Finite Difference Methods
Problem Set

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Exercises

1. (Order of Accuracy) For the diffusion equation $u_t = u_{xx}$ show that the Forward-Time/Central-Space scheme

$$\frac{v_m^{n+1} - v_m^n}{k} = \frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2}$$

is first order in time and second order in space.

2. (Stability) For the diffusion equation $u_t = u_{xx}$, show using von Neumann Analysis $v_m^n \to g^n e^{im\theta}$ that the scheme in problem 1 is stable provided $s < 1/2$ where $s = k/h^2$.

3. (Stability) For the diffusion equation $u_t = u_{xx}$ show using von Neumann Analysis that the Crank-Nicolson scheme

$$\frac{v_m^{n+1} - v_m^n}{k} = \frac{1}{2}\left[\frac{v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}}{h^2} + \frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2}\right]$$

is unconditionally stable.

4. (Stability) For the advection equation $u_t = cu_x$, determine using von Neumann Analysis $v_m^n \to g^n e^{im\theta}$ the conditions on $s = k/h$ and c under which the scheme below is stable. Show the scheme is unstable when $c < 0$ for any choice of s. Show the scheme is stable when $c > 0$ provided s taken sufficiently small.

$$\frac{v_m^{n+1} - v_m^n}{k} = c\frac{v_{m+1}^n - v_m^n}{h}$$

5. (Stability) For the advection equation $u_t = cu_x$, show using von Neumann Analysis $v_m^n \to g^n e^{im\theta}$ that the Lax-Wendroff scheme below is stable provided $s = k/h$ is taken sufficiently small.

$$\frac{v_m^{n+1} - v_m^n}{k} = c\frac{v_{m+1}^n - v_{m-1}^n}{2h} + \left(\frac{c^2 k}{2}\right)\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2}.$$

6. (Order of Accuracy) For the advection equation $u_t = cu_x$, show that the Lax-Wendroff scheme in Problem 5 is second order accurate.

7. (Bonus Problem: Order of Accuracy) Show that the (forward-backward) MacCormack scheme

$$\begin{aligned}
\tilde{v}_m^{n+1} &= v_m^n - a\lambda\left(v_{m+1}^n - v_m^n\right) + kf_m^n \\
v_m^{n+1} &= \frac{1}{2}\left(v_m^n + \tilde{v}_m^{n+1} - a\lambda\left(v_m^{n+1} - \tilde{v}_{m-1}^{n+1}\right) + kf_m^{n+1}\right)
\end{aligned}$$

is a second-order accurate scheme for the one-way wave equation $u_t + au_x + f = 0$. Show that for $f = 0$ it is identical with the Lax-Wendroff scheme.