Name: ____________________________

Practice Problems:

Introduction to Numerical Analysis, 104B

Professor: Paul J. Atzberger

Scoring:

Problem1: ______________

Problem2: ______________

Problem3: ______________

<u>**Directions**:</u> Answer each question carefully and be sure to show all of your work. You are permitted to use a calculator but please be sure to show intermediate steps in your calculations. If you have any questions please feel free to ask.

Problem 1: One-step Methods for Ordinary Differential Equation (ODEs).

Consider the ODE

$$\begin{aligned} y' &= f(y,t) \\ y(0) &= y_0 \end{aligned}$$

approximated by a one-step method

$$\begin{aligned} w_0 &= y_0 \\ w_{n+1} &= w_n + h\phi(t_n, w_n). \end{aligned}$$

(a) What are sufficient conditions on f so that the ODE is well-posed over the time $t \in [a,b]$?

(b) Euler's Method approximates the ODE by

$$\begin{aligned} w_0 &= y_0 \\ w_{n+1} &= w_n + hf(t_n, w_n). \end{aligned}$$

What is the truncation error for Euler's Method? For $\tau_{i+1}(h) = O(h^\ell)$ what is the order ℓ?

(c) In the case that $f(t, y) = \lambda y$ with $\lambda < 0$ what condition must be satisfied by h in order for Euler's Method to be asymptotically stable? Give the condition when $\lambda = -2$.

(d) The Trapezoidal Method approximates the ODE by

$$\begin{aligned} w_0 &= y_0 \\ w_{n+1} &= w_n + \frac{h}{2}\left(f(t_{n+1}, w_{n+1}) + f(t_n, w_n)\right). \end{aligned}$$

In this case what condition must be satisfied by h in order for the Trapezoidal Method to be asymptotically stable? Give the condition when $\lambda = -2$.

Problem 2: Runge-Kutta Methods for Ordinary Differential Equation (ODEs).

Consider the two-stage Runge-Kutta Method with Tableau

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

(a) In the case that $f(t, y) = -y + t$ give the first time-step $y(t_1)$ approximation that is generated by using the Runge-Kutta Method when $h = 0.1$ and $t_0 = 0, y(0) = 0$.

(b) In the case that $f(t, y) = \lambda y$ with $\lambda < 0$ what condition must be satisfied by h in order for Runge-Kutta Method to be asymptotically stable? Give the condition when $\lambda = -2$.

Problem 3: Multi-step Methods for Ordinary Differential Equation (ODEs).
Consider a Multistep Method that approximates the ODE by

$$\begin{aligned} w_{n+1} &= a_m w_n + \cdots + a_0 w_{n-m+1} \\ &+ h\left[b_{m+1} f(t_{n+1}, w_{n+1}) + b_m f(t_n, w_n) + \cdots + +b_0 f(t_{n-m+1}, w_{n-m+1})\right]. \\ w_0 &= \alpha_0, w_1 = \alpha_1. \end{aligned}$$

(a) Consider the Two-Step Adams-Bashforth Method with $a_m = 1$, $a_{m-1} = 0$, and $b_{m+1} = 0, b_m = \frac{3}{2}, b_{m-1} = -\frac{1}{2}$. In the case that $f(t, y) = -y+t$ give the time-step $y(t_2)$ approximation that is generated by using this Multistep Method with $\alpha_0 = 0, \alpha_1 = 0.1$ when $h = 0.1$, $t_0 = 0$.

(b) In the prototype problem with $f(t, y) = 0$, what condition must be satisfied in order for the Two-Step Adams-Bashforth Method to be stable?

(c) Consider the Two-Step Adams-Moulten Method with $a_m = 1$, $a_{m-1} = 0$, and $b_{m+1} = \frac{5}{12}, b_m = \frac{8}{12}, b_{m-1} = -\frac{1}{12}$. In the prototype problem with $f(t, y) = 0$ what condition must be satisfied in order for the method to be stable?