{thm}}^T(t)\rangle = \left(2k_BT\right)\Upsilon\,\delta(t-s) \tag{2.7}$$

$$\langle \mathbf{f}_{\mathrm{thm}}(s)\mathbf{F}_{\mathrm{thm}}^T(t)\rangle = -\left(2k_BT\right)\Lambda\Upsilon\,\delta(t-s). \tag{2.8}$$

We have used that $\Gamma = \Lambda^\dagger$ and $\Upsilon = \Upsilon^T$. We remark that the notation $\mathbf{g}\mathbf{h}^T$ which is used for the covariance operators should be interpreted as the tensor product. This notation is meant to suggest the analogue to the outer-product operation which holds in the discrete setting [4]. A more detailed discussion and derivation of the thermal fluctuations is given in Section 6.

It is important to mention that some care must be taken when using the above formalism in practice and when choosing operators. An important issue concerns the treatment of the material derivative of the fluid, $d\mathbf{u}/dt = \partial\mathbf{u}/\partial t + \mathbf{u} \cdot \nabla\mathbf{u}$. For stochastic systems the field $\mathbf{u}$ is often highly irregular and not defined in a point-wise sense, but rather only in the sense of a generalized function (distribution) [14, 30]. This presents issues in how to define the non-linear term arising in the material derivative, which appears to require point-wise values of $\mathbf{u}$. For such irregular velocity fields, this also calls into question the applicability of the theorems typically used to derive the differential equations from the conservation laws. For instance, for such velocity fields the fluid material body may no longer exhibit smooth deformations over time.

There are a number of ways to deal with this issue. The first is to consider a regularization of the fluid stresses, which are typically the source of irregularity, see equation 2.6.

This can be motivated by the fact that the fluid stress tensors typically considered in continuum mechanics are expected to become inaccurate at molecular length-scales. Ideally, from molecular models of the fluid the small-length scale (large wave-number) responses of the fluid could be determined and provide a justified regularization. For instance, this could provide an alternative to using responses based on Newtonian stresses for all length-scales. For the SELM formalism, this would simply correspond to using for $\mathcal{L}$ an alternative to the dissipative operator based on Newtonian stresses. The second more easily implemented approach is simply to work with the linearized material derivative, which still retains many of the essential features of the fluid dynamics and is useful for many applications [5].

In this version of SELM, we assume the advective derivative is well-defined and include its terms in the computation. We will use a time scale specific to the advective term to keep track of which terms arise due to the convective term. In this way, the advective term may be neglected in specific regimes where the approximation $d\mathbf{u}/dt = \partial\mathbf{u}/\partial t$ can be used.

3. Formulation in terms of Total Momentum and Particle Momentum equations. We now consider the regime in which the full dynamics of the fluid-structure system are retained but reformulated in terms of a field describing the total momentum of the fluid-structure system at a given spatial location. This description is more convenient to work with in practice since it results in simplifications in the stochastic driving fields. Primarily, the two noise terms in the stochastic differential equations involved become uncorrelated, simplifying the analysis. For this purpose we define

$$\mathbf{p}(\mathbf{x}(t),t) = \rho\mathbf{u}(\mathbf{x}(t),t) + \Lambda(\mathbf{X}(t))[m\mathbf{v}(t)]. \tag{3.1}$$

The operator Λ is used to give the distribution in space of the momentum associated with the structures for given configuration $\mathbf{X}(t)$. We may take a time derivative in $\mathbf{p}$ to obtain:

$$\frac{\partial\mathbf{p}}{\partial t} = \rho\frac{\partial\mathbf{u}}{\partial t} + \nabla_{\mathbf{X}}\Lambda(m\mathbf{v})\cdot\mathbf{v} + \Lambda(m\frac{\partial\mathbf{v}}{\partial t}) \tag{3.2}$$

Now,

$$\rho\frac{d\mathbf{u}}{dt} = \rho\frac{\partial\mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} = \mathcal{L}\mathbf{u} + \Lambda[\Upsilon(\mathbf{v} - \Gamma\mathbf{u})] + \lambda + \mathbf{f}_{\text{thm}} \tag{3.3}$$

Using this approach, the fluid-structure dynamics are described by

$$\frac{\partial\mathbf{p}}{\partial t} = \mathcal{L}\mathbf{u} + \Lambda[\Upsilon(\mathbf{v} - \Gamma\mathbf{u})] + \lambda + \mathbf{f}_{\text{thm}} \tag{3.4}$$

$$- \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla_{\mathbf{X}}\Lambda(m\mathbf{v})\cdot\mathbf{v} + \Lambda(m\frac{\partial\mathbf{v}}{\partial t}) \tag{3.5}$$

$$= \mathcal{L}\rho^{-1}[\mathbf{p} - \Lambda(m\mathbf{v})] - \rho^{-1}[\mathbf{p} - \Lambda(m\mathbf{v})]\cdot\nabla[\mathbf{p} - \Lambda(m\mathbf{v})] + \nabla_{\mathbf{X}}\Lambda(m\mathbf{v})\cdot\mathbf{v} \tag{3.6}$$

$$+ \Lambda[-\nabla_{\mathbf{X}}\Phi(\mathbf{X})] + \lambda + \mathbf{g}_{thm} \tag{3.7}$$

$$m\frac{d\mathbf{v}}{dt} = -\Upsilon\left(\mathbf{v} - \rho^{-1}\Gamma(\mathbf{p} - \Lambda m\mathbf{v})\right) - \nabla_{\mathbf{X}}\Phi[\mathbf{X}] + \zeta + \mathbf{F}_{\text{thm}} \tag{3.8}$$

$$\frac{d\mathbf{X}}{dt} = \mathbf{v} \tag{3.9}$$

where $\mathbf{g}_{\text{thm}} = \mathbf{f}_{\text{thm}} + \Lambda[\mathbf{F}_{\text{thm}}]$. The third term in the first equation arises from the dependence of Λ on the configuration of the structures, $\Lambda[m\mathbf{v}] = (\Lambda[X])[m\mathbf{v}]$. The Lagrange multipliers for imposed constraints are denoted by λ, ζ. For the constraints, we use rather liberally the notation with the Lagrange multipliers denoted here not necessarily assumed to be equal to the previous definition. The stochastic driving fields are again Gaussian with mean zero and δ-correlation in time [43]. The stochastic driving fields have the covariance structure given by

$$\langle\mathbf{g}_{\text{thm}}(X,s)\mathbf{g}_{\text{thm}}^T(Y,t)\rangle = -(2k_BT)\,\mathcal{L}\,\delta^3(X-Y)\delta(t-s) \tag{3.10}$$

$$\langle\mathbf{F}_{\text{thm}}(s)\mathbf{F}_{\text{thm}}^T(t)\rangle = (2k_BT)\,\Upsilon\,\delta(t-s) \tag{3.11}$$

$$\langle\mathbf{g}_{\text{thm}}(X,s)\mathbf{F}_{\text{thm}}^T(t)\rangle = 0. \tag{3.12}$$

This formulation has the convenient feature that the stochastic driving fields become independent. This is a consequence of using the field for the total momentum for which the dissipative exchange of momentum between the fluid and structure no longer arises. In the equations for the total momentum, the only source of dissipation remaining occurs from the stresses of the fluid. This approach simplifies the effort required to generate numerically the stochastic driving fields and will be used throughout.

4. Non-Dimensional Equations. Our interest in this section is to develop an appropriate dimension analysis to make precise the reduce of our stochastic equations in a limiting regime corresponding to taking the viscous coupling coefficient large, "$\Upsilon \to \infty$". Our reduction procedure will then aim to eliminate equation 3.8, which carries Υ, and replaces equations 3.4 and 3.9 with effective equations in this limiting regime.

4.1. Dimension Analysis of the Equations. To faciliate proper handling of the reduction of the SELM SPDEs, it is convenient to work with non-dimensional equations. This will have the added benefit when taking limits of precisely defining the asymptotic regime. Since the SELM SPDEs describe physical laws the behaviors of the underlying dynamical system must be invariant under any change of the fundamental units, in this case the units of *mass*, *length*, *time*, and *temperature*. The Buckingham Π theorem provides a useful guideline for the number of non-dimensional groups required to characterise distinctly the non-dimensionally equivalent dynamical systems, in particular, that for N parameters and m fundamental units there is a need for $M = N - m$ non-dimensional constants $\Pi_1, \ldots \Pi_M$ (cite). Simple counting of the parameters for the SELM SPDEs gives $N = 6$ and the number of fundamental units involved are $m = 4$ requiring $M = 2$ non-dimensional groups to be identified.

