Finite Element Spaces

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206D: Finite Element Methods
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and $L \equiv 0$ implies $\alpha_i = 0$.

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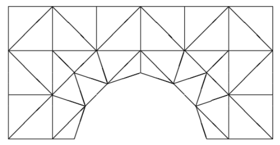
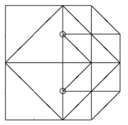


admissible triangulation



inadmissible (hanging nodes)

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For a bounded domain Ω, admissible partition, and $k \geq 1$, a piecewise infinitely differentiable function $v : \overline{\Omega} \to \mathbb{R}$ belongs to $H^k(\Omega)$ if and only if $v \in C^{k-1}(\overline{\Omega})$.

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Proof: We show this for the case $k = 1$, and for simplicity $\mathbb{R}^2$. The result follows for larger derivatives by applying the result to the $(k-1)$-order derivative functions. $(\Leftarrow)$ For $v \in C^0(\Omega)$, let $\mathcal{T} = \{T_j\}_{j=1}^M$

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Other shape spaces, partition types, and non-conforming finite elements are also possible.

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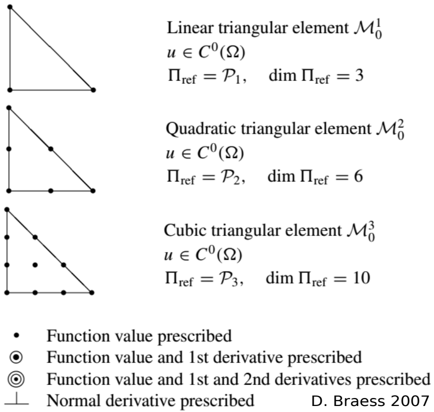

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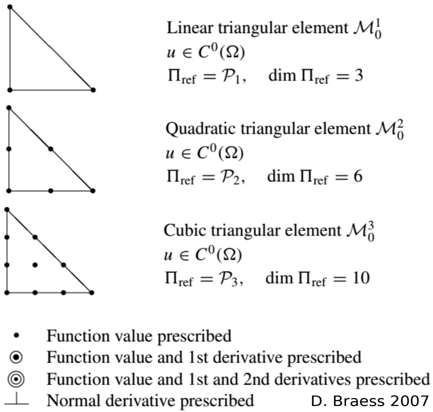

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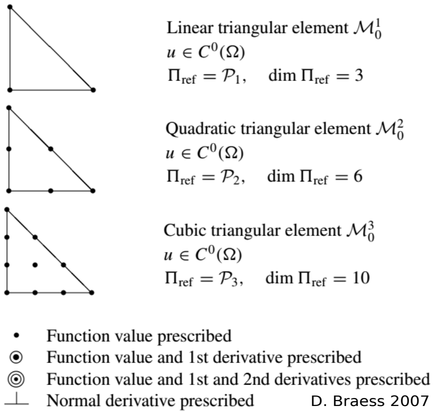

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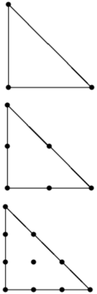

Linear triangular element $\mathcal{M}_0^1$
$u \in C^0(\Omega)$
$\Pi_{\text{ref}} = \mathcal{P}_1, \quad \dim \Pi_{\text{ref}} = 3$

Quadratic triangular element $\mathcal{M}_0^2$
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• Function value prescribed
⊙ Function value and 1st derivative prescribed
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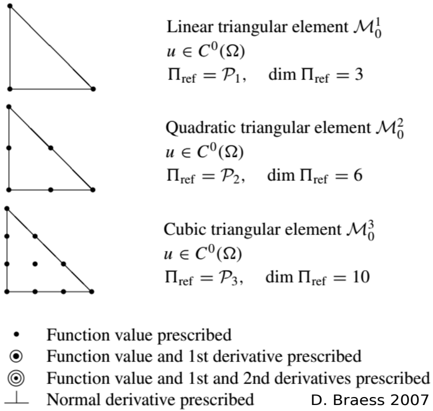

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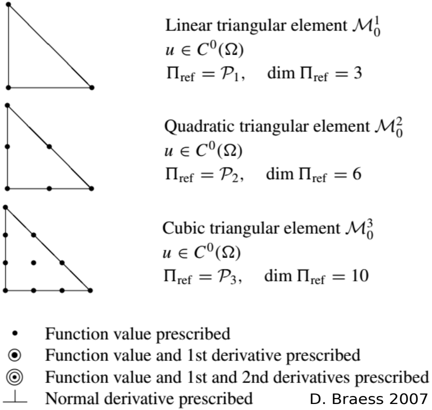

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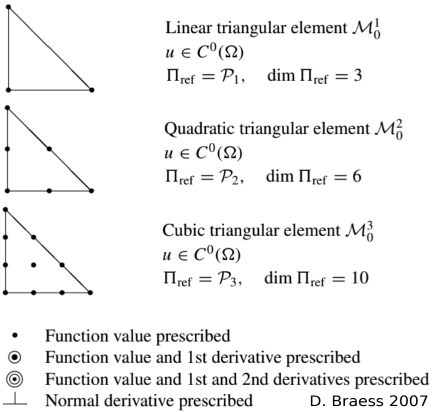

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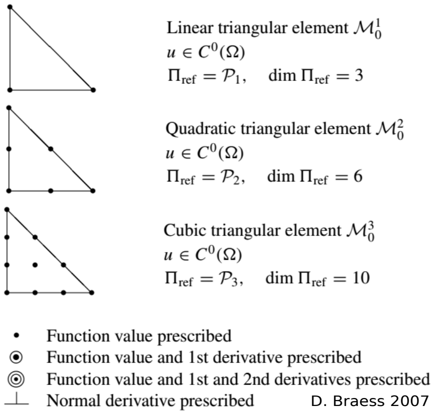

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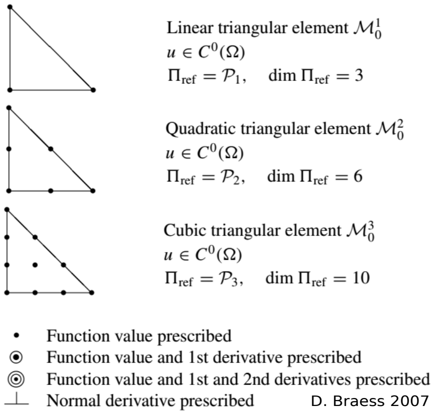

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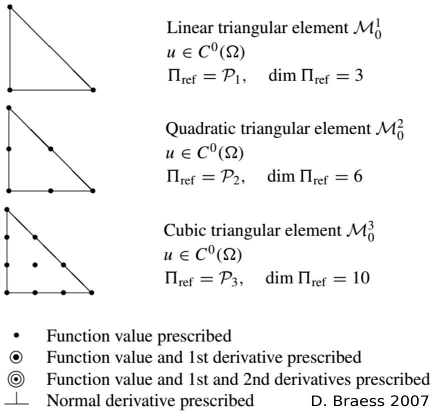

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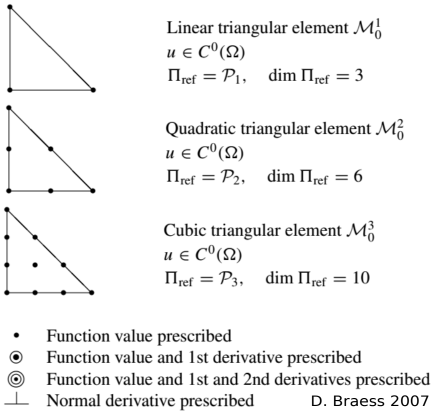

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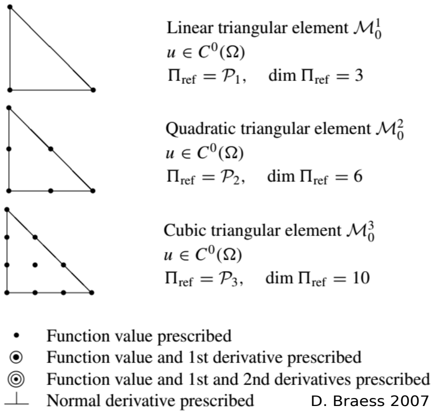
Proof: We proceed by induction. Clearly, in the case of $t = 0$ when $s = 1$ we have interpolation by the constant polynomials.

