Exercises

ODEs and Dynamical Systems — Paul J. Atzberger
MATH 243A — http://atzberger.org/

1. Consider the linear dynamical system $\dot{x} = Ax$ when

$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \beta \end{bmatrix}.$$

   (a) For what values of α, β does the matrix A have repeated roots.

   (b) For what values of α, β does this system diverges with growth only as a polynomial in time t, that is $\|x(t)\| < Cp(t)$ for some polynomial $p(t)$ and $\|x(t)\| \to \infty$. What is the smallest degree that can be used for p?

   (c) Consider if instead $A \in \mathbb{R}^{n \times n}$ with

$$A = \begin{bmatrix} \alpha & 1 & 0 & \ldots & 0 \\ 0 & \alpha & 1 & \ldots & \vdots \\ \vdots & & \ddots & \ddots & \\ 0 & & \ldots & \alpha & 1 \\ 0 & & \ldots & & \alpha \end{bmatrix}.$$

   For what α does the system diverge only with polynomial growth? What is the smallest degree that can be used for p?

   (d) For $\alpha = -\epsilon$, show that for the A in part (c) that for any $\epsilon > 0$ we have $\lim_{t\to\infty} x(t) = 0$.

   (e) For $\alpha = i\epsilon$ with $i = \sqrt{-1}$, show that for the A in part (c) that $\lim_{t\to\infty} x(t) = \infty$ and diverges with polynomial growth for almost all initial conditions except a set of measure zero.

2. Consider the following proposed differential equation $\dot{x} = f(x)$ with $f(x) = \alpha x$ for $x \leq \gamma$ and $f(x) = \beta x$ for $x > \gamma$ with $x(0) = x_0$.

   (a) Show that for any $\alpha, \beta > 0, \gamma = 1$, and $\alpha \neq \beta$ that the differential equation has no solution $x(t) \in C^1$ for initial conditions in an open interval containing $x = 1$.

   (b) If $\alpha = 0, \beta < 0, \gamma = 1$ show there are solutions that exist for all $t \geq 0$. What conditions must be satisfied by x_0?

   (c) If $\alpha > 0, \beta < 0$ and $\gamma = 1$ for a given $x_0 > 1$ over what time interval does a solution exist? Be sure to include both forward and backward in time part of the interval.

   (d) Show if $\gamma = 0$ then solutions exist and are unique for all time for any choice of α, β and initial conditions x_0.

3. Consider the following mechanical system with dynamics based on Newton's second law $m\ddot{x} = f(x)$ and potential energy $\phi(x) = \frac{k}{4}(1 - x^2)^2$.

(a) Give an equivalent differential equation that is first order with $\dot{z} = g(z)$ where $z = (q, p) = (x, mv)$ with $v = \dot{x}$.

(b) When the force is $f(x) = -\nabla\phi(x)$ show for any initial condition $x(0) = x_0$ and $v(0) = v_0$ there exists a unique solution for all time $0 < t < \infty$.

(c) Show the total energy $E(x) = \frac{m}{2}v^2 + \phi(x)$ is conserved under the dynamics, $\partial E(x(t))/\partial t = 0$.

(d) Show that each of the solutions $x(t)$ remain bounded with $\|x(t)\| < C$ for all $t > 0$. The constant here can depend on (x_0, v_0).

(e) When the system has friction $-\gamma v$ the force is modified to $f(x) = -\gamma v - \nabla\phi(x)$. Show that all solutions either converge to $x^* = -1$ or $x^* = +1$, provided $k > 0$. Hint: Consider the behavior of $\partial E(x(t))/\partial t$.

4. Consider a general differential equation $\dot{x} = f(x)$ with $x(0) = x_0$ where f is locally Lipschitz. Suppose there is a perturbation of the differential equation which gives $\dot{z} = f(z) + \delta(t)$ with $z(0) = x_0 + \delta(0)$.

(a) Show that if $|\delta(t)| \leq \delta$ that for any finite time interval $t \in [0, T]$ the difference between the solutions is at most $|z(t) - x(t)| \leq C(T)\delta \exp(TL)$ where L is the Lipschitz constant of f.

(b) Show this bound can be further refined to

$$|z(t) - x(t)| \leq \delta(0) \exp(tL) + \frac{\delta}{L}\left(\exp(tL) - 1\right).$$

Hint: Generalize Gronwall's Inequality by including in the proof the source term $t\delta$.