Before we identify our non-dimensional groups, we will identify the time-scales that will allow us to understand the behavior of the system. We will first examine Υ in 3.8. Suppose we can non-dimensionalize $\Upsilon = \Upsilon_0 \bar{\Upsilon}$, where Υ_0 is a scalar that carries the units of Υ and $\bar{\Upsilon}$ is a non-dimensional operator. We will define t_1 as the relaxation time for v under the Υ operator, which is evidently m/Υ_0. This choice of t_1 also characterizes the momentum transfer rate between the structure and the fluid, since Υ controls the dissipative force on the structure and fluid. We will take $t_2 = \sqrt{m\ell^2/k_B T}$, which describes the time-scale for the particle to diffuse over a distance ℓ.

For the purposes of our analysis, the non-dimensional groups we we use are $\kappa = \rho\ell^3/m$ and $\epsilon = t_1/t_2$. In the notation, ρ is the fluid density, m the excess particle mass over the same volume of fluid, and ℓ the particle radius. In other words, κ describes the density ratio of the particle and fluid, while ϵ encapsulates the ratio of the two time-scales involved. We will fix κ for our reduction.

Now, we assume the dynamics of the system in which we are interested function on the time-scale t_2, and that $t_1 << t_2$. In this way, the fluctuations due to the interaction between the fluid and the structure occur on a much faster time scale than the remaining terms in the system.

Since we are only interested in taking $\epsilon \to 0$, we may use a similar scaling for each of our variables other than Υ. We non-dimensionalize each of the variables using ℓ, m, and t_2: $t = t_2 \bar{t}, \mathbf{X} = \ell\bar{\mathbf{X}}, \mathbf{v} = v_0 \bar{\mathbf{v}} = \frac{\ell}{t_2} \bar{\mathbf{v}}, \mathbf{p} = p_0 \bar{\mathbf{p}} = \frac{m}{t_2\ell^2} \bar{\mathbf{p}}, \Lambda = \Lambda_0 \bar{\Lambda} = \frac{1}{\ell^3} \bar{\Lambda}, \mathcal{L} = \mathcal{L}_0 \bar{\mathcal{L}} = \frac{m}{\ell^3 t_2} \bar{\mathcal{L}}, \Phi = \Phi_0 \bar{\Phi} = \frac{m\ell^2}{t_2^2} \bar{\Phi}$. .

We also non-dimensionalize the noise terms,

$$\mathbf{g}_{\mathrm{thm}}(\mathbf{X}, s) = g_0 \bar{\mathbf{g}}_{thm} = g_0 D_{-\bar{\mathcal{L}}} \bar{\xi}(\frac{\mathbf{X}}{\ell}, \frac{s}{t_2}), \tag{4.1}$$

$$\mathbf{F}_{\mathrm{thm}}(s) = F_0 \bar{\mathbf{F}}_{thm} = F_0 D_{\bar{\Upsilon}} \bar{\eta}(\frac{s}{t_2}). \tag{4.2}$$

Here,

$$\langle \bar{\xi}(\bar{\mathbf{X}}, \bar{s})\bar{\xi}^T(\bar{\mathbf{Y}}, \bar{t})\rangle = \delta^3(\bar{\mathbf{X}} - \bar{\mathbf{Y}})\delta^3(\bar{s} - \bar{t}), \tag{4.3}$$

$$\langle \bar{\eta}(\bar{s})\bar{\eta}^T(\bar{t})\rangle = \delta^3(\bar{s} - \bar{t}), \tag{4.4}$$

$$g_0{}^2 = \frac{k_B T}{\ell^3 t_2}\mathcal{L}_0, \tag{4.5}$$

$$F_0{}^2 = \frac{k_B T}{t_2}\Upsilon_0, \tag{4.6}$$

$$2D_A D_A{}^T = A \tag{4.7}$$

where A is a symmetric positive-definite operator. Here, D_A is an operator defined with the same domain and range as A. Its existence is guranteed by the properties of A.

4.2. Other Conventions. Since $\bar{\eta}(\cdot)$ and $\bar{\xi}(\bar{\mathbf{X}}, \cdot)$ are Gaussian processes with 0 mean and δ-correlation, we will be able to use

$$\bar{\eta}(\bar{t}) = \frac{dB_{\bar{t}}}{d\bar{t}}, \bar{\xi}(\bar{\mathbf{X}}, \bar{t}) = \phi(\bar{\mathbf{X}})\frac{dB_{\bar{t}}^{(\bar{\mathbf{X}})}}{d\bar{t}}, \tag{4.8}$$

where

$$\langle \phi(\bar{\mathbf{X}})\phi^T(\bar{\mathbf{Y}})\rangle = \delta^3(\bar{\mathbf{X}} - \bar{\mathbf{Y}}). \tag{4.9}$$

Above, the superscript distinguishes the different Weiner processes at each spatial point.

For the purposes of our analysis, we will approximate $\mathbf{p}$ as a function of a finite number of spatial points. However, since the number of points we use is arbitrary, the dynamics of the system may be approximated arbitrarily well, assuming the system is sufficiently well-behaved. Moreover, during our analysis we will use the convention that $\mathbf{p} = \mathbf{p}(t)$, where we establish an implicit spatial dependence of $\mathbf{p}$ by using a greater dimension for the range of $\mathbf{p}$. For example, if we would like to use N points, we could use $\mathbf{p}_i^k(t)$ where k = 1, ..., N corresponds to the spatial point and i is the corresponding component of $\mathbf{p}$ at each point.

With the new convention, there is no need to include ϕ because of our discretization scheme. Thus the noise due to $\mathbf{g}_{thm}$ can be labeled as $D_{-\mathcal{L}}\frac{d\hat{B}_t}{dt}$.

4.3. Non-Dimensional Equations. Writing each of the terms in their non-dimensional form, removing ζ and ξ for simplicity, and removing the bars from the non-dimensional variables for convenience, we can re-write equations 3.4 3.8, and 3.9 as

$$\frac{\partial \mathbf{p}}{\partial t} = -\kappa^{-1}\nu[\mathbf{p} - \Lambda(\mathbf{v})] \cdot \nabla_{\mathbf{x}}[\mathbf{p} - \Lambda(\mathbf{v})] + \kappa^{-1}\mathcal{L}(\mathbf{p} - \Lambda[\mathbf{v}]) \tag{4.10}$$

$$+ \Lambda[-\nabla_{\mathbf{X}}\Phi(\mathbf{X})] + (\nabla_{\mathbf{X}}\Lambda[\mathbf{v}]) \cdot \mathbf{v} + D_{-\mathcal{L}}\frac{d\hat{B}_t}{dt} \tag{4.11}$$

$$\frac{d\mathbf{v}}{dt} = \frac{1}{\epsilon}\Big[-\Upsilon\mathbf{v} + \kappa^{-1}\Upsilon\Gamma(\mathbf{p} - \mathbf{\Lambda}[\mathbf{v}]) - \epsilon\nabla_{\mathbf{x}}\Phi(\mathbf{X})\Big] + \sqrt{\frac{1}{\epsilon}}D_\Upsilon\frac{dB_t}{dt} \tag{4.12}$$

$$\frac{d\mathbf{X}}{dt} = \mathbf{v} \tag{4.13}$$

It will be useful to write 4.12 as

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\epsilon}\Upsilon C(\mathbf{v} - \mathbf{v}_0) + \sqrt{\frac{1}{\epsilon}}D_\Upsilon\frac{dB_t}{dt}. \tag{4.14}$$

Here,

$$C = (I + \kappa^{-1}\Gamma\Lambda), \mathbf{v}_0 = C^{-1}\left(\kappa^{-1}\Gamma\mathbf{p} - \epsilon\Upsilon^{-1}\nabla_{\mathbf{X}}\Phi(\mathbf{X})\right) \tag{4.15}$$

5. Summary of Results. The result to first order in ϵ is given in 7.78 and 7.79. We summarize the results in the limit $\epsilon \to 0$ next.