Now if the interpolation for $t - 1$ holds, we prove it holds for t. Let p_1 be the univariate Lagrange polynomial interpolating the $t + 1$ points on the x-axis. Consider the sub-triangle neglecting the points on the x-axis. Let p_2 be the interpolating polynomial for these points with $p_2(z_i) = (f(z_i) - p_1(z_i))/y_i$, $1 \leq i \leq s - (t+1)$. The polynomial $q(x, y) = p_1(x) + yp_2(x, y)$ interpolates all points.



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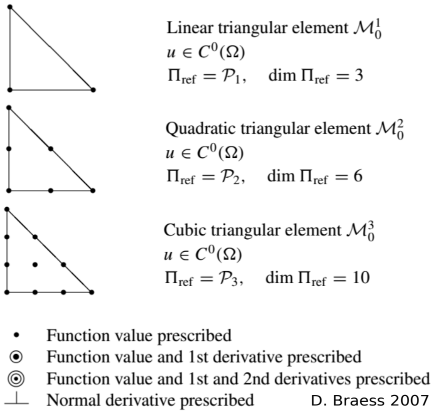
Triangular Finite Elements: Lagrange Elements

Consider partition of the domain into triangular elements $\mathcal{T}$.



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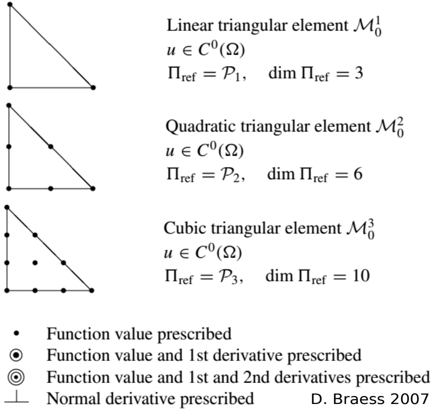

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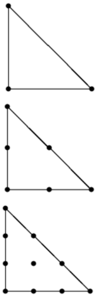

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D. Braess 2007

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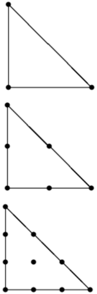

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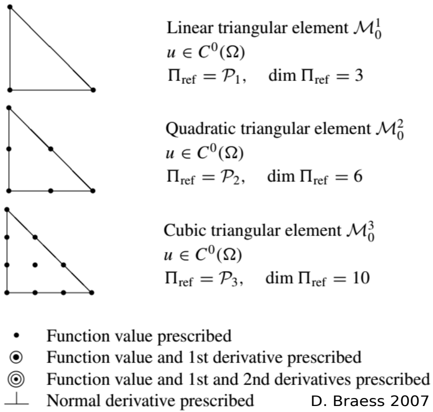

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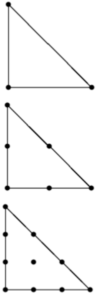

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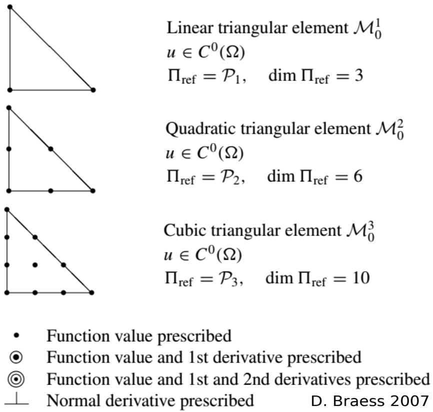

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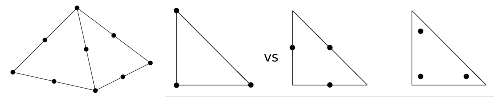

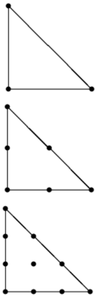

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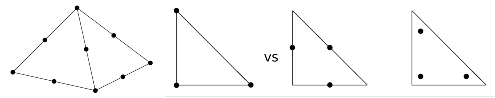
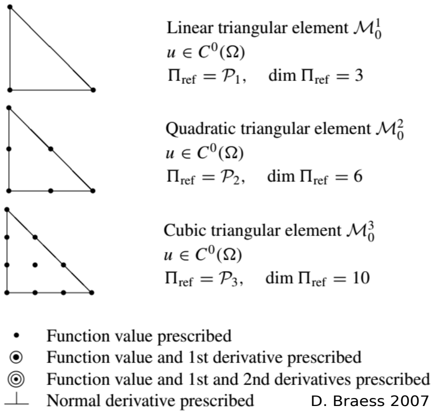


Note: Shared common nodes at vertices ensures the continuity.



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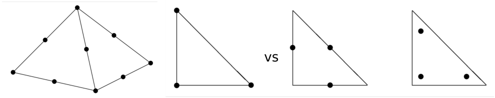

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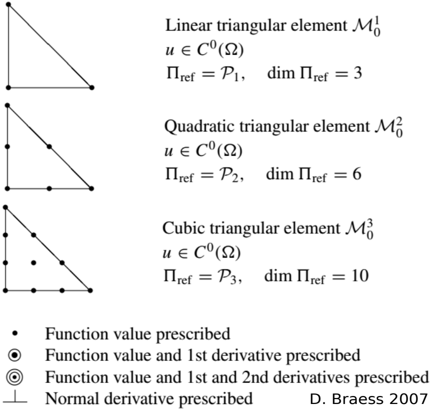
$\mathcal{M}^k_0$ is called the **conforming** P_k **element**.



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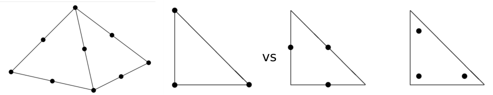

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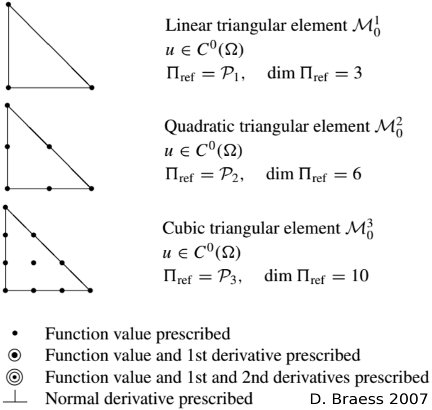

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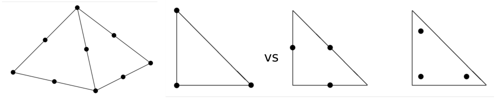

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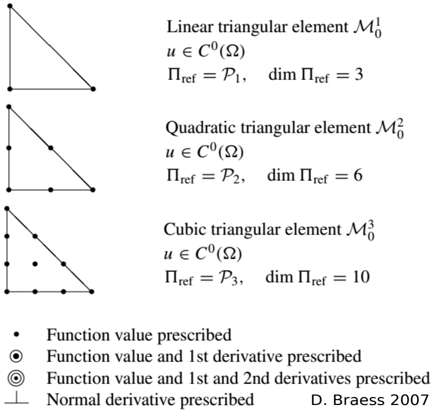
Nodal variables are $N_j(u) = u(z_j)$, so also called **Lagrange elements**.



