Sobolev Spaces

Paul J. Atzberger

206D: Finite Element Methods
University of California Santa Barbara

Basic Definitions

The $L^2(\Omega)$ for a smooth domain Ω, denotes the space of all functions f that are Lebegue square-integrable $\int_\Omega f^2 dx < \infty$.

Basic Definitions

The $L^2(\Omega)$ for a smooth domain Ω, denotes the space of all functions f that are Lebegue square-integrable $\int_\Omega f^2 dx < \infty$. We define the L^2-**inner-product** as

$$(u, v)_0 = (u, v)_{L^2} = \int_\Omega u(x)v(x)dx.$$

Basic Definitions

The $L^2(\Omega)$ for a smooth domain Ω, denotes the space of all functions f that are Lebegue square-integrable $\int_\Omega f^2 dx < \infty$. We define the L^2-**inner-product** as

$$(u, v)_0 = (u, v)_{L^2} = \int_\Omega u(x)v(x)dx.$$

This has the compatible L^2-**norm**

$$\|u\|_2 = \sqrt{(u, u)_{L^2}}.$$

Basic Definitions

The $L^2(\Omega)$ for a smooth domain Ω, denotes the space of all functions f that are Lebegue square-integrable $\int_\Omega f^2 dx < \infty$. We define the L^2-**inner-product** as

$$(u, v)_0 = (u, v)_{L^2} = \int_\Omega u(x)v(x)dx.$$

This has the compatible L^2-**norm**

$$\|u\|_2 = \sqrt{(u, u)_{L^2}}.$$

Definition:

A function $u \in L^2$ has as its **weak derivative** $v = \mathcal{D}_{\boldsymbol{\alpha}} u = \partial^{\boldsymbol{\alpha}} u$ if

$$(v, w)_{L^2} = (-1)^{|\alpha|} (u, \partial^{\boldsymbol{\alpha}} w)_{L^2}, \ \forall w \in C_0^\infty.$$

Basic Definitions

The $L^2(\Omega)$ for a smooth domain Ω, denotes the space of all functions f that are Lebegue square-integrable $\int_\Omega f^2 dx < \infty$. We define the L^2-**inner-product** as

$$(u, v)_0 = (u, v)_{L^2} = \int_\Omega u(x)v(x)dx.$$

This has the compatible L^2-**norm**

$$\|u\|_2 = \sqrt{(u, u)_{L^2}}.$$

Definition:

A function $u \in L^2$ has as its **weak derivative** $v = \mathcal{D}_{\boldsymbol{\alpha}} u = \partial^{\boldsymbol{\alpha}} u$ if

$$(v, w)_{L^2} = (-1)^{|\alpha|} (u, \partial^{\boldsymbol{\alpha}} w)_{L^2}, \ \forall w \in C_0^\infty.$$

C^∞ is the space of all functions is infinitely continuously differentiable.

Basic Definitions

The $L^2(\Omega)$ for a smooth domain Ω, denotes the space of all functions f that are Lebegue square-integrable $\int_\Omega f^2 dx < \infty$. We define the L^2-**inner-product** as

$$(u, v)_0 = (u, v)_{L^2} = \int_\Omega u(x)v(x)dx.$$

This has the compatible L^2-**norm**

$$\|u\|_2 = \sqrt{(u, u)_{L^2}}.$$

Definition:

A function $u \in L^2$ has as its **weak derivative** $v = \mathcal{D}_{\boldsymbol{\alpha}} u = \partial^{\boldsymbol{\alpha}} u$ if

$$(v, w)_{L^2} = (-1)^{|\alpha|} (u, \partial^{\boldsymbol{\alpha}} w)_{L^2}, \ \forall w \in C_0^\infty.$$

C^∞ is the space of all functions is infinitely continuously differentiable.
The $C_0^\infty \subset C^\infty$ are all functions zero outside a compact set.

Sobolev Spaces

For any integer $m \geq 0$, let H^m be the space of all functions that have weak derivatives $\partial^\alpha u$ up to order m, $|\boldsymbol{\alpha}| \leq m$.

Sobolev Spaces

For any integer $m \geq 0$, let H^m be the space of all functions that have weak derivatives $\partial^\alpha u$ up to order m, $|\boldsymbol{\alpha}| \leq m$.
We define an **inner-product on H^m** as

$$(u, v)_m = \sum_{|\alpha| \leq m} (\partial^\alpha u, \partial^\alpha v).$$

Sobolev Spaces

For any integer $m \geq 0$, let H^m be the space of all functions that have weak derivatives $\partial^\alpha u$ up to order m, $|\boldsymbol{\alpha}| \leq m$.
We define an **inner-product on** H^m as

$$(u, v)_m = \sum_{|\alpha| \leq m} (\partial^\alpha u, \partial^\alpha v).$$

We define H^m-**norm** as

$$\|u\|_m = \sqrt{(u, u)_m} = \sqrt{\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^2}^2}.$$

Sobolev Spaces

For any integer $m \geq 0$, let H^m be the space of all functions that have weak derivatives $\partial^\alpha u$ up to order m, $|\boldsymbol{\alpha}| \leq m$.