The zeroth-order approximation in ϵ is

$$\frac{\partial \mathbf{p}}{\partial t} = -\rho^{-1}\nu\{\mathbf{p} \cdot \nabla_{\mathbf{x}}\mathbf{p} - \Lambda[m\mathbf{v}_0] \cdot \nabla_{\mathbf{x}}\mathbf{p} - \mathbf{p} \cdot \nabla_{\mathbf{x}}(\Lambda[m\mathbf{v}_0]) + (\Lambda[m\mathbf{v}_0]) \cdot \nabla_{\mathbf{x}}(\Lambda[m\mathbf{v}_0]) \quad (5.1)$$

$$+ mk_BT(\nabla_{\mathbf{x}}\Lambda C^{-1}) : \Lambda^T\}$$

$$+ \rho^{-1}\mathcal{L}\left(\mathbf{p} - \Lambda[m\mathbf{v}_0]\right) + k_BT\nabla_{\mathbf{X}}\Lambda : C^{-1} + (\nabla_{\mathbf{X}}\Lambda[m\mathbf{v}_0]) \cdot \mathbf{v}_0 - \Lambda\nabla_{\mathbf{X}}\Phi(\mathbf{X}) + \lambda + \mathbf{g}_{\text{thm}}$$

$$\frac{d\mathbf{X}}{dt} = \mathbf{v}_0 \quad (5.2)$$

5.1. Limit with Neglibigle Excess Mass $m \to 0$. With $m \to 0$, we note $C \to I$, so that $\nabla_{\mathbf{X}}\Lambda : C^{-1} \to \nabla_{\mathbf{X}}\Lambda : I = \nabla_{\mathbf{X}}\Lambda$. We obtain the equations

$$\frac{\partial \mathbf{p}}{\partial t} = -\rho^{-1}\nu[\mathbf{p} \cdot \nabla_{\mathbf{x}}\mathbf{p}] + \rho^{-1}\mathcal{L}\mathbf{p} + k_BT\nabla_{\mathbf{x}} \cdot \Lambda - \Lambda\nabla_{\mathbf{X}}\Phi(\mathbf{X}) + \mathbf{g}_{\text{thm}} \quad (5.3)$$

$$\frac{d\mathbf{X}}{dt} = \mathbf{v}_0 \quad (5.4)$$

5.2. Regime IV. The description of the fluid-structure system can be further simplified by considering for the fluid the viscous limit in which $\mu \to \infty$. In this regime only the structure dynamics remain and can be shown to be given by

$$\frac{d\mathbf{X}}{dt} = H_{\text{SELM}}[-\nabla_{\mathbf{X}}\Phi(\mathbf{X})] + (\nabla_{\mathbf{X}} \cdot H_{\text{SELM}})k_BT + \mathbf{h}_{\text{thm}} \quad (5.5)$$

$$H_{\text{SELM}} = \Gamma(-\wp\mathcal{L})^{-1}\Lambda \quad (5.6)$$

$$\langle \mathbf{h}_{\text{thm}}(s)\mathbf{h}_{\text{thm}}^T(t)\rangle = (2k_BT)\, H_{\text{SELM}}\, \delta(t - s). \quad (5.7)$$

The $\wp$ denotes a projection operator imposing constraints, such as incompressibility. The adjoint property $\Lambda = \Gamma^\dagger$ and symmetry of $\wp\mathcal{L}$ yields an operator H_{SELM} which is symmetric. A more detailed discussion and derivation of the equations in this regime is given in Section 6.

5.3. Summary. This gives an overview of the SELM formalism and the associated stochastic differential equations. We remark that each of these regimes were motivated by a rather specific limit. Non-dimensional analysis of the equations can also be carried out and other limits considered to motivate working with such reduced equations. We discuss in more detail the derivation of the reduced equations in our regime of interest in Section 7. developed for the SELM formalism in Section 6.

6. Derivations for the Stochastic Eulerian Lagrangian Method. We now discuss formal derivations to motivate the stochastic differential equations used in each of the physical regimes. For this purpose, we do not present the most general derivation of the equations. For brevity, we make simplifying assumptions when convenient.

In the initial formulation of SELM, the fluid-structure system is described by

$$\rho\frac{d\mathbf{u}}{dt} = \mathcal{L}\mathbf{u} + \Lambda[\Upsilon(\mathbf{v} - \Gamma\mathbf{u})] + \lambda + \mathbf{f}_{\text{thm}} \quad (6.1)$$

$$m\frac{d\mathbf{v}}{dt} = -\Upsilon(\mathbf{v} - \Gamma\mathbf{u}) - \nabla_{\mathbf{X}}\Phi(\mathbf{X}) + \zeta + \mathbf{F}_{\text{thm}} \quad (6.2)$$

$$\frac{d\mathbf{X}}{dt} = \mathbf{v}. \quad (6.3)$$

The notation and operators appearing in these equations has been discussed in detail in Section 2. For these equations, we focus primarily on the motivation for the stochastic driving fields used for the fluid-structure system.

For the thermal fluctuations of the system, we assume Gaussian random fields with mean zero and δ-correlated in time. For such stochastic fields, the central challenge is to determine an appropriate covariance structure. For this purpose, we use the fluctuation-dissipation principle of statistical mechanics [43, 27]. For linear stochastic differential equations of the form

$$d\mathbf{Z}_t = L\mathbf{Z}_t dt + Qd\mathbf{B}_t \tag{6.4}$$

the fluctuation-dissipation principle can be expressed as

$$G = QQ^T = -(LC) - (LC)^T. \tag{6.5}$$

This relates the equilibrium covariance structure C of the system to the covariance structure G of the stochastic driving field. The operator L accounts for the dissipative dynamics of the system. For the equations 6.1 – 6.3, the dissipative operators only appear in the momentum equations. This can be shown to have the consequence that there is no thermal forcing in the equation for $\mathbf{X}(t)$, this will also be confirmed in subsequent sections. To simplify the presentation, we do not represent explicitly the stochastic dynamics of the structure configuration $\mathbf{X}$.

For the fluid-structure system it is convenient to work with the stochastic driving fields by defining

$$\mathbf{q} = [\rho^{-1}\mathbf{f}_{\text{thm}}, m^{-1}\mathbf{F}_{\text{thm}}]^T. \tag{6.6}$$

The field $\mathbf{q}$ formally is given by $\mathbf{q} = Qd\mathbf{B}_t/dt$ and determined by the covariance structure $G = QQ^T$. This covariance structure is determined by the fluctuation-dissipation principle expressed in equation 6.5 with

$$L = \begin{bmatrix} \rho^{-1}(\mathcal{L} - \Lambda\Upsilon\Gamma) & \rho^{-1}\Lambda\Upsilon \\ m^{-1}\Upsilon\Gamma & -m^{-1}\Upsilon \end{bmatrix} \tag{6.7}$$

$$C = \begin{bmatrix} \rho^{-1}k_BT\mathcal{I} & 0 \\ 0 & m^{-1}k_BT\mathcal{I} \end{bmatrix}. \tag{6.8}$$

The $\mathcal{I}$ denotes the identity operator. The covariance C was obtained by considering the fluctuations at equilibrium. The covariance C is easily found since the Gibbs-Boltzmann distribution is a Gaussian with formal density $\Psi(\mathbf{u}, \mathbf{v}) = \frac{1}{Z_0}\exp\left[-E/k_BT\right]$. The Z_0 is the normalization constant for Ψ. The energy is given by equation 2.4. For this purpose, we need only consider the energy E in the case when $\Phi = 0$. This gives the covariance structure

$$G = (2k_BT)\begin{bmatrix} -\rho^{-2}(\mathcal{L} - \Lambda\Upsilon\Gamma) & -m^{-1}\rho^{-1}\Lambda\Upsilon \\ -m^{-1}\rho^{-1}\Upsilon\Gamma & m^{-2}\Upsilon \end{bmatrix}. \tag{6.9}$$

To obtain this result we use that $\Gamma = \Lambda^{\dagger}$ and $\Upsilon = \Upsilon^{\dagger}$. From the definition of $\mathbf{q}$, it is found the covariance of the stochastic driving fields of SELM are given by equations 2.6– 2.8. This provides a description of the thermal fluctuations in the fluid-structure system.