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Triangular Finite Elements: C^1 Regularity

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More challenging to obtain elements with C^1 regularity.

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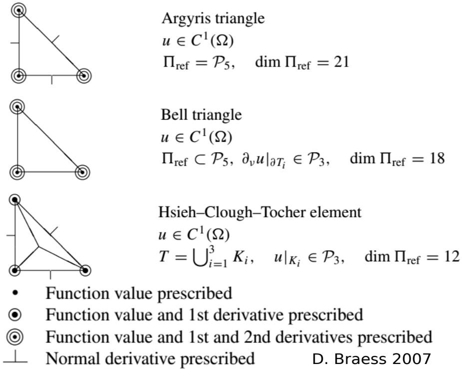

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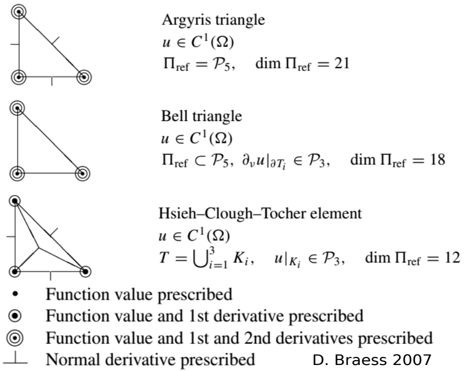
Argyris element:



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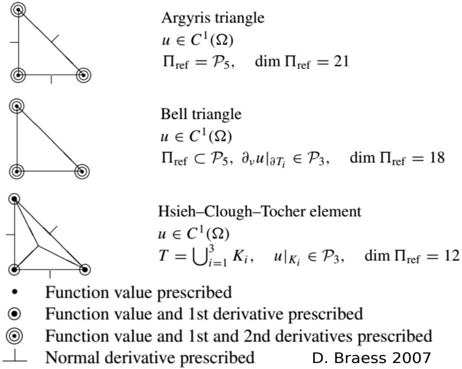
Uses $\mathcal{P}_5$ which has $\dim \mathcal{P}_5 = 21$.



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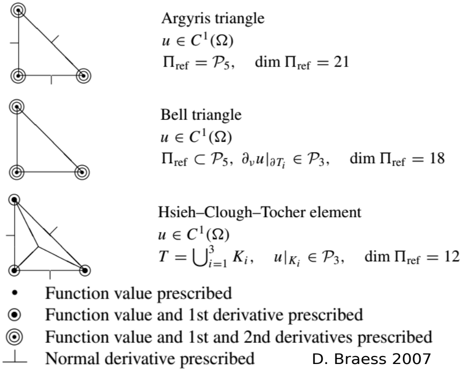

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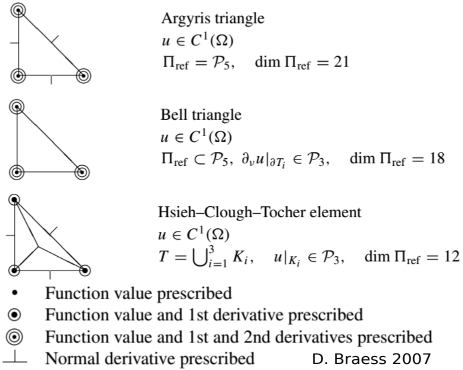

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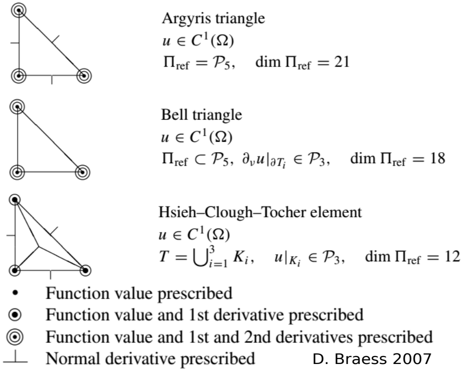

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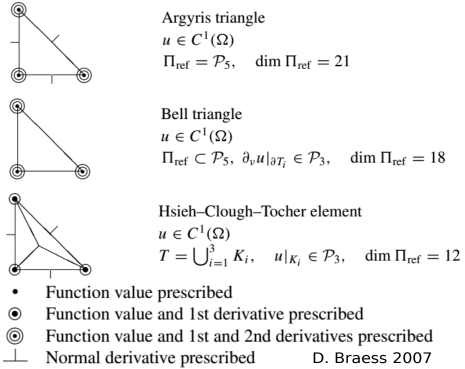

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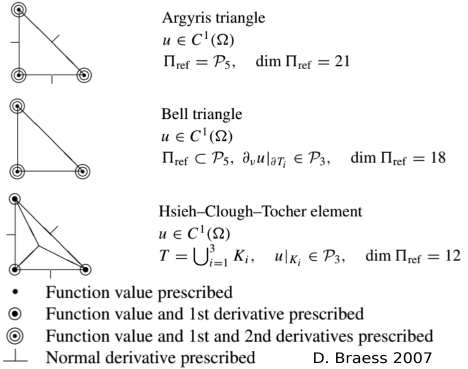

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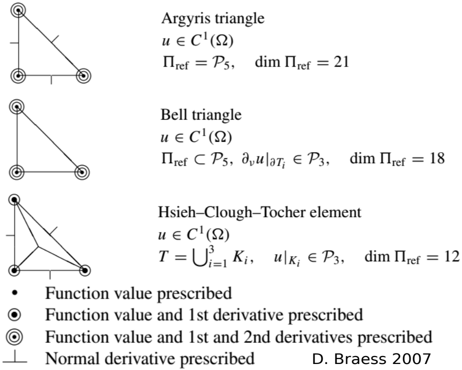

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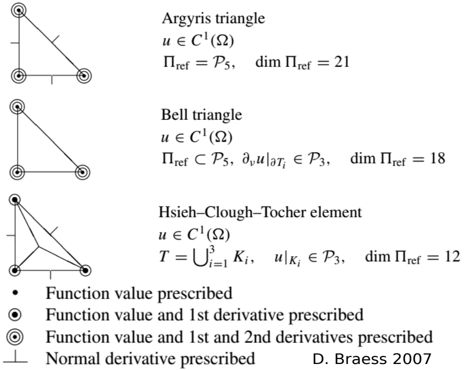

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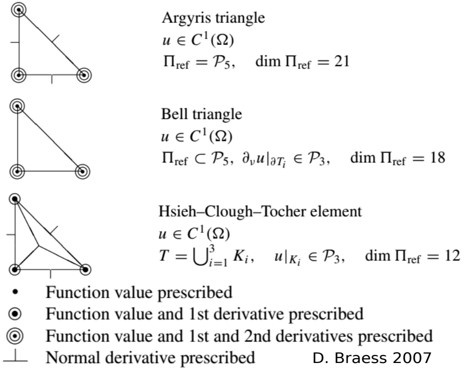

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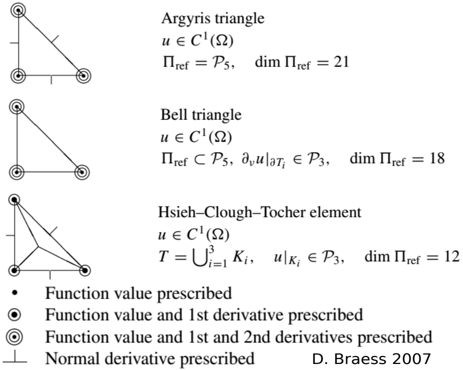
Hsieh-Clough-Tocher element: Macroelement approach.



