Exercises

ODEs and Dynamical Systems
MATH 243A

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1. Consider the ODE given by

$$\begin{aligned}\dot{x} &= (1 - x^2 - y^2)x - y \\ \dot{y} &= x + (1 - x^2 - y^2)y\end{aligned}$$

   (a) Find the Poincare' Map $g(\mathbf{x})$ for the section given by $\mathcal{S} = \{[x, y] \mid \frac{1}{2} < x < \frac{3}{2}, y = 0\}$. Hint: Express the ODE in polar coordinates.

   (b) Linearize the Poincare' Map $g(\mathbf{x})$ at $\bar{\mathbf{x}} = [1, 0]$ to obtain $Dg(\bar{\mathbf{x}})$.

   (c) Compute the eigenvalues of $Dg(\bar{\mathbf{x}})$.

   (d) Show the orbit $\gamma = \{\mathbf{x}(t) = [\cos(t), \sin(t)]\}$ is asymptotically stable.

2. Consider the gradient flow

$$\dot{\mathbf{x}} = \text{-grad } V(\mathbf{x}),$$

   for the potential $V(\mathbf{x})$ with $\mathbf{x} = (x, y)$. As a qualiative method, draw a sketch by hand of the phase portrait and sample solution trajectories for the following systems.

   (a) $V(x, y) = x^2 - y^2 - x$.
   Hint: Use completing the square.

   (b) $V(x, y) = y \sin(x)$.

   (c) $V(x, y) = x^2(1 - x)^2$.

3. Consider the ODE

$$\begin{aligned}\dot{x} &= -2x - y^3 \\ \dot{y} &= -y - x^3.\end{aligned}$$

   (a) Find a strict Liapunov Function for equilibrium $\bar{\mathbf{x}} = (0, 0)$.

   (b) Show that $\bar{\mathbf{x}}$ is asymptotically stable.