EXERCISE SET 1.1

1. Show that the following equations have at least one solution in the given intervals.
   a. $x \cos x - 2x^2 + 3x - 1 = 0$, $[0.2, 0.3]$ and $[1.2, 1.3]$
   b. $(x-2)^2 - \ln x = 0$, $[1, 2]$ and $[e, 4]$
   c. $2x \cos(2x) - (x-2)^2 = 0$, $[2, 3]$ and $[3, 4]$
   d. $x - (\ln x)^x = 0$, $[4, 5]$
2. Show that the following equations have at least one solution in the given intervals.
   a. $\sqrt{x} - \cos x = 0$, $[0, 1]$
   b. $e^x - x^2 + 3x - 2 = 0$, $[0, 1]$
   c. $-3\tan(2x) + x = 0$, $[0, 1]$
   d. $\ln x - x^2 + \frac{5}{2}x - 1 = 0$, $[\frac{1}{2}, 1]$
3. Find intervals containing solutions to the following equations.
   a. $x - 2^{-x} = 0$
   b. $2x\cos(2x) - (x+1)^2 = 0$
   c. $3x - e^x = 0$
   d. $x + 1 - 2\sin(\pi x) = 0$

4. Find intervals containing solutions to the following equations.
   a. $x - 3^{-x} = 0$
   b. $4x^2 - e^x = 0$
   c. $x^3 - 2x^2 - 4x + 2 = 0$
   d. $x^3 + 4.001x^2 + 4.002x + 1.101 = 0$
5. Find $\max_{a \le x \le b} |f(x)|$ for the following functions and intervals.
   a. $f(x) = (2 - e^x + 2x)/3, \quad [0, 1]$
   b. $f(x) = (4x - 3)/(x^2 - 2x), \quad [0.5, 1]$
   c. $f(x) = 2x\cos(2x) - (x-2)^2, \quad [2, 4]$
   d. $f(x) = 1 + e^{-\cos(x-1)}, \quad [1, 2]$
6. Find $\max_{a \le x \le b} |f(x)|$ for the following functions and intervals.
   a. $f(x) = 2x/(x^2 + 1), \quad [0, 2]$
   b. $f(x) = x^2\sqrt{(4 - x)}, \quad [0, 4]$
   c. $f(x) = x^3 - 4x + 2, \quad [1, 2]$
   d. $f(x) = x\sqrt{(3 - x^2)}, \quad [0, 1]$
7. Show that $f'(x)$ is 0 at least once in the given intervals.
   a. $f(x) = 1 - e^x + (e - 1)\sin((\pi/2)x), \quad [0, 1]$
   b. $f(x) = (x - 1)\tan x + x\sin \pi x, \quad [0, 1]$
   c. $f(x) = x\sin \pi x - (x - 2)\ln x, \quad [1, 2]$
   d. $f(x) = (x - 2)\sin x \ln(x + 2), \quad [-1, 3]$
8. Suppose $f \in C[a, b]$ and $f'(x)$ exists on (a, b). Show that if $f'(x) \neq 0$ for all x in (a, b), then there can exist at most one number p in $[a, b]$ with $f(p) = 0$.
9. Let $f(x) = x^3$.
   a. Find the second Taylor polynomial $P_2(x)$ about $x_0 = 0$.
   b. Find $R_2(0.5)$ and the actual error in using $P_2(0.5)$ to approximate $f(0.5)$.
   c. Repeat part (a) using $x_0 = 1$.
   d. Repeat part (b) using the polynomial from part (c).
10. Find the third Taylor polynomial $P_3(x)$ for the function $f(x) = \sqrt{x + 1}$ about $x_0 = 0$. Approximate $\sqrt{0.5}$, $\sqrt{0.75}$, $\sqrt{1.25}$, and $\sqrt{1.5}$ using $P_3(x)$ and find the actual errors.
11. Find the second Taylor polynomial $P_2(x)$ for the function $f(x) = e^x \cos x$ about $x_0 = 0$.