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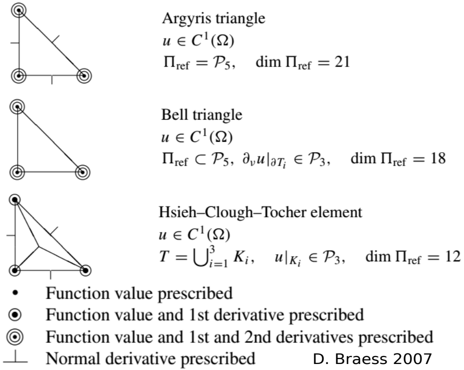

Triangular Finite Elements: C^1 Regularity

More challenging to obtain elements with C^1 regularity.

Argyris element:
Uses $\mathcal{P}_5$ which has $\dim \mathcal{P}_5 = 21$.
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However, this is only $3 \times 6 = 18$ DOF.
Determine 3 DOF from normal derivative at edge centers.

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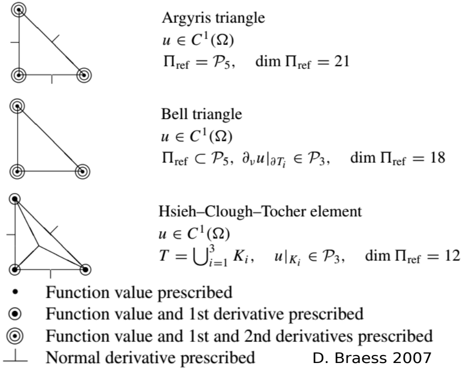

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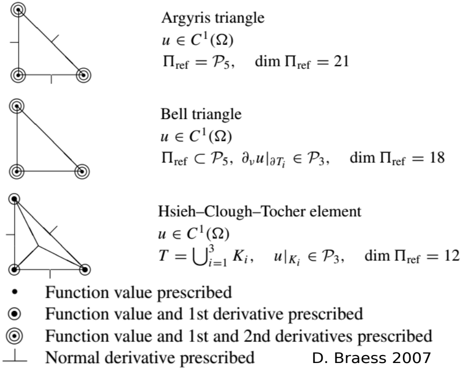

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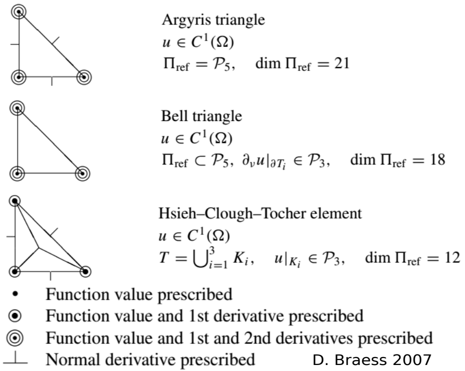

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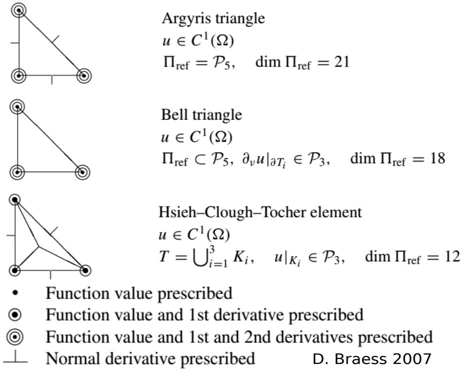

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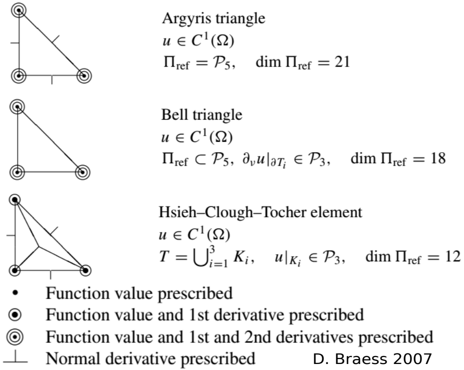

Bernstein-Bézier representation of polynomials used to handle derivatives along element boundaries.

Quadrilateral Finite Elements

Tensor Product Bases

Quadrilateral Finite Elements

Tensor Product Bases

A **tensor-product basis** generated by $\{\phi_k\}_{k=1}^{t}$ for $\mathbf{x} \in \mathbb{R}^n$

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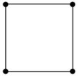

Bilinear quadrilateral element Q_1
$u \in C^0(\Omega)$
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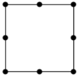

Serendipity element
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Quadrilateral Finite Elements

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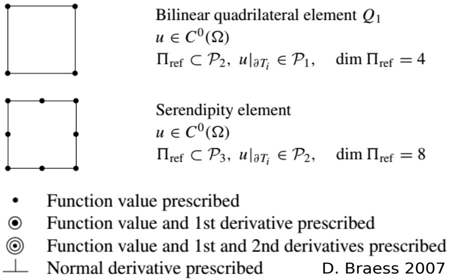

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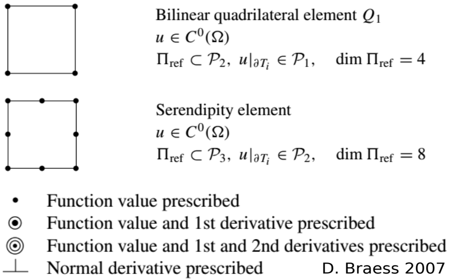

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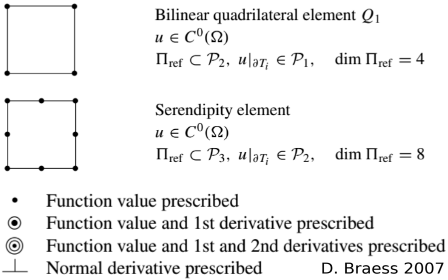

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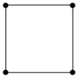
Quadrilateral Finite Elements



Bilinear quadrilateral element Q_1
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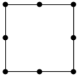

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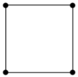

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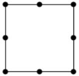

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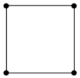

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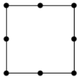

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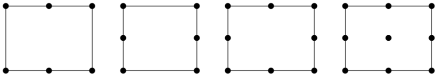

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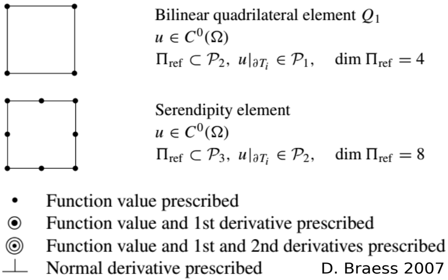