We define an **inner-product on** H^m as

$$(u, v)_m = \sum_{|\alpha| \leq m} (\partial^\alpha u, \partial^\alpha v).$$

We define H^m-**norm** as

$$\|u\|_m = \sqrt{(u, u)_m} = \sqrt{\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^2}^2}.$$

We define k-**semi-norm** as

$$|u|_k = \sqrt{\sum_{|\alpha| = k} (\partial^\alpha u, \partial^\alpha u)_0} = \sqrt{\sum_{|\alpha| = k} \|\partial^\alpha u\|_{L^2}^2}.$$

Sobolev Spaces

For any integer $m \geq 0$, let H^m be the space of all functions that have weak derivatives $\partial^\alpha u$ up to order m, $|\boldsymbol{\alpha}| \leq m$.

We define an **inner-product on** H^m as

$$(u, v)_m = \sum_{|\alpha| \leq m} (\partial^\alpha u, \partial^\alpha v).$$

We define H^m-**norm** as

$$\|u\|_m = \sqrt{(u, u)_m} = \sqrt{\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^2}^2}.$$

We define k-**semi-norm** as

$$|u|_k = \sqrt{\sum_{|\alpha| = k} (\partial^\alpha u, \partial^\alpha u)_0} = \sqrt{\sum_{|\alpha| = k} \|\partial^\alpha u\|_{L^2}^2}.$$

We refer to H^m with this inner-product as a **Sobolev space**. Also denoted by $W^{m,2}$.

Sobolev Spaces

We can define Sobolev spaces without resorting directly to the notion of weak derivatives.

Sobolev Spaces

We can define Sobolev spaces without resorting directly to the notion of weak derivatives.

Theorem

Let $\Omega \subset \mathbb{R}^n$ be an open set with piecewise smooth boundary. Let $m \geq 0$, then $C^\infty(\Omega) \bigcap H^m(\Omega)$ is dense in $H^m(\Omega)$ under the norm $\|\cdot\|_m$.

Sobolev Spaces

We can define Sobolev spaces without resorting directly to the notion of weak derivatives.

Theorem

Let $\Omega \subset \mathbb{R}^n$ be an open set with piecewise smooth boundary. Let $m \geq 0$, then $C^\infty(\Omega) \bigcap H^m(\Omega)$ is dense in $H^m(\Omega)$ under the norm $\| \cdot \|_m$.

This means that we can view H^m as the natural extension of working with smooth functions $C^\infty(\Omega)$ and inner-product $(\cdot, \cdot)_m$.

Sobolev Spaces

We can define Sobolev spaces without resorting directly to the notion of weak derivatives.

Theorem

Let $\Omega \subset \mathbb{R}^n$ be an open set with piecewise smooth boundary. Let $m \geq 0$, then $C^\infty(\Omega) \bigcap H^m(\Omega)$ is dense in $H^m(\Omega)$ under the norm $\|\cdot\|_m$.

This means that we can view H^m as the natural extension of working with smooth functions $C^\infty(\Omega)$ and inner-product $(\cdot,\cdot)_m$.
The H^m is the completion under $\|\cdot\|_m$.

Sobolev Spaces

We can define Sobolev spaces without resorting directly to the notion of weak derivatives.

Theorem

Let $\Omega \subset \mathbb{R}^n$ be an open set with piecewise smooth boundary. Let $m \geq 0$, then $C^\infty(\Omega) \bigcap H^m(\Omega)$ is dense in $H^m(\Omega)$ under the norm $\|\cdot\|_m$.

This means that we can view H^m as the natural extension of working with smooth functions $C^\infty(\Omega)$ and inner-product $(\cdot,\cdot)_m$.
The H^m is the completion under $\|\cdot\|_m$.

Definition

Denote the completion of $C_0^\infty(\Omega)$ under $\|\cdot\|_m$ by H_0^m.

Sobolev Spaces

We can define Sobolev spaces without resorting directly to the notion of weak derivatives.

Theorem

Let $\Omega \subset \mathbb{R}^n$ be an open set with piecewise smooth boundary. Let $m \geq 0$, then $C^\infty(\Omega) \bigcap H^m(\Omega)$ is dense in $H^m(\Omega)$ under the norm $\|\cdot\|_m$.

This means that we can view H^m as the natural extension of working with smooth functions $C^\infty(\Omega)$ and inner-product $(\cdot,\cdot)_m$.
The H^m is the completion under $\|\cdot\|_m$.

Definition

Denote the completion of $C_0^\infty(\Omega)$ under $\|\cdot\|_m$ by H_0^m.

We have the following relations between the function spaces

$$\begin{array}{ccccccccc} L^2(\Omega) & = & H^0(\Omega) & \supset & H^1(\Omega) & \supset & H^2(\Omega) & \cdots & \supset & H^m(\Omega) \\ & & \parallel & & \cup & & \cup & & & \cup \\ & = & H_0^0(\Omega) & \supset & H_0^1(\Omega) & \supset & H_0^2(\Omega) & \cdots & \supset & H_0^m(\Omega). \end{array}$$

Sobolev Spaces

We can also define function spaces based on $L^p(\Omega)$, C^∞, C_0^∞ similarly using the norm $\|\cdot\|_p$.

Sobolev Spaces

We can also define function spaces based on $L^p(\Omega)$, C^∞, C_0^∞ similarly using the norm $\|\cdot\|_p$.

Definition

The Sobolev space denoted by $W^{m,p}$ (also by W_p^m) is the collection of functions obtained by completing $C^\infty(\Omega) \subset L^p(\Omega)$ under the norm $\|\cdot\|_m$.