11. Find the second Taylor polynomial $P_2(x)$ for the function $f(x) = e^x \cos x$ about $x_0 = 0$.

a. Use $P_2(0.5)$ to approximate $f(0.5)$. Find an upper bound for error $|f(0.5) - P_2(0.5)|$ using the error formula and compare it to the actual error.

b. Find a bound for the error $|f(x) - P_2(x)|$ in using $P_2(x)$ to approximate $f(x)$ on the interval $[0, 1]$.

c. Approximate $\int_0^1 f(x)\, dx$ using $\int_0^1 P_2(x)\, dx$.

d. Find an upper bound for the error in (c) using $\int_0^1 |R_2(x)\, dx|$ and compare the bound to the actual error.

12. Repeat Exercise 11 using $x_0 = \pi/6$.

13. Find the third Taylor polynomial $P_3(x)$ for the function $f(x) = (x - 1) \ln x$ about $x_0 = 1$.

a. Use $P_3(0.5)$ to approximate $f(0.5)$. Find an upper bound for error $|f(0.5) - P_3(0.5)|$ using the error formula and compare it to the actual error.

b. Find a bound for the error $|f(x) - P_3(x)|$ in using $P_3(x)$ to approximate $f(x)$ on the interval $[0.5, 1.5]$.

c. Approximate $\int_{0.5}^{1.5} f(x)\, dx$ using $\int_{0.5}^{1.5} P_3(x)\, dx$.

d. Find an upper bound for the error in (c) using $\int_{0.5}^{1.5} |R_3(x)\, dx|$ and compare the bound to the actual error.

14. Let $f(x) = 2x\cos(2x) - (x - 2)^2$ and $x_0 = 0$.

a. Find the third Taylor polynomial $P_3(x)$ and use it to approximate $f(0.4)$.

b. Use the error formula in Taylor's Theorem to find an upper bound for the error $|f(0.4) - P_3(0.4)|$. Compute the actual error.

c. Find the fourth Taylor polynomial $P_4(x)$ and use it to approximate $f(0.4)$.

d. Use the error formula in Taylor's Theorem to find an upper bound for the error $|f(0.4) - P_4(0.4)|$. Compute the actual error.

15. Find the fourth Taylor polynomial $P_4(x)$ for the function $f(x) = xe^{x^2}$ about $x_0 = 0$.

a. Find an upper bound for $|f(x) - P_4(x)|$, for $0 \leq x \leq 0.4$.

b. Approximate $\int_0^{0.4} f(x)\, dx$ using $\int_0^{0.4} P_4(x)\, dx$.

c. Find an upper bound for the error in (b) using $\int_0^{0.4} P_4(x)\, dx$.

d. Approximate $f'(0.2)$ using $P_4'(0.2)$ and find the error.

16. Use the error term of a Taylor polynomial to estimate the error involved in using $\sin x \approx x$ to approximate $\sin 1°$.

17. Use a Taylor polynomial about $\pi/4$ to approximate $\cos 42°$ to an accuracy of 10^{-6}.

18. Let $f(x) = (1 - x)^{-1}$ and $x_0 = 0$. Find the nth Taylor polynomial $P_n(x)$ for $f(x)$ about x_0. Find a value of n necessary for $P_n(x)$ to approximate $f(x)$ to within 10^{-6} on $[0, 0.5]$.

19. Let $f(x) = e^x$ and $x_0 = 0$. Find the nth Taylor polynomial $P_n(x)$ for $f(x)$ about x_0. Find a value of n necessary for $P_n(x)$ to approximate $f(x)$ to within 10^{-6} on $[0, 0.5]$.

20. Find the nth Maclaurin polynomial $P_n(x)$ for $f(x) = \arctan x$.

21. The polynomial $P_2(x) = 1 - \frac{1}{2}x^2$ is to be used to approximate $f(x) = \cos x$ in $[-\frac{1}{2}, \frac{1}{2}]$. Find a bound for the maximum error.