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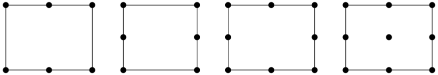

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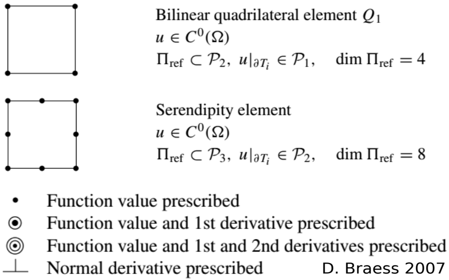
\end{aligned}
$$



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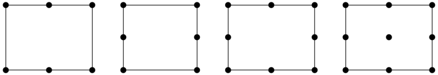

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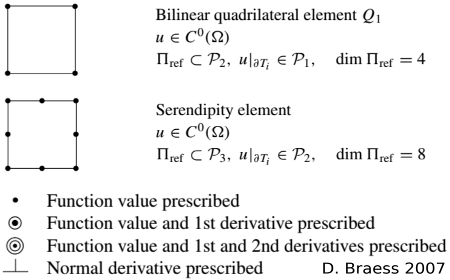
Nodal locations are vertices of rectangle and edge mid-points.



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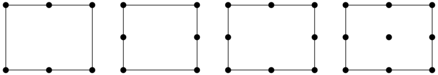

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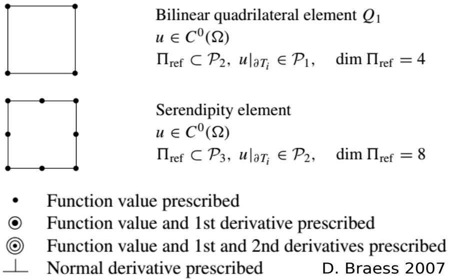
Nodal locations are vertices of rectangle and edge mid-points.

9-**Point Element:**



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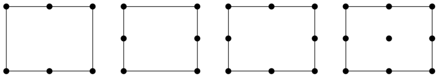

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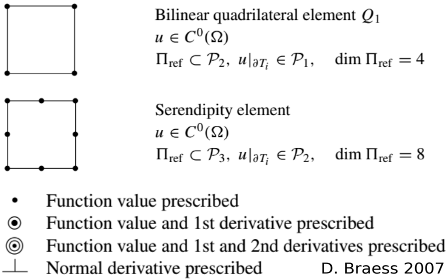

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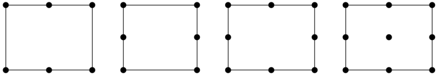

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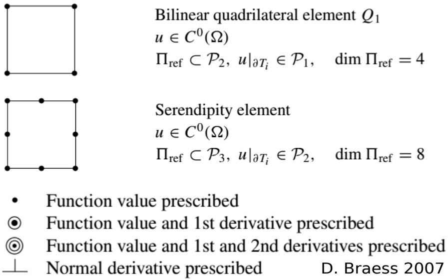

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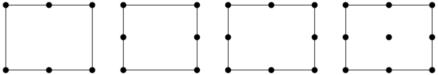

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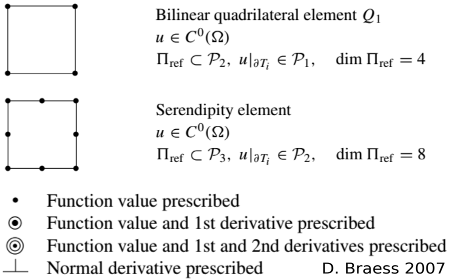

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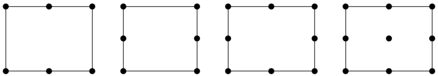

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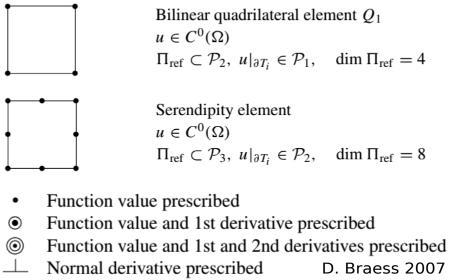

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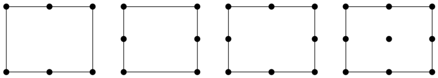

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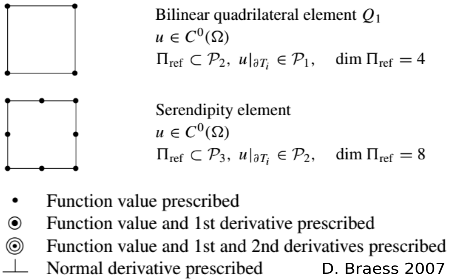

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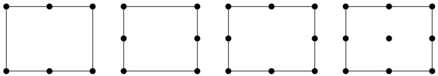

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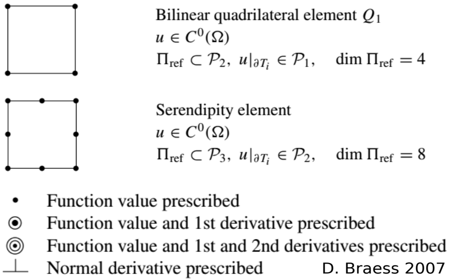

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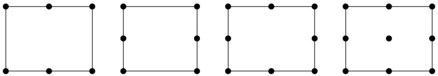

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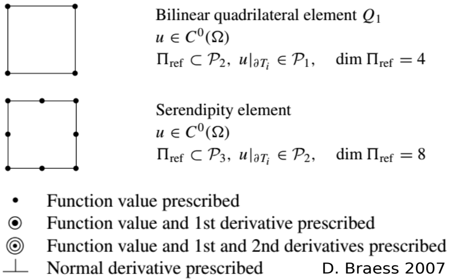

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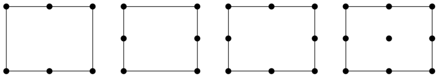

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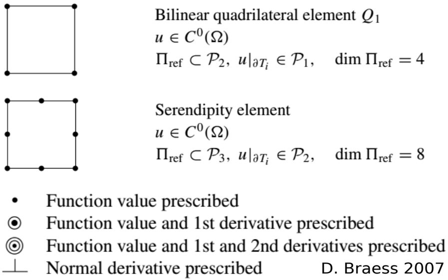

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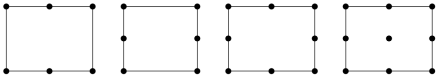

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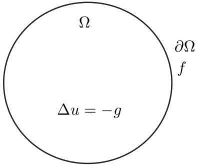

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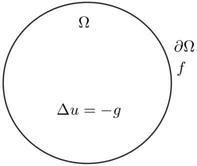

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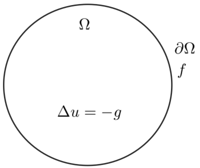

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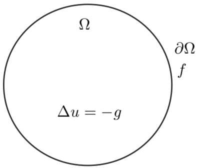

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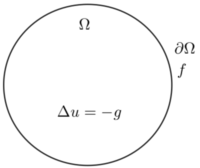

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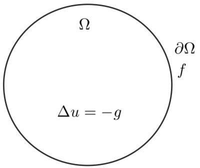

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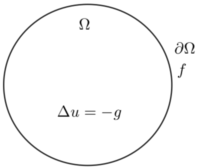

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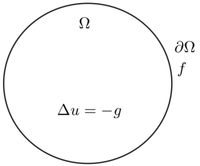

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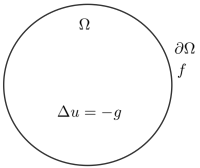

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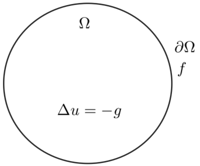

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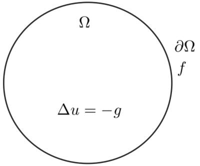

Discretization:

Application to Elliptic PDEs

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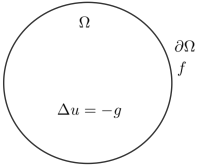
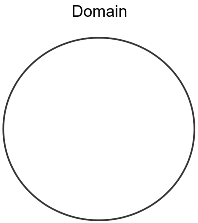


Discretization:

Divide domain into triangular elements T_j.