Sobolev Spaces

We can also define function spaces based on $L^p(\Omega)$, C^∞, C_0^∞ similarly using the norm $\|\cdot\|_p$.

Definition

The Sobolev space denoted by $W^{m,p}$ (also by W_p^m) is the collection of functions obtained by completing $C^\infty(\Omega) \subset L^p(\Omega)$ under the norm $\|\cdot\|_m$.

Similarly, we obtain $W_0^{m,p}$ by completing $C_0^\infty(\Omega) \subset L^p(\Omega)$ under $\|\cdot\|_m$.

Sobolev Spaces

Definition

Consider a given domain Ω and compact sets $K \subset \Omega$. We define the set of **locally integrable** functions as

$$L^1_{\text{loc}}(\Omega) := \{v | v \in L^1(K), \ \forall K \subset \Omega^o\}$$

Sobolev Spaces

Definition

Consider a given domain Ω and compact sets $K \subset \Omega$. We define the set of **locally integrable** functions as

$$L^1_{\text{loc}}(\Omega) := \{v | v \in L^1(K), \ \forall K \subset \Omega^o\}$$

These functions can behave poorly near the boundary of Ω as illustrated by $v(x) = \phi(1/\text{dist}(x, \partial\Omega))$ where $\phi(x) = e^{e^x}$ which still yields $v \in L^1_{\text{loc}}(\Omega)$.

Sobolev Spaces

Definition

Consider a given domain Ω and compact sets $K \subset \Omega$. We define the set of **locally integrable** functions as

$$L^1_{\mathrm{loc}}(\Omega) := \{v | v \in L^1(K), \; \forall K \subset \Omega^o\}$$

These functions can behave poorly near the boundary of Ω as illustrated by $v(x) = \phi(1/\mathrm{dist}(x, \partial\Omega))$ where $\phi(x) = e^{e^x}$ which still yields $v \in L^1_{\mathrm{loc}}(\Omega)$.

Definition

The $p = \infty$ norm is given by

$$\|v\|_{L^\infty(\Omega)} := \text{ess-sup}\{|v(x)| \mid x \in \Omega\}$$

Sobolev Spaces

Definition

Consider a given domain Ω and compact sets $K \subset \Omega$. We define the set of **locally integrable** functions as

$$L^1_{\text{loc}}(\Omega) := \{v | v \in L^1(K), \ \forall K \subset \Omega^o\}$$

These functions can behave poorly near the boundary of Ω as illustrated by $v(x) = \phi(1/\text{dist}(x, \partial\Omega))$ where $\phi(x) = e^{e^x}$ which still yields $v \in L^1_{\text{loc}}(\Omega)$.

Definition

The $p = \infty$ norm is given by

$$\|v\|_{L^\infty(\Omega)} := \text{ess-sup}\{|v(x)| \mid x \in \Omega\}$$

If $U = \text{ess-sup}(v)$ then $v(x) \leq U$ for almost every $x \in \Omega$ (except set of measure zero).

Sobolev Spaces

Definition

Consider a given domain Ω and compact sets $K \subset \Omega$. We define the set of **locally integrable** functions as

$$L^1_{\text{loc}}(\Omega) := \{v | v \in L^1(K), \ \forall K \subset \Omega^o\}$$

These functions can behave poorly near the boundary of Ω as illustrated by $v(x) = \phi(1/\text{dist}(x, \partial\Omega))$ where $\phi(x) = e^{e^x}$ which still yields $v \in L^1_{\text{loc}}(\Omega)$.

Definition

The $p = \infty$ norm is given by

$$\|v\|_{L^\infty(\Omega)} := \text{ess-sup}\{|v(x)| \mid x \in \Omega\}$$

If $U = \text{ess-sup}(v)$ then $v(x) \leq U$ for almost every $x \in \Omega$ (except set of measure zero).
Example: Let $f(x) = 3$ on the rationals $\mathbb{Q}$ and $f(x) = 2$ on the positive irrationals $\mathbb{R}^+ \setminus \mathbb{Q}$ and $f(x) = -1$ on the negative irrationals $\mathbb{R}^- \setminus \mathbb{Q}$.

Sobolev Spaces

Definition

Consider a given domain Ω and compact sets $K \subset \Omega$. We define the set of **locally integrable** functions as

$$L^1_{\text{loc}}(\Omega) := \{v | v \in L^1(K),\ \forall K \subset \Omega^o\}$$

These functions can behave poorly near the boundary of Ω as illustrated by $v(x) = \phi(1/\text{dist}(x, \partial\Omega))$ where $\phi(x) = e^{e^x}$ which still yields $v \in L^1_{\text{loc}}(\Omega)$.

Definition

The $p = \infty$ norm is given by

$$\|v\|_{L^\infty(\Omega)} := \text{ess-sup}\{|v(x)| \mid x \in \Omega\}$$

If $U = \text{ess-sup}(v)$ then $v(x) \leq U$ for almost every $x \in \Omega$ (except set of measure zero).
Example: Let $f(x) = 3$ on the rationals $\mathbb{Q}$ and $f(x) = 2$ on the positive irrationals $\mathbb{R}^+ \setminus \mathbb{Q}$ and $f(x) = -1$ on the negative irrationals $\mathbb{R}^- \setminus \mathbb{Q}$. We have $\text{ess-sup}\{f(x) \mid x \in \Omega\} = 2$

Sobolev Spaces

Definition

Consider a given domain Ω and compact sets $K \subset \Omega$. We define the set of **locally integrable** functions as

$$L^1_{\text{loc}}(\Omega) := \{v | v \in L^1(K), \ \forall K \subset \Omega^o\}$$

These functions can behave poorly near the boundary of Ω as illustrated by $v(x) = \phi(1/\text{dist}(x, \partial\Omega))$ where $\phi(x) = e^{e^x}$ which still yields $v \in L^1_{\text{loc}}(\Omega)$.