22. Use the Intermediate Value Theorem 1.11 and Rolle's Theorem 1.7 to show that the graph of $f(x) = x^3 + 2x + k$ crosses the x-axis exactly once, regardless of the value of the constant k.

23. A Maclaurin polynomial for e^x is used to give the approximation 2.5 to e. The error bound in this approximation is established to be $E = \frac{1}{6}$. Find a bound for the error in E.

23. A Maclaurin polynomial for e^x is used to give the approximation 2.5 to e. The error bound in this approximation is established to be $E = \frac{1}{6}$. Find a bound for the error in E.

24. The *error function* defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

gives the probability that any one of a series of trials will lie within x units of the mean, assuming that the trials have a normal distribution with mean 0 and standard deviation $\sqrt{2}/2$. This integral cannot be evaluated in terms of elementary functions, so an approximating technique must be used.

a. Integrate the Maclaurin series for e^{-x^2} to show that

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}.$$

b. The error function can also be expressed in the form

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{1 \cdot 3 \cdot 5 \cdots (2k+1)}.$$

Verify that the two series agree for $k = 1, 2, 3$, and 4. [*Hint:* Use the Maclaurin series for e^{-x^2}.]

c. Use the series in part (a) to approximate erf(1) to within 10^{-7}.

d. Use the same number of terms as in part (c) to approximate erf(1) with the series in part (b).

e. Explain why difficulties occur using the series in part (b) to approximate $\operatorname{erf}(x)$.

THEORETICAL EXERCISES

25. The nth Taylor polynomial for a function f at x_0 is sometimes referred to as the polynomial of degree at most n that "best" approximates f near x_0.

a. Explain why this description is accurate.

b. Find the quadratic polynomial that best approximates a function f near $x_0 = 1$ if the tangent line at $x_0 = 1$ has equation $y = 4x - 1$ and if $f''(1) = 6$.

26. Prove the Generalized Rolle's Theorem, Theorem 1.10, by verifying the following.

a. Use Rolle's Theorem to show that $f'(z_i) = 0$ for $n - 1$ numbers in $[a, b]$ with $a < z_1 < z_2 < \cdots < z_{n-1} < b$.

b. Use Rolle's Theorem to show that $f''(w_i) = 0$ for $n - 2$ numbers in $[a, b]$ with $z_1 < w_1 < z_2 < w_2 \cdots w_{n-2} < z_{n-1} < b$.

c. Continue the arguments in parts (a) and (b) to show that for each $j = 1, 2, \ldots, n - 1$, there are $n - j$ distinct numbers in $[a, b]$, where $f^{(j)}$ is 0.

d. Show that part (c) implies the conclusion of the theorem.

27. Example 3 stated that for all x we have $|\sin x| \le |x|$. Use the following to verify this statement.

a. Show that for all $x \ge 0$, $f(x) = x - \sin x$ is nondecreasing, which implies that $\sin x \le x$ with equality only when $x = 0$.

b. Use the fact that the sine function is odd to reach the conclusion.

28. A function $f : [a, b] \to \mathbb{R}$ is said to satisfy a *Lipschitz condition* with Lipschitz constant L on $[a, b]$ if, for every $x, y \in [a, b]$, we have $|f(x) - f(y)| \le L|x - y|$.

a. Show that if f satisfies a Lipschitz condition with Lipschitz constant L on an interval $[a, b]$, then $f \in C[a, b]$.

b. Show that if f has a derivative that is bounded on $[a, b]$ by L, then f satisfies a Lipschitz condition with Lipschitz constant L on $[a, b]$.

c. Give an example of a function that is continuous on a closed interval but does not satisfy a Lipschitz condition on the interval.