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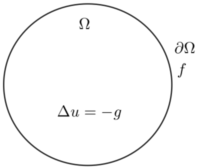

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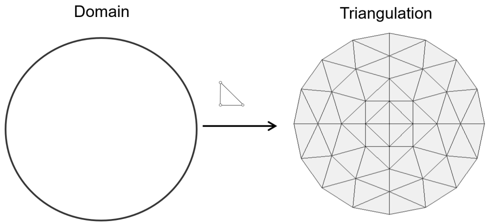

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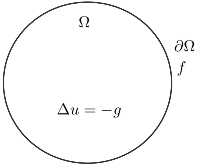

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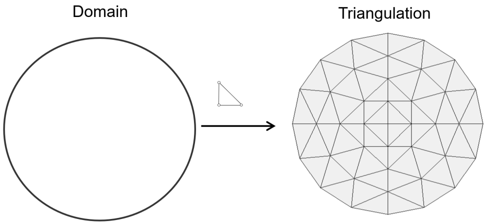
Divide domain into triangular elements T_j.

Denote triangle vertices as $\mathbf{x}_i$.



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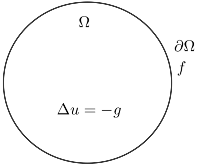

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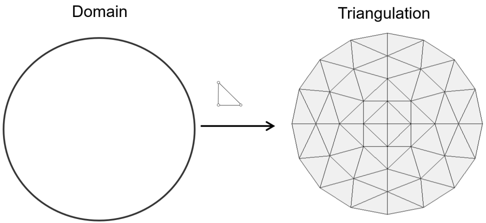
Use for shape space $\mathcal{P}_1$.



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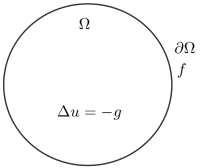

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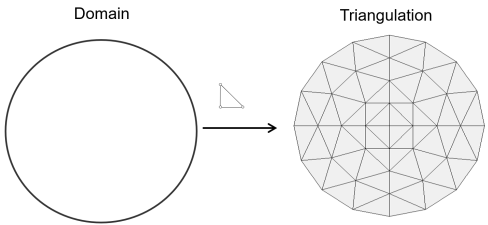

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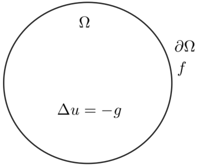

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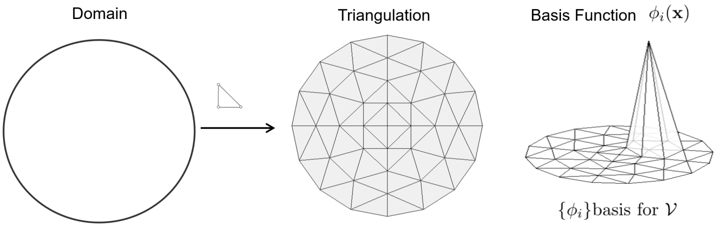

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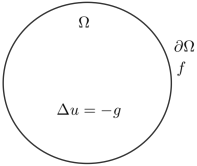

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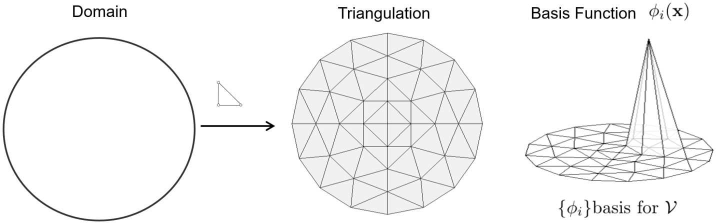
Nodal basis $\{\phi_i\}$ are 2D "hat functions."



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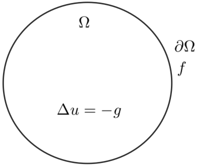

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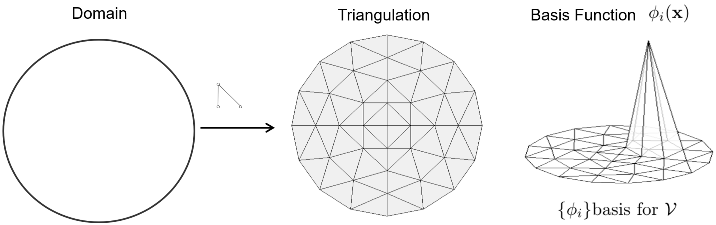
Functions in $v \in \mathcal{S}$ can be represented as

$$v(x) = \sum_{i=1}^{n} v(\mathbf{x}_i)\phi_i(\mathbf{x}) \in H^1.$$



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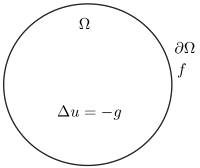

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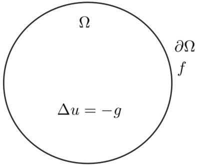

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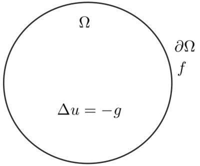

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Can increase accuracy by refining the mesh.

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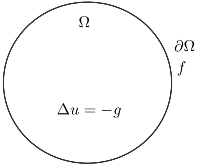

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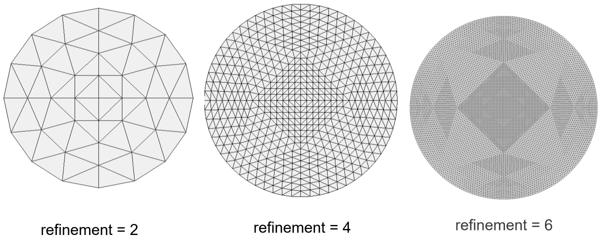

Application to Elliptic PDEs

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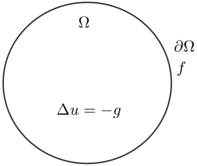

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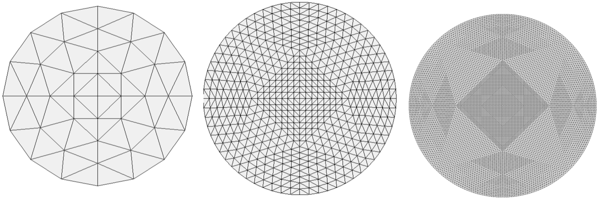

refinement = 2 refinement = 4 refinement = 6

Application to Elliptic PDEs

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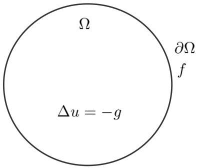

