Finite Element Methods: Supplemental Exercises

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1. (Regularity of Elements) Consider the following elements:

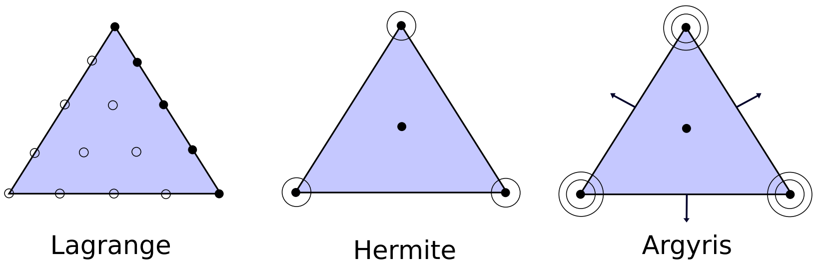

Figure 1:

Show that each of the elements have the stated regularity as follows:

(a) Lagrange triangular element based on $\mathcal{P}_k$ with $k+1$ distinct nodes along each edge is C^0.

(b) Hermite triangular element based on $\mathcal{P}_3$ is C^0.

(c) Argyris triangular element based on $\mathcal{P}_5$ is C^1 in the normal direction across edges.

2. (1D Finite Element Methods) Consider the differential equation

$$\begin{aligned} u''(x) &= -f(x), & x \in [0, L], \\ u(0) &= u'(L) = 0, & x \text{ on boundary}. \end{aligned}$$

(a) Formulate the variational problem associated with this differential equation.

(b) For the linear Lagrange elements and quadradic Lagrange elements state precisely the elements $(\mathcal{K}, \mathcal{P}, \mathcal{N})$ and the Ritz-Galerkin approximation.

(c) Implement the Linear Lagrange finite element method and the Quadratic Lagrange finite element method. Do this for an arbitrary distribution of distinct nodes. Compute numerically the *stiffness matrix* and *mass matrix* for the differential equation and boundary conditions above. State for each method the matrices when $n = 4$ for a uniform node distribution.

(d) Consider the case when $f(x) = -\sin(5\pi x/2L)$. Derive the exact solution $u^*(x)$ for the differential equation.

(e) For uniformly spaced nodes in the case $f(x) = -\sin(5\pi x/2L)$, compute as $h \to 0$ for each finite element method the following measures of error (i) $\max_i |u_S(x_i) - u^*(x_i)|$, (ii) $\|u_S - u^*\|_{L^2}$, (iii) $\|u_S - u^*\|_E$, and (iv) $\|u_S - u^*\|_{W_2^1}$. The u^* is the exact solution, u_S is the Ritz-Galerkin approximation. Make a log-log plot of the error vs h^{-1} for each measure of error above.

(f) When the log-log plot is approximately linear the error satisfies $\log(\text{error}) \sim -\alpha \log(h^{-1}) + \tilde{c}$ and scales like error $\sim h^{\alpha}$. By fitting a line to the log-log plots as $h \to 0$ estimate the slope α. State the rates of convergence with respect to each measure of error above for each finite element method.