7. Stochastic Reduction Procedure.

7.1. Set-Up and Summary of Derivation. The next step in our reduction procedure is to use the Kolmogorov backward equation to find a PDE corresponding to the system of SDEs we found. Specifically, given a sufficiently smooth function with compact support $\theta : \mathbb{R}^N \to \mathbb{R}$, we define $f(t, \mathbf{X}, \mathbf{p}, \mathbf{v}) = E^{\mathbf{X}, \mathbf{p}, \mathbf{v}}[\theta(\mathbf{X}_t, \mathbf{p}_t, \mathbf{v}_t)]$. Then

$$\frac{\partial f}{\partial t} = \mathcal{A}_\epsilon(f), \tag{7.1}$$

with initial conditions

$$f(0, \mathbf{X}, \mathbf{p}, \mathbf{v}) = \theta(\mathbf{X}, \mathbf{p}, \mathbf{v}). \tag{7.2}$$

Here, $\mathcal{A}_\epsilon$ is the infinitesimal generator corresponding to the system of SDEs 4.10, 4.12, and 4.13:

$$\mathcal{A}_\epsilon = L_{slow} + \frac{1}{\epsilon}L_{fast} \tag{7.3}$$

$$L_{slow} = \{-\kappa^{-1}\nu(\mathbf{p} - \Lambda[\mathbf{v}]) \cdot \nabla_{\mathbf{x}}(\mathbf{p} - \Lambda[\mathbf{v}]) + [\kappa^{-1}\mathcal{L}\,(\mathbf{p} - \Lambda[\mathbf{v}]) + \Lambda[-\nabla_{\mathbf{X}}\Phi(\mathbf{X})] \tag{7.4}$$
$$+ (\nabla_{\mathbf{X}}\Lambda[\mathbf{v}]) \cdot \mathbf{v}\} \cdot \nabla_{\mathbf{p}} - \mathcal{L} : \nabla_{\mathbf{p}}^2 + \mathbf{v} \cdot \nabla_{\mathbf{X}}$$
$$L_{fast} = [-\Upsilon C(\mathbf{v} - \mathbf{v}_0)] \cdot \nabla_{\mathbf{v}} + \Upsilon : \nabla_{\mathbf{v}}^2 \tag{7.5}$$

Here, we use the double dot product, $A : B = A_{ij}B_{ij}$. We will use Einstein summation convention whenever a double index is present, unless otherwise noted.

Next, we will perform a perturbative analysis on a part of $\mathcal{A}_\epsilon$. However, as we will see, to perform the perturbative analysis we will need a specific solvability condition to hold. In order to satisfy this condition, we will write

$$\mathcal{A}_\epsilon = \bar{L}_1 + L_1 + \frac{1}{\epsilon}L_2. \tag{7.6}$$

and perform the analysis on the operator $L_\epsilon = L_1 + \frac{1}{\epsilon}L_2$ (we will take $L_2 = L_{fast}$). Specifically, the solvability condition will require L_1 to have mean 0 with respect to the $\mathbf{v}$ distribution. The operator by which we will replace $L_1 + \frac{1}{\epsilon}L_2$ will be the first-order perturbation in ϵ, while $\bar{L}_1$ represents the zeroth-order approximation in ϵ.

For situations with no slip (i.e. the limit $\epsilon \to 0$), we may use $\bar{L}_1$ as the effective operator replacing $\mathcal{A}_\epsilon$. Physically, we can think of $\mathbf{v}$ as being spread over a distribution in quasi-static equilibrium, and we may interpret $\bar{L}_1$ as the configuration average of L_{slow} over the $\mathbf{v}$ distribution. That is, the fast variable $\mathbf{v}$ fluctuates so quickly that we may assume it is effectively in an equilibrium with $\mathbf{p}$ and $\mathbf{X}$ fixed.

We can obtain the quasi-static equilibrium $\mathbf{v}$-distribution, ψ from the Kolmogorov forward equation,

$$\epsilon\frac{d\psi}{dt} = [\epsilon L_{slow} + L_2]^*\psi \tag{7.7}$$

Taking $\epsilon \to 0$ yields

$$L_2^*\psi = 0 \tag{7.8}$$

The solution to the above equation is a Gaussian, unique up to a multiplicative constant. One might expect to obtain such a distribution from Boltzmann-Gibbs statistics. Normalizing, we obtain

$$\psi(v) = \frac{\sqrt{|C|}}{(2\pi)^{N/2}}e^{-\frac{1}{2}(\mathbf{v}-\mathbf{v}_0)^T C(\mathbf{v}-\mathbf{v}_0)}. \tag{7.9}$$

We will also use ψ later in our perturbative analysis, although we will see analytically where the condition 7.8 arises.

Taking the configuration average of $\mathbf{v}$ over L_{slow}, we obtain

$$\bar{L}_1 = \{-\kappa^{-1}\nu[\mathbf{p} \cdot \nabla_{\mathbf{x}}\mathbf{p} - \Lambda\mathbf{v_0} \cdot \nabla_{\mathbf{x}}\mathbf{p} - \mathbf{p} \cdot \nabla(\Lambda\mathbf{v}_0) + (\Lambda\mathbf{v}_0) \cdot \nabla(\Lambda\mathbf{v}_0) + (\nabla_{\mathbf{x}}\Lambda C^{-1}) : \Lambda^T] \tag{7.10}$$

$$+ \kappa^{-1}\mathcal{L}(\mathbf{p} - \Lambda\mathbf{v}_0) + \nabla_{\mathbf{X}}\Lambda : C^{-1} + (\nabla_{\mathbf{X}}\Lambda[\mathbf{v}_0]) \cdot \mathbf{v}_0 - \Lambda\nabla_{\mathbf{X}}\Phi(\mathbf{X})\} \cdot \nabla_{\mathbf{p}} + \mathbf{v}_0 \cdot \nabla_{\mathbf{X}} - \mathcal{L} : \nabla_{\mathbf{p}}^2 \tag{7.11}$$

The corresponding equations of motion are

$$\frac{\partial \mathbf{p}}{\partial t} = -\kappa^{-1}\nu[\mathbf{p}\cdot\nabla_{\mathbf{x}}\mathbf{p} - \Lambda\mathbf{v_0}\cdot\nabla_{\mathbf{x}}\mathbf{p} - \mathbf{p}\cdot\nabla_{\mathbf{x}}(\Lambda\mathbf{v}_0) + (\Lambda\mathbf{v}_0)\cdot\nabla_{\mathbf{x}}(\Lambda\mathbf{v}_0) + (\nabla_{\mathbf{x}}\Lambda C^{-1}):\Lambda^T]$$
(7.12)

$$+ \kappa^{-1}\mathcal{L}(\mathbf{p} - \Lambda\mathbf{v}_0) + \nabla_{\mathbf{X}}\Lambda:C^{-1} + (\nabla_{\mathbf{X}}\Lambda[\mathbf{v}_0])\cdot\mathbf{v}_0 - \Lambda\nabla_X\Phi(\mathbf{X}) + D_{-\mathcal{L}}\frac{d\hat{B}_t}{dt} \qquad (7.13)$$

$$\frac{d\mathbf{X}}{dt} = \mathbf{v}_0 \tag{7.14}$$

Note that both $\nabla_{\mathbf{x}}$ and Λ are as a linear operators. This also allows us to compute the configuration average (using 7.27) $\overline{(\Lambda\tilde{\mathbf{v}})\cdot\nabla(\Lambda\tilde{\mathbf{v}})} = \overline{\Lambda_{ij}\tilde{v}_j\partial_{x_i}\Lambda_{mn}\tilde{v}_n} = \partial_{x_i}\Lambda_{mn}C_{nj}^{-1}\Lambda_{ji}^T = \nabla_{\mathbf{x}}(\Lambda C^{-1}) : \Lambda^T$.