29. Suppose $f \in C[a, b]$ and x_1 and x_2 are in $[a, b]$.

a. Show that a number ξ exists between x_1 and x_2 with

$$f(\xi) = \frac{f(x_1) + f(x_2)}{2} = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

b. Suppose c_1 and c_2 are positive constants. Show that a number ξ exists between x_1 and x_2 with

$$f(\xi) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2}.$$

c. Give an example to show that the result in part (b) does not necessarily hold when c_1 and c_2 have opposite signs with $c_1 \ne -c_2$.

29. Suppose $f \in C[a, b]$ and x_1 and x_2 are in $[a, b]$.

a. Show that a number ξ exists between x_1 and x_2 with

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b. Suppose c_1 and c_2 are positive constants. Show that a number ξ exists between x_1 and x_2 with

$$f(\xi) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2}.$$

c. Give an example to show that the result in part (b) does not necessarily hold when c_1 and c_2 have opposite signs with $c_1 \neq -c_2$.

30. Let $f \in C[a, b]$, and let p be in the open interval (a, b).

a. Suppose $f(p) \neq 0$. Show that a $\delta > 0$ exists with $f(x) \neq 0$, for all x in $[p - \delta, p + \delta]$, with $[p - \delta, p + \delta]$ a subset of $[a, b]$.

b. Suppose $f(p) = 0$ and $k > 0$ is given. Show that a $\delta > 0$ exists with $|f(x)| \leq k$, for all x in $[p - \delta, p + \delta]$, with $[p - \delta, p + \delta]$ a subset of $[a, b]$.

DISCUSSION QUESTION

1. In your own words, describe the Lipschitz condition. Give several examples of functions that satisfy this condition or give several examples of functions that do not satisfy this condition.

EXERCISE SET 1.2

1. Compute the absolute error and relative error in approximations of p by p^*.

   a. $p = \pi$, $p^* = 22/7$ **b.** $p = \pi$, $p^* = 3.1416$
   c. $p = e$, $p^* = 2.718$ **d.** $p = \sqrt{2}$, $p^* = 1.414$

2. Compute the absolute error and relative error in approximations of p by p^*.

   a. $p = e^{10}$, $p^* = 22000$ **b.** $p = 10^{\pi}$, $p^* = 1400$
   c. $p = 8!$, $p^* = 39900$ **d.** $p = 9!$, $p^* = \sqrt{18\pi}(9/e)^9$

3. Suppose p^* must approximate p with relative error at most 10^{-3}. Find the largest interval in which p^* must lie for each value of p.

   a. 150 **b.** 900
   c. 1500 **d.** 90

4. Find the largest interval in which p^* must lie to approximate p with relative error at most 10^{-4} for each value of p.

   a. π **b.** e
   c. $\sqrt{2}$ **d.** $\sqrt[3]{7}$

5. Perform the following computations (i) exactly, (ii) using three-digit chopping arithmetic, and (iii) using three-digit rounding arithmetic. (iv) Compute the relative errors in parts (ii) and (iii).

   a. $\frac{4}{5} + \frac{1}{3}$ **b.** $\frac{4}{5} \cdot \frac{1}{3}$
   c. $\left(\frac{1}{3} - \frac{3}{11}\right) + \frac{3}{20}$ **d.** $\left(\frac{1}{3} + \frac{3}{11}\right) - \frac{3}{20}$

6. Use three-digit rounding arithmetic to perform the following calculations. Compute the absolute error and relative error with the exact value determined to at least five digits.

   a. $133 + 0.921$ **b.** $133 - 0.499$
   c. $(121 - 0.327) - 119$ **d.** $(121 - 119) - 0.327$

7. Use three-digit rounding arithmetic to perform the following calculations. Compute the absolute error and relative error with the exact value determined to at least five digits.