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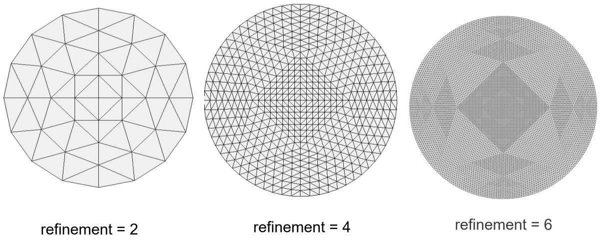

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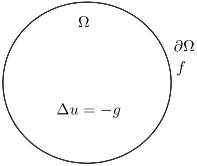

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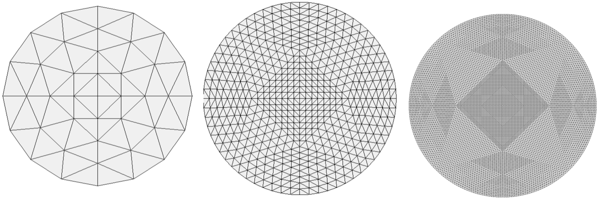

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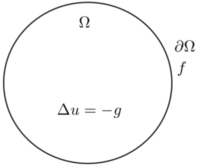

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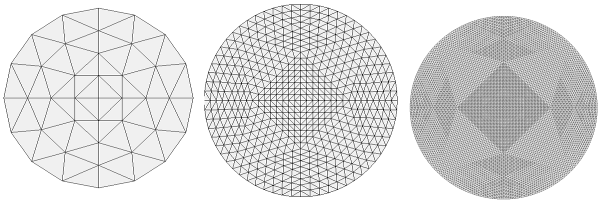

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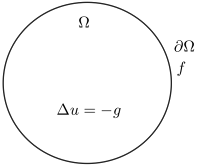

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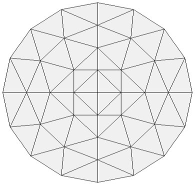
Quality of the triangle shapes is important.

Quality impacts condition number of the stiffness matrix K.



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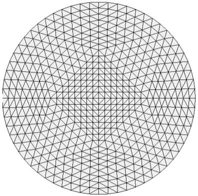
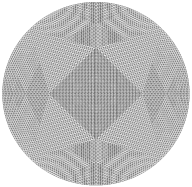


refinement = 4



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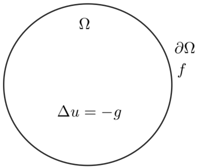

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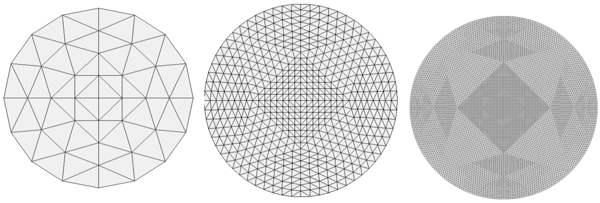
Convergence expected sufficiently uniform refinements.



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Application to Elliptic PDEs

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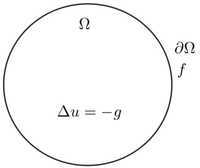

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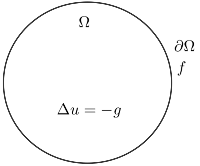


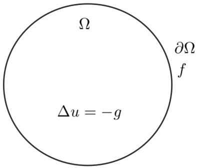
Example:

Application to Elliptic PDEs

Poisson Equation as Model Problem :

Consider Poisson equation with Dirichlet boundary conditions and Ritz-Galerkin (RG) Approximation:

$$\left\{ \begin{array}{ll} \Delta u = -g, & x \in \Omega \\ u = f, & x \in \partial\Omega. \end{array} \right\} \rightarrow \left\{ \begin{array}{l} a(u,v) = -(g,v), \; v \in \mathcal{S} \\ a(u,v) = \int_\Omega \nabla u \cdot \nabla v d\mathbf{x}. \end{array} \right\} \text{(RG-Approximation)}$$



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Consider PDE with

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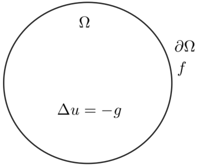

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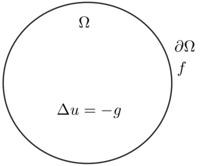

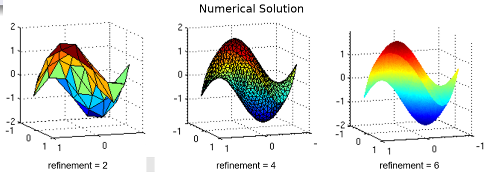

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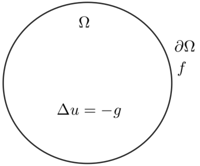

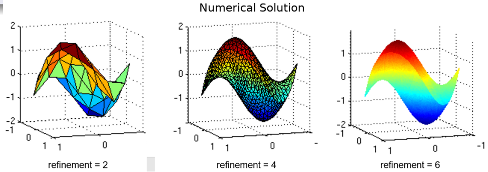

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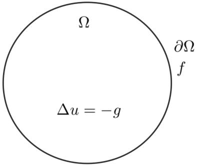

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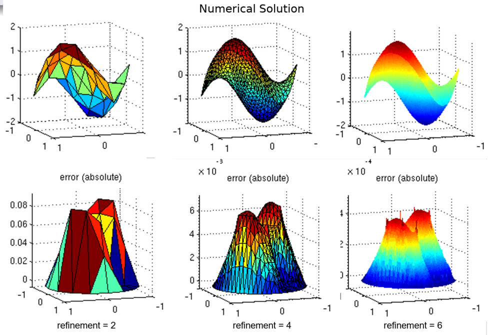

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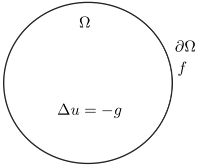

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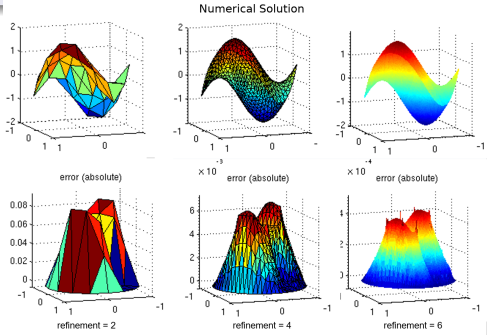

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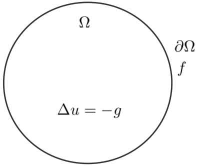

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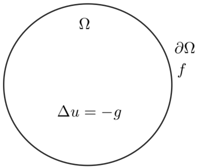

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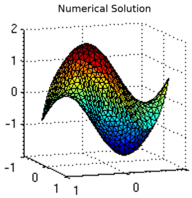

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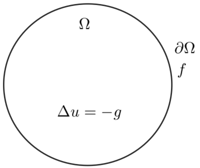

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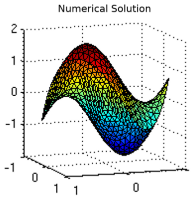

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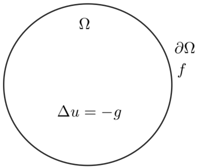