Here, we use $\nabla_{\mathbf{X}}\Lambda : C^{-1} = [\partial_{X_k}\Lambda_{i,j}C_{jk}^{-1}]$ and $\nabla_{\mathbf{x}}(\Lambda)$
re-labeling each variable with its physical dimensions (we assumed $\mathbf{v}_0$ has dimensions of ℓ/t_2 and that C has no dimensions) and using our original notation for $\mathbf{g}_{thm}$, we obtain the equations

$$\frac{\partial \mathbf{p}}{\partial t} = -\rho^{-1}\nu\{\mathbf{p}\cdot\nabla_{\mathbf{x}}\mathbf{p} - \Lambda[m\mathbf{v}_0]\cdot\nabla_{\mathbf{x}}\mathbf{p} - \mathbf{p}\cdot\nabla_{\mathbf{x}}(\Lambda[m\mathbf{v}_0]) + (\Lambda[m\mathbf{v}_0])\cdot\nabla_{\mathbf{x}}(\Lambda[m\mathbf{v}_0]) \quad (7.15)$$

$$+ mk_BT(\nabla_{\mathbf{x}}\Lambda C^{-1}):\Lambda^T\} \tag{7.16}$$

$$+ \rho^{-1}\mathcal{L}\left(\mathbf{p} - \Lambda[m\mathbf{v}_0]\right) + k_BT\nabla_{\mathbf{X}}\Lambda:C^{-1} + (\nabla_{\mathbf{X}}\Lambda[m\mathbf{v}_0])\cdot\mathbf{v}_0 - \Lambda\nabla_{\mathbf{X}}\Phi(\mathbf{X}) + \lambda + \mathbf{g}_{\text{thm}}$$
(7.17)

$$\frac{d\mathbf{X}}{dt} = \mathbf{v}_0 \tag{7.18}$$

Since we are assuming $\epsilon \to 0$, we may use $\mathbf{v}_0 = \rho^{-1}C^{-1}\Gamma\mathbf{p}$.

7.2. ϵ-Expansion of the Coefficients and Required Assumptions. Previously, we found the PDEs that describe the evolution of the probability density function of the above SDEs using the Kolmogorov Backwards Equations. We will next isolate the operator that encapsulates the fast dynamics of the system that satisfies particular solvability conditions on which we will perform a perturbative analysis.

In our reduction, we will replace the infinitesimal generator of a given stochastic process $L_\epsilon = L_1 + \frac{1}{\epsilon}L_2$ by an effective first-order term in the limit $\epsilon \to 0$. We will require certain assumptions to hold for this reduction, which will be important in practice.

To carry out the reduction, we consider the the stochastic process generated by $\frac{\partial f}{\partial t} = L_\epsilon f$. First, we make a change of variable $t = \tau/\epsilon$ to obtain

$$\frac{\partial f}{\partial \tau} = \frac{1}{\epsilon}L_1 + \frac{1}{\epsilon^2}L_2 \tag{7.19}$$

and expand

$$f(\tau, \mathbf{X}, \mathbf{p}, \mathbf{v}) = f_0(\tau, \mathbf{X}, \mathbf{p}, \mathbf{v}) + \epsilon f_1(\tau, \mathbf{X}, \mathbf{p}, \mathbf{v}) + \epsilon^2 f_2(\tau, \mathbf{X}, \mathbf{p}, \mathbf{v}) + \ldots$$

We find the following relationships must hold by comparing powers of ϵ:

$$L_2 f_0 = 0 \text{ (from } \frac{1}{\epsilon^2}) \tag{7.20}$$

$$L_1 f_0 + L_2 f_1 = 0 \text{ (from } \frac{1}{\epsilon}) \tag{7.21}$$

$$\frac{\partial f_0}{\partial \tau} = L_1 f_1 + L_2 f_2 \text{ (from 1)} \tag{7.22}$$

In our analysis, our choice of L_2 keeps $\mathbf{X}$ and $\mathbf{p}$ fixed. Assuming a system with L_2 as its infinitesimal generator is ergodic, we will now show that $f_0 = f_0(\tau, \mathbf{X}, \mathbf{p})$. Since a solution

to 7.20 is a steady state of L_2, we consider the PDE governing a function $g(r, \mathbf{X}, \mathbf{p}, \mathbf{v})$ with any initial conditions $g(0, \mathbf{X}, \mathbf{p}, \mathbf{v}(0)) = \theta_{\mathbf{X}, \mathbf{p}}(\mathbf{v}(0))$,

$$L_2 g = \frac{\partial g}{\partial r}.$$

In the limit $r \to \infty$, g will reach a steady-state $L_2 g = 0$, which will be independent of $\mathbf{v}(0)$. By uniqueness, $g = f_0$.

We would like to use 7.21 to solve for f_1. We notice a solvability condition from the requirement $L_1 f_0 \in \mathcal{R}(L_2)$, or equivalently $L_1 f_0 \in \mathcal{N}(L_2^*)^\perp$. We can re-express this condition as follows: $\forall \psi \in \mathcal{N}(L_2^*)$ (i.e. $L_2^* \psi = 0$), we must have

$$\langle L_1 f_0, \psi \rangle = 0. \tag{7.23}$$

We find a ψ with the property $\int_{\mathbb{R}^N} \psi d\mathbf{v} = 1$ (given in 7.9). Checking the solvability condition with this choice of ψ is sufficient since ψ is unique up to a multiplicative constant. Supposing the solvability condition is satisfied, we can solve for

$$f_1 = -L_2^{-1} L_1 f_0. \tag{7.24}$$

From 7.22, we know

$$\frac{\partial f_0}{\partial \tau} = -L_1 L_2^{-1} L_1 f_0 + L_2 f_2.$$

Rearranging, we find that

$$\frac{\partial}{\partial \tau} + L_1 L_2^{-1} L_1 \in \mathcal{N}(L_2^*)^\perp.$$

Thus,

$$\langle \frac{\partial}{\partial \tau} + L_1 L_2^{-1} L_1, \psi \rangle = 0.$$

But since f_0 is independent of $\mathbf{v}$,

$$\int_{\mathbb{R}} \frac{\partial f_0}{\partial \tau} \psi d\mathbf{v} = \frac{\partial f_0}{\partial \tau} \int_{\mathbb{R}} \psi d\mathbf{v} = \frac{\partial f_0}{\partial \tau}.$$

So

$$\frac{\partial f_0}{\partial \tau} = -[\int_{\mathbb{R}^N} \psi L_1 L_2^{-1} L_1 d\mathbf{v}] f_0. \tag{7.25}$$

We use the resulting operator $L_0 = -[\int_{\mathbb{R}^N} \psi L_1 L_2^{-1} L_1 d\mathbf{v}]$ as the approximate generator to first order in ϵ. After making back the change of variable from τ to t, we see L_ϵ is approximated by ϵL_0 in the limit $\epsilon \to 0$. Thus we replace 7.6 by the effective operator

$$\mathcal{A}_0 = \bar{L}_1 + \epsilon L_0 \tag{7.26}$$

7.3. Preparing $\mathcal{A}_\epsilon$ for the Reduction. Given ψ, we must choose an appropriate L_1 which will satisfy the solvability condition given by 7.23. Now,

$$< L_1 f, \psi >= \int_{\mathbb{R}^N} L_1(f) \psi = \overline{L_1(f)}.$$

Therefore, our choice of L_1 must have configuration average 0. To achieve this, we split L_{slow} appearing in 7.4 due to the slow variables as follows:

$$L_1 = L_{slow} - \overline{L_{slow}}; \ \bar{L}_1 = \overline{L_{slow}}$$

It will be useful to use the notation $\tilde{\mathbf{v}} = \mathbf{v} - \mathbf{v}_0$.

We will now compute $\overline{\tilde{v}_k \tilde{v}_j}$.

First, we make a change of variable. We may assume that C, the covariance matrix, is positive-definite. Thus C has a square root matrix, which we may pick to be symmetric as well. Let $\tilde{\mathbf{v}} = C^{-\frac{1}{2}}\boldsymbol{\alpha}$. Noting the Jacobian $\frac{\partial \tilde{\mathbf{v}}}{\partial \boldsymbol{\alpha}} = |C|^{-\frac{1}{2}}$, we find:

$$
\begin{aligned}
\overline{\tilde{v}_k \tilde{v}_j} &= \int_{\mathbb{R}^N} \frac{\sqrt{|C|}}{(2\pi)^{N/2}} v_k v_j e^{-\frac{1}{2}v_l C_{ln} v_n} dv_1...dv_N \\
&= \int_{\mathbb{R}^N} \frac{1}{(2\pi)^{N/2}} C_{tj}^{-\frac{1}{2}} C_{sk}^{-\frac{1}{2}} \alpha_t \alpha_s e^{-\frac{1}{2}\alpha_l \alpha_l} d\alpha_1...d\alpha_N \\
&= C_{tj}^{-\frac{1}{2}} C_{sk}^{-\frac{1}{2}} \delta_s^t = C_{kj}^{-1}.
\end{aligned}
\tag{7.27}
$$