Definition

The $p = \infty$ norm is given by

$$\|v\|_{L^\infty(\Omega)} := \text{ess-sup}\{|v(x)| \mid x \in \Omega\}$$

If $U = \text{ess-sup}(v)$ then $v(x) \leq U$ for almost every $x \in \Omega$ (except set of measure zero).
Example: Let $f(x) = 3$ on the rationals $\mathbb{Q}$ and $f(x) = 2$ on the positive irrationals $\mathbb{R}^+ \setminus \mathbb{Q}$ and $f(x) = -1$ on the negative irrationals $\mathbb{R}^- \setminus \mathbb{Q}$. We have $\text{ess-sup}\{f(x) \mid x \in \Omega\} = 2$ and $\text{ess-inf}\{f(x) \mid x \in \Omega\} = -\text{ess-sup}\{-f(x) \mid x \in \Omega\} = -1$.

Sobolev Spaces

Definition

For $1 \leq p < \infty$, we define the **Sobolev norm** as

$$\|v\|_{W_p^k(\Omega)} := \left(\sum_{|\alpha| \leq k} \|D_w^\alpha v\|_{L^p(\Omega)}^p \right)^{1/p},$$

Sobolev Spaces

Definition

For $1 \leq p < \infty$, we define the **Sobolev norm** as

$$\|v\|_{W_p^k(\Omega)} := \left(\sum_{|\alpha| \leq k} \|D_w^\alpha v\|_{L^p(\Omega)}^p \right)^{1/p},$$

We assume k is a non-negative integer, $v \in L^1_{\mathrm{loc}}(\Omega)$, and $D_w^\alpha v$ exists for all $|\alpha| \leq k$.

Sobolev Spaces

Definition

For $1 \leq p < \infty$, we define the **Sobolev norm** as

$$\|v\|_{W_p^k(\Omega)} := \left(\sum_{|\alpha| \leq k} \|D_w^\alpha v\|_{L^p(\Omega)}^p \right)^{1/p},$$

We assume k is a non-negative integer, $v \in L^1_{\text{loc}}(\Omega)$, and $D_w^\alpha v$ exists for all $|\alpha| \leq k$.

For $p = \infty$, we define the **Sobolev norm** as

$$\|v\|_{W_\infty^k(\Omega)} := \max_{|\alpha| \leq k} \|D_w^\alpha v\|_{L^\infty(\Omega)}.$$

Sobolev Spaces

Definition

The **Sobolev space** is defined as

$$W_p^k(\Omega) := \{v \in L^1_{\text{loc}}(\Omega) \mid \|v\|_{W_p^k(\Omega)} < \infty\}$$

Sobolev Spaces

Definition

The **Sobolev space** is defined as

$$W_p^k(\Omega) := \{v \in L^1_{\mathrm{loc}}(\Omega) \mid \|v\|_{W_p^k(\Omega)} < \infty\}$$

Definition

For $1 \leq p < \infty$, we define the **Sobolev semi-norm** as

$$|v|_{W_p^k}(\Omega) := \left(\sum_{|\alpha|=k} \|D_w^\alpha v\|_{L^p(\Omega)}^p\right)^{1/p},$$

Sobolev Spaces

Definition

The **Sobolev space** is defined as

$$W_p^k(\Omega) := \{v \in L^1_{\text{loc}}(\Omega) \mid \|v\|_{W_p^k(\Omega)} < \infty\}$$

Definition

For $1 \leq p < \infty$, we define the **Sobolev semi-norm** as

$$|v|_{W_p^k(\Omega)} := \left(\sum_{|\alpha|=k} \|D_w^\alpha v\|_{L^p(\Omega)}^p\right)^{1/p},$$

For $p = \infty$, the **Sobolev semi-norm** as

$$|v|_{W_\infty^k(\Omega)} := \max_{|\alpha|=k} \|D_w^\alpha v\|_{L^\infty(\Omega)}.$$

Sobolev Spaces

The general Sobolev spaces also satisfy inclusion relations.

Sobolev Spaces

The general Sobolev spaces also satisfy inclusion relations.

Theorem

For k, m are non-negative integers with $k \leq m$ and p any real number with $1 \leq p \leq \infty$, we have

$$W_p^m(\Omega) \subset W_p^k(\Omega).$$

Sobolev Spaces

The general Sobolev spaces also satisfy inclusion relations.

Theorem

For k, m are non-negative integers with $k \leq m$ and p any real number with $1 \leq p \leq \infty$, we have

$$W_p^m(\Omega) \subset W_p^k(\Omega).$$

Theorem

For k any non-negative integer and p, q any real numbers with $1 \leq p \leq q \leq \infty$, we have

$$W_q^k(\Omega) \subset W_p^k(\Omega).$$

Sobolev Spaces

The general Sobolev spaces also satisfy inclusion relations.

