Exercises

ODEs and Dynamical Systems
MATH 243A

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1. Prove that for any $A \in \mathbb{C}^{n \times n}$ with $n < \infty$ that the matrix exponential $S(t) = \exp(tA) = \sum_{k=0}^{\infty} \frac{1}{k!}(tA)^k$ has series that converges to a well-defined operator $S(t)$ for any choice of A and $t < \infty$.

2. Draw the phase-portrait for the ODE given by
$$\dot{x} = Ax, \quad A = \begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix}.$$
Show the solution when $x(0) = [1, 0]^T$.

3. Compute the matrix exponentials for the following
$$\text{(a)}\ A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}, \quad \text{(b)}\ A = \begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix}, \quad \text{(c)}\ A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \text{(d)}\ A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 3 \end{bmatrix}.$$

4. If the matrix A is diagonalizable show that
$$\det(\exp(A)) = \exp(\mathrm{tr}(A)),$$
where $\det(\cdot)$ denotes the determinant and $\mathrm{tr}(\cdot)$ denotes the trace.

5. Let $\phi(x_0; t)$ be the solution map for $\dot{x} = Ax$ with $x(0) = x_0$. Show that for any t that
$$\lim_{x_0' \to x_0} \phi(x_0'; t) = \phi(x_0; t).$$
This shows continuity for finite dimensional linear systems in their dependence on the initial conditions.

6. Put the following matrices into the Jordan canonical form
$$\text{(a)}\ A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad \text{(b)}\ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \text{(c)}\ A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}, \quad \text{(d)}\ A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

7. Give the solution for $x_0 = \mathbf{e}_1$ for $\dot{x} = Ax$ for each of the matrices in problem 6. Here, the $\mathbf{e}_1$ denotes the standard basis element with components $[\mathbf{e}_1]_i = \delta_{1i}$.

8. State the stable, unstable, and center spaces for the dynamics given by $\dot{x} = Ax$ for the matrices in problem $3ac$ and $6ad$.