We then see

$$
\begin{aligned}
\overline{v_k v_j} &= \overline{(v_{0k} + \tilde{v}_k)(v_{0j} + \tilde{v}_j)} \\
&= \overline{\tilde{v}_k \tilde{v}_j} + \overline{\tilde{v}_k v_{0j}} + \overline{v_{0k}\tilde{v}_j} + \overline{v_{0k}v_{0j}} \\
&= C_{kj}^{-1} + v_{0k}v_{0j}.
\end{aligned}
$$

So that the configuration average

$$
\overline{(\nabla_{\mathbf{X}}\Lambda[\mathbf{v}])\cdot\mathbf{v}} = \overline{[\partial_{X_k}\Lambda_{i,j}v_j v_k]} = [\partial_{X_k}\Lambda_{i,j}(C_{k,j}^{-1} + v_{0k}v_{0j})]
\tag{7.28}
$$
$$
= \nabla_{\mathbf{X}}\Lambda : C^{-1} + (\nabla_{\mathbf{X}}\Lambda[\mathbf{v}_0])\cdot\mathbf{v}_0
\tag{7.29}
$$

From the above computation and $\overline{\tilde{\mathbf{v}}} = 0$ we may obtain 7.10.

to obtain L_1, we compute

$$
\begin{aligned}
v_k v_j - \overline{v_k v_j} &= \tilde{v}_k\tilde{v}_j + v_{0k}\tilde{v}_j + \tilde{v}_k v_{0j} + v_{0k}v_{0j} - (v_{0k}v_{0j} + \overline{\tilde{v}_k\tilde{v}_j}) \\
&= \tilde{v}_k\tilde{v}_j + v_{0k}\tilde{v}_j + \tilde{v}_k v_{0j} - \overline{\tilde{v}_k\tilde{v}_j}
\end{aligned}
$$

It will be useful to maintain $\overline{\tilde{v}_k\tilde{v}_j}$ in L_1 in its current form.

For the purposes of a cleaner presentation, we shall use $\mathbf{y} = (\mathbf{X}, \mathbf{p})$.

We find L_1 is given by

$$
L_1 = \tilde{v}_i \frac{\partial}{\partial x_i} + \left[(-\kappa^{-1}\mathcal{L}\Lambda\tilde{v})_i + \partial_{X_k}\Lambda_{i,j}(\tilde{v}_k\tilde{v}_j + v_{0k}\tilde{v}_j + \tilde{v}_k v_{0j} - \overline{\tilde{v}_k\tilde{v}_j})\right]\frac{\partial}{\partial p_i}
$$
$$
- \kappa^{-1}\nu[-\Lambda_{nj}\tilde{v}_j\partial_{x_n}p_i - p_n\partial_{x_n}\Lambda_{ij}\tilde{v}_j + \Lambda_{nm}\tilde{v}_m\partial_{x_n}\Lambda_{ij}v_{0j} + \Lambda_{nm}v_{0m}\partial_{x_n}\Lambda_{ij}\tilde{v}_j
\tag{7.30}
$$
$$
+ \Lambda_{mk}\partial_{x_m}\Lambda_{ij}(\tilde{v}_k\tilde{v}_j - \overline{\tilde{v}_k\tilde{v}_j})]\frac{\partial}{\partial p_i}
\tag{7.31}
$$
$$
= \left[R_{ij}\tilde{v}_j + S_{ijk}(\tilde{v}_k\tilde{v}_j - \overline{\tilde{v}_k\tilde{v}_j})\right]\frac{\partial}{\partial y_i}.
\tag{7.32}
$$

where

$$
R = \begin{pmatrix} \mathcal{I} \\ -\kappa^{-1}\mathcal{L}\Lambda + (\nabla_{\mathbf{X}}\Lambda)\cdot\mathbf{v}_0 + \nabla_{\mathbf{X}}(\Lambda\mathbf{v}_0) - R_2 \end{pmatrix} = \begin{pmatrix} \mathcal{I} \\ B \end{pmatrix}
\tag{7.33}
$$
$$
R_2 = \kappa^{-1}\nu[-\nabla_{\mathbf{x}}\mathbf{p}\cdot\Lambda - \mathbf{p}\cdot\nabla_{\mathbf{x}}(\Lambda\cdot) + \nabla_{\mathbf{x}}(\Lambda\mathbf{v}_0)\Lambda + \Lambda\mathbf{v}_0\cdot\nabla_{\mathbf{x}}\Lambda]
\tag{7.34}
$$
$$
S = \begin{pmatrix} 0 \\ \nabla_{\mathbf{X}}\Lambda - \kappa^{-1}\nu[(\nabla_x\Lambda)\cdot\Lambda^T] \end{pmatrix} = \begin{pmatrix} 0 \\ P \end{pmatrix}
\tag{7.35}
$$

The next step of the reduction, applying L_2^{-1} to L_1, could be done term-by-term using elementary calculus. However, we prefer a more general and elegant approach, which we shall discuss next.

7.4. Inversion of L_2. In summary, we will perform a change of basis on $\tilde{\mathbf{v}}$ so that L_2 can be written as a sum of operators $L_2^{(i)}$ each involving only the ith coordinate. We will then show that L_1 can be written as a linear combination of eigenfunctions of L_2, each of which is a product of the eigenfunctions of the various $L_2^{(i)}$.

7.4.1. Change of Variable. From 7.4 we have

$$L_2 = -(\Upsilon C \tilde{\mathbf{v}}) \cdot \nabla_{\tilde{\mathbf{v}}} + \Upsilon : \nabla_{\tilde{\mathbf{v}}}^2 = -(\Upsilon C \tilde{\mathbf{v}})_i \frac{\partial f}{\partial v_i} + \Upsilon_{ij} \frac{\partial^2}{\partial v_i v_j} \tag{7.36}$$

Since C is symmetric and positive-definite, we may find a square root $C^{\frac{1}{2}}$ which is also symmetric and positive-definite. By using a change of variable $\tilde{\mathbf{v}} = C^{-\frac{1}{2}}\boldsymbol{\alpha}$, we may re-write

$$L_2 = C_{ni}^{\frac{1}{2}} \Upsilon_{ij} C_{jm}^{\frac{1}{2}} [-\alpha_m \partial_{\alpha_n} + \partial_{\alpha_m} \partial_{\alpha_n}]. \tag{7.37}$$

We will use

$$\Xi = C^{\frac{1}{2}} \Upsilon C^{\frac{1}{2}}, \tag{7.38}$$

so that we may write

$$L_2 = \Xi_{nm}[-\alpha_m \partial_{\alpha_n} + \partial_{\alpha_m} \partial_{\alpha_n}]. \tag{7.39}$$

Note Ξ is symmetric since $C^{\frac{1}{2}}$ and Υ are symmetric, and Ξ is positive-definite since Υ is positive-definite. Therefore, we may find a unitary decomposition $\Xi = QDQ^{-1}$ where D is diagonal and $Q^T = Q^{-1}$. Note that in general we may have $\Xi^{\frac{1}{2}} \neq \Lambda^{\frac{1}{2}} C^{\frac{1}{2}}$.

Now, take $\boldsymbol{\alpha} = Q\mathbf{w}$ to obtain

$$L_2 = D_{nm}[-w_m \partial_{w_n} + \partial_{w_m} \partial_{w_n}]. \tag{7.40}$$

To summarize, we have simplified the form of L_2 using the change of basis

$$\tilde{\mathbf{v}} = C^{-\frac{1}{2}} Q\mathbf{w} = \hat{Q}\mathbf{w}. \tag{7.41}$$

7.4.2. Generic Formulation. We recall the Hermite equation $-up'(w) + p''(w) = -\lambda p(w)$, whose eigenfunctions are the (probabilists') Hermite polynomials $He_\lambda(w)$ with eigenvalues λ corresponding to the degree of the polynomial. We see that for each i, the Hermite polynomials $He_\lambda(w_i)$ are eigenfunctions of $L_2^{(i)}$ (the term in L_2 involving terms w_i) with eigenvalues $-\lambda D_i$. Thus $\Pi_k He_{\lambda_k}(w_k)$ is an eigenfunction of L_2 with eigenvalue $-\sum_k \lambda_k D_k$. Moreover, any non-trivial product $\Pi_k He_{\lambda_k}(w_k)$ (i.e. at least some $\lambda_k \neq 0$), is in $\mathcal{R}(L_2)$, since its corresponding eigenvalue is non-zero.