Theorem

For k, m are non-negative integers with $k \leq m$ and p any real number with $1 \leq p \leq \infty$, we have

$$W_p^m(\Omega) \subset W_p^k(\Omega).$$

Theorem

For k any non-negative integer and p, q any real numbers with $1 \leq p \leq q \leq \infty$, we have

$$W_q^k(\Omega) \subset W_p^k(\Omega).$$

Theorem

For k, m non-negative integers with $k < m$ and and p, q any real numbers with $1 \leq p < q \leq \infty$, we have

$$W_q^m(\Omega) \subset W_p^k(\Omega).$$

Poincaré-Friedrichs Inequality

Theorem

Poincaré-Friedrichs Inequality: Consider the domain $\Omega \subset [0, s]^n$ is contained within a cube of side-length s. Then

$$\|v\|_0 \leq s|v|_1, \ \forall v \in H_0^1(\Omega).$$

Poincaré-Friedrichs Inequality

Theorem

Poincaré-Friedrichs Inequality: Consider the domain $\Omega \subset [0, s]^n$ is contained within a cube of side-length s. Then

$$\|v\|_0 \leq s|v|_1, \; \forall v \in H_0^1(\Omega).$$

This shows the 1-semi-norm bounds the 0-norm.

Poincaré-Friedrichs Inequality

Theorem

Poincaré-Friedrichs Inequality: Consider domain $\Omega \subset Q = [0,s]^n$, Q is cube of side-length s. Then

$$\|v\|_0 \leq s|v|_1, \ \forall v \in H_0^1(\Omega).$$

Proof:

Poincaré-Friedrichs Inequality

Theorem

Poincaré-Friedrichs Inequality: Consider domain $\Omega \subset Q = [0, s]^n$, Q is cube of side-length s. Then

$$\|v\|_0 \leq s|v|_1, \; \forall v \in H_0^1(\Omega).$$

Proof: Since $v \in H_0^1$ and using a point on the boundary $(0, x_2, x_3, \ldots, x_n)$ we can express v as

$$v(x_1, x_2, \ldots, x_n) \;=\; v(0, x_2, \ldots, x_n) + \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz = \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz$$

Poincaré-Friedrichs Inequality

Theorem

Poincaré-Friedrichs Inequality: Consider domain $\Omega \subset Q = [0, s]^n$, Q is cube of side-length s. Then

$$\|v\|_0 \leq s|v|_1, \; \forall v \in H_0^1(\Omega).$$

Proof: Since $v \in H_0^1$ and using a point on the boundary $(0, x_2, x_3, \ldots, x_n)$ we can express v as

$$v(x_1, x_2, \ldots, x_n) \;\; = \;\; v(0, x_2, \ldots, x_n) + \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz = \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz$$

By the Cauchy-Swartz inequality we have

$$|v(\mathbf{x})|^2 \leq \left(\int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz \right)^2 \;\; = \;\; \int_0^{x_1} 1^2 dz \int_0^{x_1} |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz$$

Poincaré-Friedrichs Inequality

Theorem

Poincaré-Friedrichs Inequality: Consider domain $\Omega \subset Q = [0,s]^n$, Q is cube of side-length s. Then

$$\|v\|_0 \leq s|v|_1, \ \forall v \in H_0^1(\Omega).$$

Proof: Since $v \in H_0^1$ and using a point on the boundary $(0, x_2, x_3, \ldots, x_n)$ we can express v as

$$v(x_1, x_2, \ldots, x_n) \ = \ v(0, x_2, \ldots, x_n) + \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz = \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz$$

By the Cauchy-Swartz inequality we have

$$\begin{aligned}
|v(\mathbf{x})|^2 \leq \left(\int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz\right)^2 \ &= \ \int_0^{x_1} 1^2 dz \int_0^{x_1} |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz \\
&\leq \ s \int_0^{x_1} |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz
\end{aligned}$$

Poincaré-Friedrichs Inequality

Theorem

Poincaré-Friedrichs Inequality: Consider domain $\Omega \subset Q = [0, s]^n$, Q is cube of side-length s. Then

$$\|v\|_0 \leq s|v|_1, \; \forall v \in H_0^1(\Omega).$$

Proof: Since $v \in H_0^1$ and using a point on the boundary $(0, x_2, x_3, \ldots, x_n)$ we can express v as

$$v(x_1, x_2, \ldots, x_n) \;=\; v(0, x_2, \ldots, x_n) + \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz = \int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz$$

By the Cauchy-Swartz inequality we have

$$\begin{aligned} |v(\mathbf{x})|^2 \leq \left(\int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz \right)^2 &= \int_0^{x_1} 1^2 dz \int_0^{x_1} |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz \\ &\leq s \int_0^{x_1} |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz \end{aligned}$$

We integrate over the cube $Q = [0, s]^n$ with v, $\partial^1 v$ extended to vanish outside of Ω.

