Name: ________________________

Practice Problems:

Professor: Paul J. Atzberger

Introduction to Numerical Analysis

Scoring:

Problem1: ____________

Problem2: ____________

Problem3: ____________

Problem4: ____________

Directions: Answer each question carefully and be sure to show the steps of your work.

Problem 1: Difference Formulas.

a) Use the two-point forward-difference formula to estimate the value of the derivative $f(x) = e^x$ using the points $x_0 = 0$ and $x_1 = x_0 + h = \ln(1 - \frac{1}{4})$.

b) For a general smooth function $f(x)$ and two-point forward-difference formula, show using Taylor expansion the accuracy is $O(h)$.
Hint: Taylor expand $f(x_0 + h)$ about x_0 and consider for the difference formula how the leading order term scales as $h \to 0$.

c) The midpoint difference formula is

$$\frac{1}{2h}\left[f(x_0+h)-f(x_0-h)\right].$$

Estimate the value of the derivative $f(x) = e^x$ again using the points $x_0 = 0$ and $h = \ln(1 - \frac{1}{4})$.

d) For a general smooth function $f(x)$ and midpoint difference formula, show using Taylor expansion the accuracy is $O(h^2)$.

e) Compare the estimates in problem (a) and (c) to the exact solution $f'(x) = e^x$ at $x_0 = 0$. In particular, for each what is the relative error? Which estimate is more accurate? Does this agree with theory? If not, state the reason why.

Problem 2: Numerical Integration.

a) Use the Trapezoidal Rule and composite integration to estimate the integral

$$I = \int_0^1 e^x dx$$

with the nodes $x_j = jh$ where $h = \frac{1}{4}$ and $j = 0, 1, \ldots, 4$.

b) Use the Simpson's Rule and composite integration to estimate the integral

$$I = \int_0^1 e^x dx$$

with the nodes $x_j = jh$ where $h = \frac{1}{4}$ and $j = 0, 1, \ldots, 4$.

c) What is the degree of accuracy of the Trapezoidal Rule? What is the degree of accuracy of the Simpson's Rule?

d) Compare the estimate in (a) and (b) with the exact solution. Give for each the relative error of the estimate. Which is more accurate? Does this agree with theory? If not, state the reason why.

Problem 3: Gaussian Quadrature.

a) The Legendre polynomials

$$L_0(x) = 1 \tag{1}$$
$$L_1(x) = x \tag{2}$$
$$L_2(x) = \frac{1}{2}\left(3x^2 - 1\right) \tag{3}$$
$$L_3(x) = \frac{1}{2}\left(5x^3 - 3x\right) \tag{4}$$

are orthogonal with the inner-product

$$\langle L_i, L_j \rangle = \int_{-1}^{1} L_i(x) L_j(x) dx. \tag{5}$$

Partially verify this statement by showing the orthogonality $\langle L_i, L_j \rangle = 0$ for the specific indices $(0,3), (1,3), (2,3)$. Hint: Use that x^k for odd powers integrates to zero on $[-1, 1]$.

b) Find the three roots $x_0 < x_1 < x_2$ of $L_3(x)$. Hint: Use that $L_3(x) = \frac{x}{2}\left(5x^2 - 3\right)$.

c) The Gaussian quadrature for approximating $I = \int_{-1}^{1} f(x)dx$ with $n = 3$ points is given by

$$\tilde{I} = \sum_{j=0}^{n-1} w_j f(x_j).$$

The weights are $w_0 = 5/9, w_1 = 8/9, w_2 = 5/9$ and the nodes x_j are the roots of $L_3(x)$. Show the Gaussian quadrature integrates exactly the polynomial $f(x) = x^4$, (i.e. show that $\tilde{I} = I$).

Problem 4: Numerical Methods for Ordinary Differential Equation (ODEs).

Consider the ODE

$$\begin{aligned} y' &= -ky \\ y(0) &= y_0 \end{aligned}$$

approximated by Euler's Method

$$\begin{aligned} w_0 &= y_0 \\ w_{n+1} &= w_n + hf(w_n). \end{aligned}$$

a) What is the truncation error for Euler's Method? For $\tau_{i+1}(h) = O(h^\ell)$, what order ℓ is Euler's Method?

b) Consider the two stage Runge-Kutta Method with Tableau

$$\begin{array}{c|cc} 0 & & \\ \frac{2}{3} & \frac{2}{3} & \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array} \tag{6}$$

What is the truncation error of the RK-Method? For $\tau_{i+1}(h) = O(h^\ell)$, what order ℓ?