   a. $\dfrac{\frac{13}{14} - \frac{6}{7}}{2e - 5.4}$ **b.** $-10\pi + 6e - \dfrac{3}{62}$
   c. $\left(\frac{2}{9}\right) \cdot \left(\frac{9}{7}\right)$ **d.** $\dfrac{\sqrt{13} + \sqrt{11}}{\sqrt{13} - \sqrt{11}}$

7. Use three-digit rounding arithmetic to perform the following calculations. Compute the absolute error and relative error with the exact value determined to at least five digits.

a. $\dfrac{\frac{13}{14} - \frac{6}{7}}{2e - 5.4}$ **b.** $-10\pi + 6e - \dfrac{3}{62}$

c. $\left(\dfrac{2}{9}\right) \cdot \left(\dfrac{9}{7}\right)$ **d.** $\dfrac{\sqrt{13} + \sqrt{11}}{\sqrt{13} - \sqrt{11}}$

8. Repeat Exercise 7 using four-digit rounding arithmetic.

9. Repeat Exercise 7 using three-digit chopping arithmetic.

10. Repeat Exercise 7 using four-digit chopping arithmetic.

11. The first three nonzero terms of the Maclaurin series for the arctangent function are $x - (1/3)x^3 + (1/5)x^5$. Compute the absolute error and relative error in the following approximations of π using the polynomial in place of the arctangent:

a. $4\left[\arctan\left(\dfrac{1}{2}\right) + \arctan\left(\dfrac{1}{3}\right)\right]$

b. $16 \arctan\left(\dfrac{1}{5}\right) - 4 \arctan\left(\dfrac{1}{239}\right)$

12. The number e can be defined by $e = \sum_{n=0}^{\infty}(1/n!)$, where $n! = n(n-1)\cdots 2\cdot 1$ for $n \neq 0$ and $0! = 1$. Compute the absolute error and relative error in the following approximations of e:

a. $\displaystyle\sum_{n=0}^{5} \frac{1}{n!}$

b. $\displaystyle\sum_{n=0}^{10} \frac{1}{n!}$

13. Let

$$f(x) = \frac{x \cos x - \sin x}{x - \sin x}.$$

a. Find $\lim_{x \to 0} f(x)$.

b. Use four-digit rounding arithmetic to evaluate $f(0.1)$.

c. Replace each trigonometric function with its third Maclaurin polynomial and repeat part (b).

d. The actual value is $f(0.1) = -1.99899998$. Find the relative error for the values obtained in parts (b) and (c).

14. Let

$$f(x) = \frac{e^x - e^{-x}}{x}.$$

a. Find $\lim_{x \to 0}(e^x - e^{-x})/x$.

b. Use three-digit rounding arithmetic to evaluate $f(0.1)$.

c. Replace each exponential function with its third Maclaurin polynomial and repeat part (b).

d. The actual value is $f(0.1) = 2.003335000$. Find the relative error for the values obtained in parts (b) and (c).

15. Use four-digit rounding arithmetic and the formulas (1.1), (1.2), and (1.3) to find the most accurate approximations to the roots of the following quadratic equations. Compute the absolute errors and relative errors.

   a. $\frac{1}{3}x^2 - \frac{123}{4}x + \frac{1}{6} = 0$

   b. $\frac{1}{3}x^2 + \frac{123}{4}x - \frac{1}{6} = 0$

   c. $1.002x^2 - 11.01x + 0.01265 = 0$

   d. $1.002x^2 + 11.01x + 0.01265 = 0$

16. Use four-digit rounding arithmetic and the formulas (1.1), (1.2), and (1.3) to find the most accurate approximations to the roots of the following quadratic equations. Compute the absolute errors and relative errors.