Poincaré-Friedrichs Inequality

Theorem

Poincaré-Friedrichs Inequality: Consider the domain $\Omega \subset [0,s]^n$ is contained within a cube of side-length s. Then

$$\|v\|_0 \leq s|v|_1, \;\forall v \in H_0^1(\Omega).$$

Proof:

$$\begin{aligned}
|v(\mathbf{x})|^2 \leq \left(\int_0^{x_1} \partial^1 v(z, x_2, \ldots, x_n) dz\right)^2 &= \int_0^{x_1} 1^2 dz \int_0^{x_1} |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz \\
&\leq s \int_0^{s} |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz
\end{aligned}$$

Poincaré-Friedrichs Inequality

Theorem

Poincaré-Friedrichs Inequality: Consider the domain $\Omega \subset [0,s]^n$ is contained within a cube of side-length s. Then
$$\|v\|_0 \leq s|v|_1, \ \forall v \in H_0^1(\Omega).$$

Proof:

$$\begin{aligned} |v(\mathbf{x})|^2 \leq \left(\int_0^{x_1} \partial^1 v(z,x_2,\ldots,x_n)dz\right)^2 &= \int_0^{x_1} 1^2 dz \int_0^{x_1} |\partial^1 v(z,x_2,\ldots,x_n)|^2 dz \\ &\leq s\int_0^{s} |\partial^1 v(z,x_2,\ldots,x_n)|^2 dz \end{aligned}$$

We now integrate both sides over $\int_0^s$ and note RHS independent of x_1

Poincaré-Friedrichs Inequality

Theorem

Poincaré-Friedrichs Inequality: Consider the domain $\Omega \subset [0,s]^n$ is contained within a cube of side-length s. Then

$$\|v\|_0 \leq s|v|_1, \ \forall v \in H_0^1(\Omega).$$

Proof:

$$\begin{aligned} |v(\mathbf{x})|^2 \leq \left(\int_0^{x_1} \partial^1 v(z,x_2,\ldots,x_n)dz\right)^2 &= \int_0^{x_1} 1^2 dz \int_0^{x_1} |\partial^1 v(z,x_2,\ldots,x_n)|^2 dz \\ &\leq s \int_0^s |\partial^1 v(z,x_2,\ldots,x_n)|^2 dz \end{aligned}$$

We now integrate both sides over $\int_0^s$ and note RHS independent of x_1

$$\int_0^s |v(\mathbf{x})|^2 dx_1 \leq s^2 \int_0^s |\partial^1 v(z,x_2,\ldots,x_n)|^2 dz$$

Poincaré-Friedrichs Inequality

Theorem

Poincaré-Friedrichs Inequality: Consider the domain $\Omega \subset [0, s]^n$ is contained within a cube of side-length s. Then

$$\|v\|_0 \leq s|v|_1, \ \forall v \in H_0^1(\Omega).$$

Proof:

$$\int_0^s |v(\mathbf{x})|^2 dx_1 \quad \leq \quad s^2 \int_0^s |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz$$

Poincaré-Friedrichs Inequality

Theorem

Poincaré-Friedrichs Inequality: Consider the domain $\Omega \subset [0, s]^n$ is contained within a cube of side-length s. Then

$$\|v\|_0 \leq s|v|_1, \ \forall v \in H_0^1(\Omega).$$

Proof:

$$\int_0^s |v(\mathbf{x})|^2 dx_1 \quad \leq \quad s^2 \int_0^s |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz$$

We integrate over the other components to obtain

Poincaré-Friedrichs Inequality

Theorem

Poincaré-Friedrichs Inequality: Consider the domain $\Omega \subset [0, s]^n$ is contained within a cube of side-length s. Then

$$\|v\|_0 \leq s|v|_1, \ \forall v \in H_0^1(\Omega).$$

Proof:

$$\int_0^s |v(\mathbf{x})|^2 dx_1 \ \ \leq \ \ s^2 \int_0^s |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz$$

We integrate over the other components to obtain

$$\|v\|_0^2 = \int_Q |v(\mathbf{x})|^2 d\mathbf{x} \ \ \leq \ \ s^2 \int_Q |\partial^1 v(\mathbf{x})|^2 d\mathbf{x} = s^2 |v|_1^2.$$

Poincaré-Friedrichs Inequality

Theorem

Poincaré-Friedrichs Inequality: Consider the domain $\Omega \subset [0, s]^n$ is contained within a cube of side-length s. Then

$$\|v\|_0 \leq s|v|_1, \ \forall v \in H_0^1(\Omega).$$

Proof:

$$\int_0^s |v(\mathbf{x})|^2 dx_1 \quad \leq \quad s^2 \int_0^s |\partial^1 v(z, x_2, \ldots, x_n)|^2 dz$$

We integrate over the other components to obtain

$$\|v\|_0^2 = \int_Q |v(\mathbf{x})|^2 d\mathbf{x} \quad \leq \quad s^2 \int_Q |\partial^1 v(\mathbf{x})|^2 d\mathbf{x} = s^2 |v|_1^2.$$

$$\Rightarrow \ \|v\|_0 \leq s|v|_1.$$

■

Poincaré-Friedrichs Inequality

We can also apply the inequality using the derivatives $\tilde{v} = \partial^\alpha u$ to obtain

Poincaré-Friedrichs Inequality

We can also apply the inequality using the derivatives $\tilde{v} = \partial^\alpha u$ to obtain

$$|\partial^\alpha u|_0 \leq s|\partial^1 \partial^\alpha u|_0, \ |\alpha| \leq m - 1, \ u \in H_0^m(\Omega).$$

Poincaré-Friedrichs Inequality

We can also apply the inequality using the derivatives $\tilde{v} = \partial^\alpha u$ to obtain

$$|\partial^\alpha u|_0 \leq s|\partial^1 \partial^\alpha u|_0, \; |\alpha| \leq m - 1, \; u \in H_0^m(\Omega).$$