Next, we will see that any polynomial $p(\mathbf{w})$ that satisfies the solvability condition $\overline{p(\mathbf{w})} = 0$ can be written as a sum of non-trivial polynomials of the form $\Pi_k He_{\lambda_k}(w_k)$. Consider the vector space $\mathcal{V}_1$ of such polynomials up to degree M. The solvability condition on p determines exactly one dimension, since a constant can be added to any polynomial to satisfy the solvability condition.

Let $\mathcal{V}_2$ be the vector space generated by linear combinations of non-trivial polynomials of degree up to M of the form $\Pi_k He_{\lambda_k}(w_k)$. Since the non-trivial Hermitian polynomials are orthogonal to constants, the non-triviality condition on the elements in $\mathcal{V}_2$ implies they satisfy the solvability condition $\overline{\Pi_k He_{\lambda_k}(w_k)} = 0$. Thus $\mathcal{V}_2$ is a subspace of $\mathcal{V}_1$. We see $\dim \mathcal{V}_1 = \sum_{d=1}^M \binom{N+d-1}{d} = \dim \mathcal{V}_2$, so we conclude $\mathcal{V}_1 = \mathcal{V}_2$. Thus we can express $p(\mathbf{w})$ as a linear combination of non-trivial terms $\Pi_k He_{\lambda_k}(w_k)$. In practice, Gaussian elimination could be used to write $p(\mathbf{w})$ in this form, starting with the highest-degree monomials and working downwards.

Now, since a non-trivial term $\Pi_k He_{\lambda_k}(w_k)$ is an eigenfunction of L_2, we see

$$L_2^{-1}[\Pi_k He_{\lambda_k}(w_k)] = -\left[\sum_k \lambda_k D_k\right]^{-1} \Pi_k He_{\lambda_k}(w_k). \tag{7.42}$$

Thus, noting L_2^{-1} is linear, the change of basis allows us to find $L_2^{-1} p(\mathbf{w})$.

7.5. Computation for Perturbative Analysis. First, we re-write 7.9 in terms of $\mathbf{w}$. Including the Jacobian, we obtain the distribution

$$\psi(\mathbf{w}) = \frac{1}{(2\pi)^{N/2}} e^{-\frac{1}{2}\mathbf{w}\mathbf{w}^T} \tag{7.43}$$

Next, we will compute 7.25.

Plugging 7.41 into 7.32 and computing $\overline{w_k w_j}$ using 7.43,

$$L_1 = \left[\hat{R}_{ij} w_j + \hat{S}_{ijk}(w_k w_j - \delta_{kj}) \right] \frac{\partial}{\partial y_i} = L_1^{(1)} + L_1^{(2)}, \text{ where} \tag{7.44}$$

$$\hat{R}_{ij} = R_{ik}\hat{Q}_{kj}, \ \hat{S}_{ijk} = S_{ilm}\hat{Q}_{lj}\hat{Q}_{mk}, \tag{7.45}$$

$$L_1^{(1)} = \hat{R}_{ij} w_j \frac{\partial}{\partial y_i}, \text{ and } L_1^{(2)} = \hat{S}_{ijk}(w_k w_j - \delta_{kj}) \frac{\partial}{\partial y_i}. \tag{7.46}$$

Notice that each term in $\mathbf{w}$ is a Hermite polynomial, as we would anticipate.

Some of the terms in our analysis will have zero contribution to the result. In particular, since $\psi(\mathbf{w})$ is an even function in each of its coordinates, any polynomial in $L_1 L_2^{-1} L_1$ with odd multi-index will make a zero contribution to 7.25, since it must be odd in some component of $\mathbf{w}$. Thus we may treat the terms $L_1^{(1)} L_2^{-1} L_1^{(1)}$ and $L_1^{(2)} L_2^{-1} L_1^{(2)}$ separtely. We will write

$$I_i = - \int_{\mathbb{R}^N} \psi(\mathbf{w}) L_1^{(i)} L_2^{-1} L_1^{(i)} d\mathbf{w} \text{ for } i = 1, 2. \tag{7.47}$$

7.5.1. Computing I_1. Using 7.42, integrating, and noting C (and therefore Q and D) are uniform in $\mathbf{X}$, we compute

$$I_1 = - \int_{\mathbb{R}^N} \psi(\mathbf{w}) L_1^{(1)} L_2^{-1} L_1^{(1)} d\mathbf{w} \tag{7.48}$$

$$= - \int_{\mathbb{R}^N} \psi(\mathbf{w}) \hat{R}_{nm} w_m \partial_{y_n} L_2^{-1} [\hat{R}_{ij} w_j \partial_{y_i}] d\mathbf{w} \tag{7.49}$$

$$= \int_{\mathbb{R}^N} \psi(\mathbf{w}) \hat{R}_{nm} w_m \partial_{y_n} [\hat{R}_{ij} D_{jk}^{-1} w_k \partial_{y_i}] d\mathbf{w} = \hat{R}_{nk} D_{jk}^{-1} \partial_{y_n} [\hat{R}_{ij} \partial_{y_i}] \tag{7.50}$$

$$= R_{nr} C_{rs}^{-\frac{1}{2}} Q_{sk} D_{jk}^{-1} \partial_{y_n} [R_{ip} C_{pq}^{-\frac{1}{2}} Q_{qj} \partial_{y_i}] = R_{nr} \Upsilon_{rp}^{-1} \partial_{y_n} [R_{ip} \partial_{y_i}] \tag{7.51}$$

$$= R_{nr} \Upsilon_{rp}^{-1} R_{pi}^T \partial_{y_n y_i}^2 + \Upsilon_{np}^{-1} [\partial_{X_n} B_{ip}] \partial_{p_i} \tag{7.52}$$

$$= (R\Upsilon^{-1} R^T) : \nabla_{\mathbf{y}}^2 + [(\nabla_{\mathbf{x}} B) : \Upsilon^{-1}] \cdot \nabla_{\mathbf{p}} \tag{7.53}$$

$$= \Upsilon^{-1} : \nabla_{\mathbf{X}}^2 + (B\Upsilon^{-1} B^T) : \nabla_{\mathbf{p}}^2 + [(\nabla_{\mathbf{x}} B) : \Upsilon^{-1}] \cdot \nabla_{\mathbf{p}} \tag{7.54}$$

Above, we used that R is a function of $\mathbf{X}$ and not $\mathbf{p}$, except in the first N rows where R is constant.

7.5.2. Computing I_2. Letting $E_{mn} = (D_m + D_n)^{-1}$, using $\hat{S}_{ijk} = S_{imn}\hat{Q}_{mj}\hat{Q}_{nk}$, and writing the sum over m and n explicitly to avoid confusion,

$$I_2 = - \int_{\mathbb{R}^N} \psi(\mathbf{w}) L_1^{(2)} L_2^{-1} L_1^{(2)} d\mathbf{w} \tag{7.55}$$

$$= - \int_{\mathbb{R}^N} \psi(\mathbf{w}) \hat{S}_{ijk}(w_j w_k - \delta_{jk}) \partial_{y_i} L_2^{-1} [\hat{S}_{lmn}(w_m w_n - \delta_{mn}) \partial_{y_l}] d\mathbf{w} \tag{7.56}$$

$$= \int_{\mathbb{R}^N} \psi(\mathbf{w}) \hat{S}_{ijk}(w_j w_k - \delta_{jk}) \partial_{y_i} \left[\sum_{mn} \hat{S}_{lmn} E_{mn}(w_m w_n - \delta_{mn}) \partial_{y_l} \right] d\mathbf{w} \tag{7.57}$$

$$= \sum_{mn} \hat{S}_{ijk} \hat{S}_{lmn} E_{mn} \int_{\mathbb{R}^N} \psi(\mathbf{w})(w_j w_k - \delta_{jk}) \left[(w_m w_n - \delta_{mn}) \right] d\mathbf{w} \partial_{p_i p_l}^2 \tag{7.58}$$

We note above $\hat{S}_{ijk}$ only yields non-zero terms when $i > N$.

An elegant way to evaluate the integral above (and more general similar integrals that may arise in a more general context) is to integrate by parts in one of the variables. Specifically,

$$\int_{\mathbb{R}^N} \psi(\mathbf{w}) w_j w_k w_m w_n d\mathbf{w} \tag{7.59}$$

$$= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^{N-1}} \left(\int_{\mathbb{R}} w_j e^{-\frac{1}{2} w_j^2} dw_j w_k w_m w_n \right) e^{-\frac{1}{2} \sum_{a \neq j} w_a^2} dw_1 ... d\hat{w}_j ... dw_N \tag{7.60}$$

$$= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{1}{2} w_j^2} \partial_{w_j} (w_k w_m w_n) dw_j e^{-\frac{1}{2} \sum_{a \neq j} w_a^2} dw_1 ... d\hat{w}_j ... dw_N \tag{7.61}$$

$$= \delta_{jk} \delta_{mn} + \delta_{jm} \delta_{kn} + \delta_{jn} \delta_{km} \tag{7.62}$$

where $d\hat{w}_j$ indicates a missing variable.