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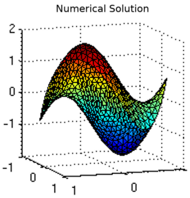

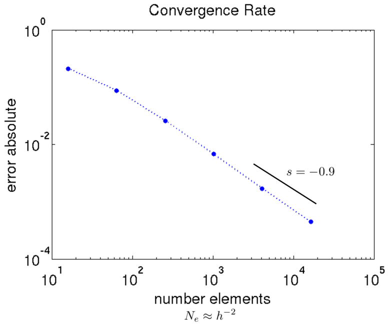

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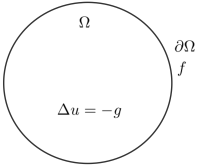

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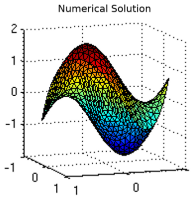
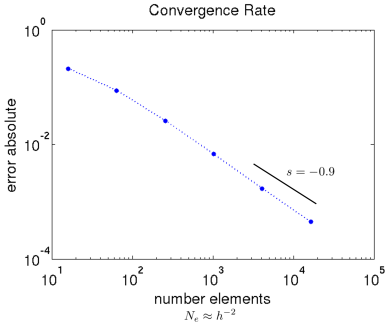
Log-log plots yield information on convergence rate





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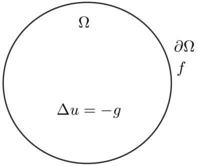

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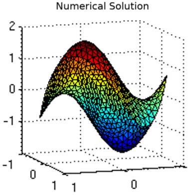
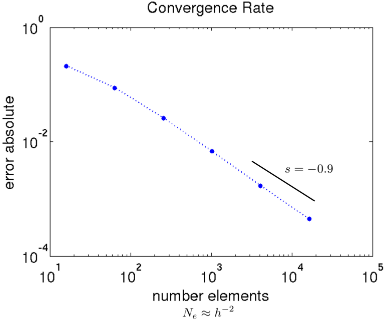
Log-log plots yield information on convergence rate $\epsilon = Ch^r$





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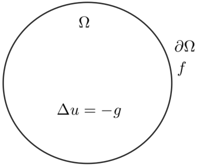

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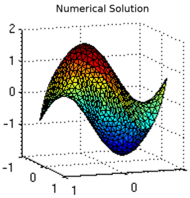
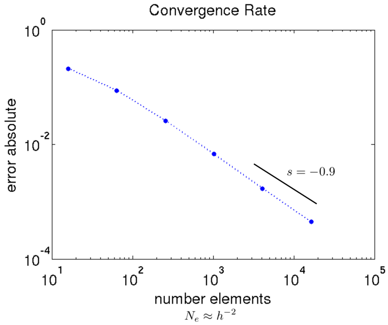
Log-log plots yield information on convergence rate $\epsilon = Ch^r \rightarrow \log(\epsilon) = \log(h)r + \log(C)$





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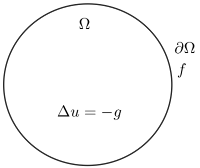

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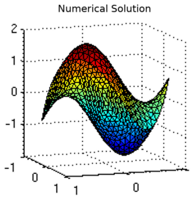
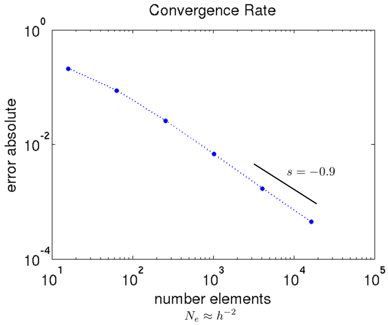
Log-log plots yield information on convergence rate

$\epsilon = Ch^r \rightarrow \log(\epsilon) = \log(h)r + \log(C) \Rightarrow \; -r/2 = s \sim -0.9$





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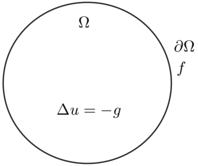

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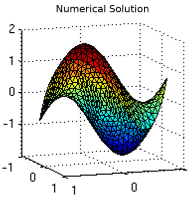
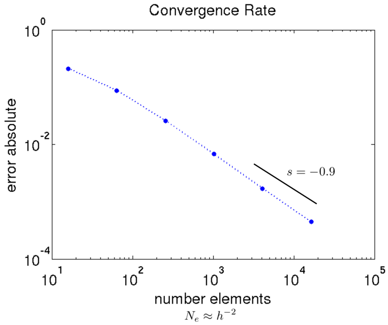
Log-log plots yield information on convergence rate
$\epsilon = Ch^r \rightarrow \log(\epsilon) = \log(h)r + \log(C) \Rightarrow -r/2 = s \sim -0.9 \rightarrow r \sim 1.8.$





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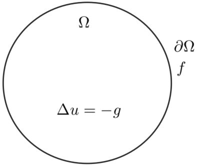

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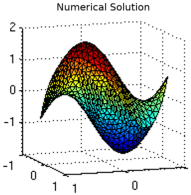
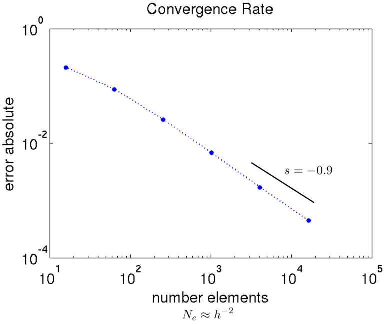
Indicates 2^{nd}-order convergence rate.

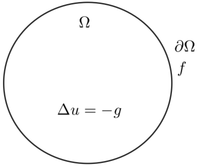




Poisson Equation as Model Problem :

Consider Poisson equation with Dirichlet boundary conditions and Ritz-Galerkin (RG) Approximation:

$$\left\{ \begin{array}{ll} \Delta u = -g, & x \in \Omega \\ u = f, & x \in \partial\Omega. \end{array} \right\} \to \left\{ \begin{array}{l} a(u,v) = -(g,v), \;\; v \in \mathcal{S} \\ a(u,v) = \int_\Omega \nabla u \cdot \nabla v d\mathbf{x}. \end{array} \right\} \text{(RG-Approximation)}$$



Example:

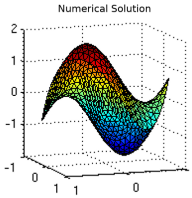
Consider PDE with

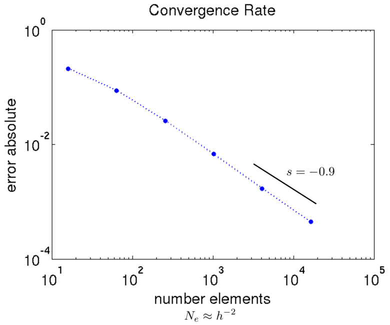
$$g(x,y) = \pi^2 \sin(\pi x) + \pi^2 \cos(\pi x)$$
$$f(x,y) = \sin(\pi x) + \cos(\pi x).$$

Solution is

$$u(x,y) = \sin(\pi x) + \cos(\pi x).$$

Study the error vs mesh refinement $N \sim h^{-2}$.





Log-log plots yield information on convergence rate

$\epsilon = Ch^r \to \log(\epsilon) = \log(h)r + \log(C) \Rightarrow \; -r/2 = s \sim -0.9 \to r \sim 1.8.$

Indicates 2^{nd}-order convergence rate.