By induction we obtain

Poincaré-Friedrichs Inequality

We can also apply the inequality using the derivatives $\tilde{v} = \partial^\alpha u$ to obtain

$$|\partial^\alpha u|_0 \leq s|\partial^1 \partial^\alpha u|_0, \; |\alpha| \leq m-1, \; u \in H_0^m(\Omega).$$

By induction we obtain

Theorem

Poincaré-Friedrichs Inequality II: Consider the domain $\Omega \subset [0,s]^n$ is contained within a cube of side-length s. Then

$$|v|_m \leq \|v\|_m \leq (1+s)^m |v|_m, \; \forall v \in H_0^m(\Omega).$$

Poincaré-Friedrichs Inequality

We can also apply the inequality using the derivatives $\tilde{v} = \partial^\alpha u$ to obtain

$$|\partial^\alpha u|_0 \leq s|\partial^1 \partial^\alpha u|_0, \; |\alpha| \leq m - 1, \; u \in H_0^m(\Omega).$$

By induction we obtain

Theorem

Poincaré-Friedrichs Inequality II: Consider the domain $\Omega \subset [0, s]^n$ is contained within a cube of side-length s. Then

$$|v|_m \leq \|v\|_m \leq (1 + s)^m |v|_m, \; \forall v \in H_0^m(\Omega).$$

When Ω is bounded, the m-semi-norm $|v|_m$ is in fact a proper norm on $H_0^m(\Omega)$.

Poincaré-Friedrichs Inequality

We can also apply the inequality using the derivatives $\tilde{v} = \partial^\alpha u$ to obtain

$$|\partial^\alpha u|_0 \leq s|\partial^1 \partial^\alpha u|_0, \ |\alpha| \leq m-1, \ u \in H_0^m(\Omega).$$

By induction we obtain

Theorem

Poincaré-Friedrichs Inequality II: Consider the domain $\Omega \subset [0,s]^n$ is contained within a cube of side-length s. Then

$$|v|_m \leq \|v\|_m \leq (1+s)^m |v|_m, \ \forall v \in H_0^m(\Omega).$$

When Ω is bounded, the m-semi-norm $|v|_m$ is in fact a proper norm on $H_0^m(\Omega)$.
The norm $|v|_m$ is equivalent to $\|v\|_m$ (convergence in one implies convergence in other).

Sobolev Inequality

Theorem

Sobolev Inequality: Consider a domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, $k > 0$ with k an integer, and p real number with $1 \leq p < \infty$ such that

Sobolev Inequality

Theorem

Sobolev Inequality: Consider a domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, $k > 0$ with k an integer, and p real number with $1 \leq p < \infty$ such that

$$\begin{aligned} k &\geq n, \text{ when } p = 1 \\ k &> n/p, \text{ when } p > 1. \end{aligned}$$

Sobolev Inequality

Theorem

Sobolev Inequality: Consider a domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, $k > 0$ with k an integer, and p real number with $1 \leq p < \infty$ such that

$$\begin{aligned} k &\geq n, \text{ when } p = 1 \\ k &> n/p, \text{ when } p > 1. \end{aligned}$$

We then have there is a constant C so that

Sobolev Inequality

Theorem

Sobolev Inequality: Consider a domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, $k > 0$ with k an integer, and p real number with $1 \leq p < \infty$ such that

$$\begin{aligned} k &\geq n, \text{ when } p = 1 \\ k &> n/p, \text{ when } p > 1. \end{aligned}$$

We then have there is a constant C so that for all $u \in W_p^k(\Omega)$

Sobolev Inequality

Theorem

Sobolev Inequality: Consider a domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, $k > 0$ with k an integer, and p real number with $1 \leq p < \infty$ such that

$$\begin{aligned} k &\geq n, \text{ when } p = 1 \\ k &> n/p, \text{ when } p > 1. \end{aligned}$$

We then have there is a constant C so that for all $u \in W_p^k(\Omega)$

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{W_p^k(\Omega)}.$$

Sobolev Inequality

Theorem

Sobolev Inequality: Consider a domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, $k > 0$ with k an integer, and p real number with $1 \leq p < \infty$ such that

$$\begin{aligned} k &\geq n, \text{ when } p = 1 \\ k &> n/p, \text{ when } p > 1. \end{aligned}$$

We then have there is a constant C so that for all $u \in W_p^k(\Omega)$

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{W_p^k(\Omega)}.$$

Also, for the equivalence class of u in $L^\infty(\Omega)$, there is a representative that is a continuous function.

Sobolev Inequality

Theorem

Sobolev Inequality: Consider a domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, $k > 0$ with k an integer, and p real number with $1 \leq p < \infty$ such that

$$\begin{aligned} k &\geq n, \text{ when } p = 1 \\ k &> n/p, \text{ when } p > 1. \end{aligned}$$

We then have there is a constant C so that for all $u \in W_p^k(\Omega)$

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{W_p^k(\Omega)}.$$

Also, for the equivalence class of u in $L^\infty(\Omega)$, there is a representative that is a continuous function.

Significance: Shows that if a function has enough weak derivatives then in fact it can be viewed as equivalent to a continuous, bounded function.

