Exercises

ODEs and Dynamical Systems — Paul J. Atzberger
MATH 243A — http://atzberger.org/

1. Consider the ODE

$$\begin{aligned}\dot{x} &= -2x - y^3 \\ \dot{y} &= -y - x^3\end{aligned}$$

   (a) For the equilibrium $\bar{\mathbf{x}} = (0,0)$ give the linearized system based on $Df(\bar{\mathbf{x}})$.

   (b) Show the equilibrium is a sink.

   (c) Show if $\bar{\mathbf{x}}$ is a sink, show there can not be any other equilibria in a neighborhood. In other words, show there is a disk with $\delta > 0$ containing $\bar{\mathbf{x}}$ and no other equilibria in the disk.

2. Consider the ODE

$$\begin{aligned}\dot{\theta} &= 1 \\ \dot{r} &= \begin{cases} r^3 \sin(1/r), & r > 0 \\ 0, & r = 0. \end{cases}\end{aligned}$$

   This gives dynamics in $\mathbb{R}^2$ given by $\mathbf{x}(t) = [r(t)\cos(\theta(t)), r(t)\sin(\theta(t))]$.

   (a) Show this system has a stable equilibrium for $\bar{\mathbf{x}} = (0,0)$.

   (b) Show while the equilibrium is stable it is not asymptotically stable.

   (c) Show that for any disk with radius $\delta > 0$ containing $\bar{\mathbf{x}}$, there is always a closed trajectory within it looping around the equilibrium $\bar{\mathbf{x}}$.

3. Consider the non-linear ODE

$$\dot{\mathbf{x}} = -\alpha\left(x^2 + y^2\right)\mathbf{x} + [-y, x],$$

   where $\mathbf{x} = [x, y]$.

   (a) Show the linearization $Df(\bar{\mathbf{x}})$ does not have eigenvalues with negative real part for any α, where $\bar{\mathbf{x}} = (0,0)$.

   (b) Show for $\alpha \neq 0$ with $\alpha > 0$ the non-linear system still has an asymptotically stable equilibrium at $\bar{\mathbf{x}}$.

   (c) Show for $\alpha = 0$ the equilibrium $\bar{\mathbf{x}} = (0,0)$ is not asymptotically stable, but still stable.