   a. $x^2 - \sqrt{7}x + \sqrt{2} = 0$

   b. $\pi x^2 + 13x + 1 = 0$

   c. $x^2 + x - e = 0$

   d. $x^2 - \sqrt{35}x - 2 = 0$

17. Repeat Exercise 15 using four-digit chopping arithmetic.

18. Repeat Exercise 16 using four-digit chopping arithmetic.

19. Use the 64-bit-long real format to find the decimal equivalent of the following floating-point machine numbers.

   a. 0 10000001010 1001001100000000000000000000000000000000000000000000

   b. 1 10000001010 1001001100000000000000000000000000000000000000000000

   c. 0 01111111111 0101001100000000000000000000000000000000000000000000

   d. 0 01111111111 0101001100000000000000000000000000000000000000000001

20. Find the next largest and smallest machine numbers in decimal form for the numbers given in Exercise 19.

21. Suppose two points (x_0, y_0) and (x_1, y_1) are on a straight line with $y_1 \neq y_0$. Two formulas are available to find the x-intercept of the line:

$$x = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0} \quad \text{and} \quad x = x_0 - \frac{(x_1 - x_0) y_0}{y_1 - y_0}.$$

a. Show that both formulas are algebraically correct.

b. Use the data $(x_0, y_0) = (1.31, 3.24)$ and $(x_1, y_1) = (1.93, 4.76)$ and three-digit rounding arithmetic to compute the x-intercept both ways. Which method is better, and why?

22. The Taylor polynomial of degree n for $f(x) = e^x$ is $\sum_{i=0}^{n} (x^i / i!)$. Use the Taylor polynomial of degree nine and three-digit chopping arithmetic to find an approximation to e^{-5} by each of the following methods.

a. $e^{-5} \approx \sum_{i=0}^{9} \frac{(-5)^i}{i!} = \sum_{i=0}^{9} \frac{(-1)^i 5^i}{i!}$

b. $e^{-5} = \frac{1}{e^5} \approx \frac{1}{\sum_{i=0}^{9} \frac{5^i}{i!}}.$

c. An approximate value of e^{-5} correct to three digits is 6.74×10^{-3}. Which formula, (a) or (b), gives the most accuracy, and why?

23. The two-by-two linear system

$$ax + by = e,$$
$$cx + dy = f,$$

where a, b, c, d, e, f are given, can be solved for x and y as follows:

$$\text{set } m = \frac{c}{a}, \quad \text{provided } a \neq 0;$$
$$d_1 = d - mb;$$
$$f_1 = f - me;$$
$$y = \frac{f_1}{d_1};$$
$$x = \frac{(e - by)}{a}.$$

Solve the following linear systems using four-digit rounding arithmetic.

a. $1.130x - 6.990y = 14.20$
$1.013x - 6.099y = 14.22$

b. $8.110x + 12.20y = -0.1370$
$-18.11x + 112.2y = -0.1376$

24. Repeat Exercise 23 using four-digit chopping arithmetic.

25. **a.** Show that the polynomial nesting technique described in Example 6 can also be applied to the evaluation of

$$f(x) = 1.01e^{4x} - 4.62e^{3x} - 3.11e^{2x} + 12.2e^{x} - 1.99.$$

b. Use three-digit rounding arithmetic, the assumption that $e^{1.53} = 4.62$, and the fact that $e^{nx} = (e^x)^n$ to evaluate $f(1.53)$ as given in part (a).

c. Redo the calculation in part (b) by first nesting the calculations.

d. Compare the approximations in parts (b) and (c) to the true three-digit result $f(1.53) = -7.61$.

APPLIED EXERCISES

26. The opening example to this chapter described a physical experiment involving the temperature of a gas under pressure. In this application, we were given $P = 1.00$ atm, $V = 0.100$ m^3, $N = 0.00420$ mol, and $R = 0.08206$. Solving for T in the ideal gas law gives

$$T = \frac{PV}{NR} = \frac{(1.00)(0.100)}{(0.00420)(0.08206)} = 290.15\text{ K} = 17°\text{C}.$$

In the laboratory, it was found that T was 15°C under these conditions, and when the pressure was doubled and the volume halved, T was 19°C. Assume that the data are rounded values accurate to the places given, and show that both laboratory figures are within the bounds of accuracy for the ideal gas law.

