Exercises

ODEs and Dynamical Systems — Paul J. Atzberger
MATH 243A — http://atzberger.org/

1. Consider the dynamical system $\dot{x} = f(x)$ with equilibrium point x_0 so $f(x_0) = 0$.
   (a) Derive an approximating differential equation for the local behaviors, $\tilde{x}(t) = x(t) - x_0$, by truncating the Taylor expansion at third order to obtain the form $\dot{\tilde{x}} = a\tilde{x} + b\tilde{x}^2 + c\tilde{x}^3$.
   (b) When $f'(x_0) \neq 0$ and $f'(x_0) < 0$ what can you say about the stability?
   (c) Suppose that $f'(x_0) = 0$ and $f''(x_0) \neq 0$ what can you now say about the stability?
   (d) Suppose that $f'(x_0) = 0$, $f''(x_0) = 0$, and $f'''(x_0) \neq 0$ with $f'''(x_0) < 0$, what can you now say about stability?

2. Consider the forced harmonic oscillator with dynamics $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{h}(t)$ where

$$A = \begin{bmatrix} 0 & 1/m \\ -k & -\gamma/m \end{bmatrix}.$$

   The $m > 0$ is the mass, $k > 0$ the spring stiffness, and $\gamma \geq 0$ the friction coefficient. The $\mathbf{x} = (q, p)$ with q the configuration and $p = mv$ the momentum. The forcing $\mathbf{h} = [0, f(t)]$.
   (a) Show the total energy is preserved in the case that $\gamma = 0$ and $f = 0$, where $E(q,p) = \frac{1}{2m}p^2 + \frac{1}{2}kq^2$.
   (b) Show when $f \neq 0$ that the total energy changes by $dE(q,p)/dt = -\gamma v^2 + vf(t)$ which is the rate work is done on the system.
   (c) Derive the general solution for this linear system when $f = 0$.

3. For the forced harmonic oscillator consider the following.
   (a) For fixed m, k for what values of γ does A have purely real eigenvalues (overdamped regime)? For what values of γ does A have complex eigenvalues (underdamped regime)?
   (b) Concerning the stability, answer the following when $f = 0$.
      i. What type of stability occurs for $\mathbf{x}_0 = (0,0)$ when $\gamma = 0$?
      ii. What type of stability occurs for $\mathbf{x}_0 = (0,0)$ when $\gamma > 0$?
      iii. Do the different regimes of $\gamma > 0$ make a difference?
   (c) Let $f(t) = \cos(\alpha t)$, show there is a value for α so that the system has a periodic solution. State if there are restrictions on γ.
   (d) Bonus (optional): At what frequency α is the amplitude of the oscillation maximum?

4. Consider the dynamical system $\dot{x} = \beta x + \cos(\alpha t)$.
   (a) Derive the general solution for this differential equation.
   Hint: Can use integrating factor and integration by parts twice.
   (b) When $\alpha = 1$ and $\beta = 1$, show there is a unique periodic solution.

(c) For $\alpha = 1$ and $\beta = 1$, derive the Poincare' map $g(x) = \phi_T(x)$ using that the period is $T = 2\pi\alpha^{-1}$.

(d) Show there is only one fixed point for the Poincare' map.

(e) Perform analysis of the stability of the periodic solution using the Poincare' map.

(f) For $\alpha = 1$ and $\beta = -1$, derive the Poincare' map $g(x) = \phi_T(x)$ using that the period is $T = 2\pi\alpha^{-1}$.

(g) Perform analysis of the stability of the periodic solution using the Poincare' map.