Thus,

$$\int_{\mathbb{R}^N} \psi(\mathbf{w})(w_j w_k - \delta_{jk})(w_m w_n - \delta_{mn}) d\mathbf{w} \tag{7.63}$$

$$= \delta_{jk} \delta_{mn} + \delta_{jm} \delta_{kn} + \delta_{jn} \delta_{km} - 2\delta_{jk} \delta_{mn} + \delta_{jk} \delta_{mn} \tag{7.64}$$

$$= \delta_{jm} \delta_{kn} + \delta_{jn} \delta_{km}. \tag{7.65}$$

Finally,

$$I_2 = \sum_{mn} \hat{S}_{ijk} \hat{S}_{lmn} E_{mn} (\delta_{jm} \delta_{kn} + \delta_{jn} \delta_{km}) \partial_{p_i p_l}^2 \tag{7.66}$$

$$= \sum_{mn} (\hat{S}_{imn} \hat{S}_{lmn} + \hat{S}_{imn} \hat{S}_{lnm}) E_{mn} \partial_{p_i p_l}^2 \tag{7.67}$$

Note for fixed m and n (no sum) we can also write

$$\hat{S}_{imn} \hat{S}_{lmn} + \hat{S}_{imn} \hat{S}_{lnm} = S_{iab} S_{lcd} \hat{Q}_{am} \hat{Q}_{bn} (\hat{Q}_{cm} \hat{Q}_{dn} + \hat{Q}_{cn} \hat{Q}_{dm}) \tag{7.68}$$

Using the form 7.68 in 7.67, we have

$$I_2 = S_{iab} S_{lcd} \left[\sum_{mn} \hat{Q}_{am} \hat{Q}_{bn} E_{mn} (\hat{Q}_{cm} \hat{Q}_{dn} + \hat{Q}_{cn} \hat{Q}_{dm}) \right] \partial_{p_i p_l}^2 \tag{7.69}$$

It may be useful to re-write 7.69 in the following way. We may interchange m and n. Averaging the original and new forms of I_2, we find

$$I_2 = \frac{1}{2} S_{iab} S_{lcd} \left[\sum_{mn} (\hat{Q}_{am} \hat{Q}_{bn} + \hat{Q}_{an} \hat{Q}_{bm}) E_{mn} (\hat{Q}_{cm} \hat{Q}_{dn} + \hat{Q}_{cn} \hat{Q}_{dm}) \right] \partial_{p_i p_l}^2 \tag{7.70}$$

This way, since the tensor applied to $\partial_{p_i p_n}^2$ is symmetric, we see I_2 may be written as

$$I_2 = \frac{1}{2} [\sigma \sigma^T]_{il} \partial_{p_i p_l}^2 = \frac{1}{2} \sigma \sigma^T : \nabla_{\mathbf{P}}^2 \tag{7.71}$$

where σ is a matrix.

7.6. Final Expression. We insert the results from the reduction into 7.26, using $P = \nabla_{\mathbf{X}} \Lambda$ to obtain

$$\mathcal{A}_0 = H_1^{\mathbf{X}} \cdot \nabla_{\mathbf{X}} + H_1^{\mathbf{P}} \cdot \nabla_{\mathbf{p}} + H_2^{\mathbf{X}} : \nabla_{\mathbf{X}}^2 + H_2^{\mathbf{P}} : \nabla_{\mathbf{p}}^2 \text{ where} \tag{7.72}$$

$$H_1^{\mathbf{X}} = \mathbf{v}_0, \tag{7.73}$$

$$H_1^{\mathbf{P}} = -\kappa^{-1}\nu[\mathbf{p} \cdot \nabla_{\mathbf{x}}\mathbf{p} - \Lambda\mathbf{v_0} \cdot \nabla_{\mathbf{x}}\mathbf{p} - \mathbf{p} \cdot \nabla_{\mathbf{x}}(\Lambda\mathbf{v}_0) + (\Lambda\mathbf{v}_0) \cdot \nabla_{\mathbf{x}}(\Lambda\mathbf{v}_0) + (\nabla_{\mathbf{x}}\Lambda C^{-1}) : \Lambda^T] \tag{7.74}$$

$$+ \kappa^{-1}\mathcal{L}(\mathbf{p} - \Lambda\mathbf{v}_0) + \nabla_{\mathbf{X}}\Lambda : C^{-1} + (\nabla_{\mathbf{X}}\Lambda[\mathbf{v}_0]) \cdot \mathbf{v}_0 - \Lambda\nabla_{\mathbf{X}}\Phi(\mathbf{X}) + \epsilon(\nabla_{\mathbf{X}}B) : \Upsilon^{-1} \tag{7.75}$$

$$H_2^{\mathbf{X}} = \epsilon\Upsilon^{-1} \tag{7.76}$$

$$H_2^{\mathbf{P}} = -\mathcal{L} + \epsilon(B\Upsilon^{-1}B^T + \frac{1}{2}\sigma\sigma^T) \tag{7.77}$$

Next, we may write the equations of motion of the system (assuming the convention with $\mathbf{p}$ a continuum, but remaining in non-dimensional coordinates):

$$\frac{d\mathbf{p}}{dt} = H_1^{\mathbf{P}} + \mathbf{g}_{thm}^{\epsilon} \tag{7.78}$$

$$\frac{d\mathbf{X}}{dt} = H_1^{\mathbf{X}} + \mathbf{N}_{thm}^{\epsilon} \tag{7.79}$$

where

$$\mathbf{g}_{thm}^{\epsilon} = D_{H_2^{\mathbf{P}}}\xi(\mathbf{X}, t) \text{ and } \mathbf{N}_{thm}^{\epsilon} = D_{H_2^{\mathbf{X}}}\eta(t) \tag{7.80}$$

In the limit $\epsilon \to 0$, we obtain

$$\frac{d\mathbf{p}}{dt} = -\kappa^{-1}\nu[\mathbf{p} \cdot \nabla_{\mathbf{x}}\mathbf{p} - \Lambda\mathbf{v_0} \cdot \nabla_{\mathbf{x}}\mathbf{p} - \mathbf{p} \cdot \nabla_{\mathbf{x}}(\Lambda\mathbf{v}_0) + (\Lambda\mathbf{v}_0) \cdot \nabla_{\mathbf{x}}(\Lambda\mathbf{v}_0) + (\nabla_{\mathbf{x}}\Lambda C^{-1}) : \Lambda^T] \tag{7.81}$$

$$+ \kappa^{-1}\mathcal{L}(\mathbf{p} - \Lambda\mathbf{v}_0) + \nabla_{\mathbf{X}}\Lambda : C^{-1} + (\nabla_{\mathbf{X}}\Lambda[\mathbf{v}_0]) \cdot \mathbf{v}_0 - \Lambda\nabla_{\mathbf{X}}\Phi(\mathbf{X}) \tag{7.82}$$

$$\frac{d\mathbf{X}}{dt} = \mathbf{v}_0 \tag{7.83}$$

We may add back the units, understanding C has no dimensions and $\mathbf{v}_0$ has units of ℓ/t_2, to obtain equations 5.1 and 5.2.

8. Conclusions. An approach for obtaining systematic reductions of the fluid-structure equations in the limit of strong viscous coupling were developed. A dimension analysis of the stochastic fluid-structure interactions equations was performed. This analysis showed the importance of formulating the reduced equations in terms of the total momentum field of the fluid and immersed structures. The reductions were then developed based the Infitesmal Generators of the stochastic process and a singular perturbation analysis of the Backward Kolomogorov PDEs. Analysis was presented primarily in the physical regime where the viscous coupling that controls the momentum exchange between the fluid and structures became large. This established a few new SELM stochastic equations for modeling fluid-structure interactions taking into account inertial effects. We expect the SELM descriptions presented to be useful for a number of applications requiring accounting for inertial effects in fluid-structure interactions, such as in developing models for spectroscopy experiments.

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