THEORETICAL EXERCISES

27. The binomial coefficient

$$\binom{m}{k} = \frac{m!}{k!\,(m-k)!}$$

describes the number of ways of choosing a subset of k objects from a set of m elements.

a. Suppose decimal machine numbers are of the form

$$\pm 0.d_1d_2d_3d_4 \times 10^n, \quad \text{with } 1 \le d_1 \le 9,\ 0 \le d_i \le 9,$$

$$\text{if } i = 2, 3, 4 \quad \text{and} \quad |n| \le 15.$$

What is the largest value of m for which the binomial coefficient $\binom{m}{k}$ can be computed for all k by the definition without causing overflow?

b. Show that $\binom{m}{k}$ can also be computed by

$$\binom{m}{k} = \left(\frac{m}{k}\right)\left(\frac{m-1}{k-1}\right)\cdots\left(\frac{m-k+1}{1}\right).$$

27. The binomial coefficient

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describes the number of ways of choosing a subset of k objects from a set of m elements.

a. Suppose decimal machine numbers are of the form

$$\pm 0.d_1d_2d_3d_4 \times 10^n, \quad \text{with } 1 \le d_1 \le 9,\ 0 \le d_i \le 9,$$

$$\text{if } i = 2, 3, 4 \quad \text{and} \quad |n| \le 15.$$

What is the largest value of m for which the binomial coefficient $\binom{m}{k}$ can be computed for all k by the definition without causing overflow?

b. Show that $\binom{m}{k}$ can also be computed by

$$\binom{m}{k} = \left(\frac{m}{k}\right)\left(\frac{m-1}{k-1}\right)\cdots\left(\frac{m-k+1}{1}\right).$$

c. What is the largest value of m for which the binomial coefficient $\binom{m}{3}$ can be computed by the formula in part (b) without causing overflow?

d. Use the equation in (b) and four-digit chopping arithmetic to compute the number of possible five-card hands in a 52-card deck. Compute the actual and relative errors.

28. Suppose that $fl(y)$ is a k-digit rounding approximation to y. Show that

$$\left|\frac{y - fl(y)}{y}\right| \le 0.5 \times 10^{-k+1}.$$

[*Hint:* If $d_{k+1} < 5$, then $fl(y) = 0.d_1d_2\ldots d_k \times 10^n$. If $d_{k+1} \ge 5$, then $fl(y) = 0.d_1d_2\ldots d_k \times 10^n + 10^{n-k}$.]

29. Let $f \in C[a, b]$ be a function whose derivative exists on (a, b). Suppose f is to be evaluated at x_0 in (a, b), but instead of computing the actual value $f(x_0)$, the approximate value, $\tilde{f}(x_0)$, is the actual value of f at $x_0 + \epsilon$; that is, $\tilde{f}(x_0) = f(x_0 + \epsilon)$.

a. Use the Mean Value Theorem 1.8 to estimate the absolute error $|f(x_0) - \tilde{f}(x_0)|$ and the relative error $|f(x_0) - \tilde{f}(x_0)|/|f(x_0)|$, assuming $f(x_0) \ne 0$.

b. If $\epsilon = 5 \times 10^{-6}$ and $x_0 = 1$, find bounds for the absolute and relative errors for

i. $f(x) = e^x$

ii. $f(x) = \sin x$

c. Repeat part (b) with $\epsilon = (5 \times 10^{-6})x_0$ and $x_0 = 10$.

DISCUSSION QUESTIONS

1. Discuss the difference between the arithmetic performed by a computer and traditional arithmetic. Why is it so important to recognize the difference?
2. Provide several real-life examples of catastrophic errors that have occurred from the use of finite digital arithmetic and explain what went wrong.
3. Discuss the various different ways to round numbers.
4. Discuss the difference between a number written in standard notation and one that is written in normalized decimal floating-point form. Provide several examples.