Sobolev Inequality

Theorem

Sobolev Inequality: Consider a domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, $k > 0$ with k an integer, and p real number with $1 \leq p < \infty$ such that

$$\begin{aligned} k &\geq n, \text{ when } p = 1 \\ k &> n/p, \text{ when } p > 1. \end{aligned}$$

We then have there is a constant C so that for all $u \in W_p^k(\Omega)$

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{W_p^k(\Omega)}.$$

Also, for the equivalence class of u in $L^\infty(\Omega)$, there is a representative that is a continuous function.

Significance: Shows that if a function has enough weak derivatives then in fact it can be viewed as equivalent to a continuous, bounded function.
Also, shows that if we have convergence in $\|\cdot\|_{W_p^k(\Omega)}$ then also converges in $\|\cdot\|_{L^\infty(\Omega)}$.

Trace Theorems (boundary conditions)

When working with L^p functions how do we characterize values on the boundary which are sets of measure zero.

Trace Theorems (boundary conditions)

When working with L^p functions how do we characterize values on the boundary which are sets of measure zero.

Example: Consider $\Omega = \{(x, y) | x^2 + y^2 < 1\} = \{(r, \theta) | r < 1, 0 \leq \theta < 2\pi\}$.

Trace Theorems (boundary conditions)

When working with L^p functions how do we characterize values on the boundary which are sets of measure zero.

Example: Consider $\Omega = \{(x,y) | x^2 + y^2 < 1\} = \{(r,\theta) | r < 1, 0 \leq \theta < 2\pi\}$.

Lemma

Let Ω be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4} \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{W_2^1(\Omega)}^{1/2}.$$

Trace Theorems (boundary conditions)

When working with L^p functions how do we characterize values on the boundary which are sets of measure zero.

Example: Consider $\Omega = \{(x,y) | x^2 + y^2 < 1\} = \{(r,\theta) | r < 1, 0 \leq \theta < 2\pi\}$.

Lemma

Let Ω be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4} \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{W_2^1(\Omega)}^{1/2}.$$

Proof (sketch):

Trace Theorems (boundary conditions)

When working with L^p functions how do we characterize values on the boundary which are sets of measure zero.

Example: Consider $\Omega = \{(x,y) | x^2 + y^2 < 1\} = \{(r,\theta) | r < 1, 0 \leq \theta < 2\pi\}$.

Lemma

Let Ω be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4} \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{W_2^1(\Omega)}^{1/2}.$$

Proof (sketch):
For $u \in C^1(\overline{\Omega})$, consider the restriction to $\partial\Omega$ when $r \leq 1$,

Trace Theorems (boundary conditions)

When working with L^p functions how do we characterize values on the boundary which are sets of measure zero.

Example: Consider $\Omega = \{(x,y) | x^2 + y^2 < 1\} = \{(r,\theta) | r < 1, 0 \leq \theta < 2\pi\}$.

Lemma

Let Ω be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4} \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{W_2^1(\Omega)}^{1/2}.$$

Proof (sketch):
For $u \in C^1(\overline{\Omega})$, consider the restriction to $\partial\Omega$ when $r \leq 1$,

$$u(1,\theta)^2 = \int_0^1 \frac{\partial}{\partial r}\left(r^2 u(r,\theta)^2\right) dr = \int_0^1 2\left(r^2 u u_r + r u^2\right)\right)(r,\theta) dr$$

Trace Theorems (boundary conditions)

When working with L^p functions how do we characterize values on the boundary which are sets of measure zero.

Example: Consider $\Omega = \{(x,y) | x^2 + y^2 < 1\} = \{(r,\theta) | r < 1, 0 \leq \theta < 2\pi\}$.

Lemma

Let Ω be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4} \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{W_2^1(\Omega)}^{1/2}.$$

Proof (sketch):
For $u \in C^1(\overline{\Omega})$, consider the restriction to $\partial\Omega$ when $r \leq 1$,

$$\begin{aligned} u(1,\theta)^2 &= \int_0^1 \frac{\partial}{\partial r}\left(r^2 u(r,\theta)^2\right) dr = \int_0^1 2\left(r^2 u u_r + r u^2\right)\right)(r,\theta) dr \\ &= \int_0^1 2\left(r^2 u \nabla u \cdot \frac{(x,y)}{r} + r u^2\right)(r,\theta) dr \leq \int_0^1 2\left(r^2 |u||\nabla u| + r u^2\right)(r,\theta) dr \end{aligned}$$

Lemma

Let Ω be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4}\|u\|_{L^2(\Omega)}^{1/2}\|u\|_{W_2^1(\Omega)}^{1/2}.$$

Proof (sketch):

$$u(1,\theta)^2 \leq \int_0^1 2\left(r^2|u||\nabla u| + ru^2\right)\right)(r,\theta)dr \leq \int_0^1 2\left(|u||\nabla u| + u^2\right)\right)(r,\theta)rdr.$$

Lemma

Let Ω be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4}\|u\|_{L^2(\Omega)}^{1/2}\|u\|_{W_2^1(\Omega)}^{1/2}.$$

Proof (sketch):

$$u(1,\theta)^2 \quad \leq \quad \int_0^1 2\left(r^2|u||\nabla u| + ru^2)\right)(r,\theta)dr \leq \int_0^1 2\left(|u||\nabla u| + u^2)\right)(r,\theta)rdr.$$

Using polar coordinates and integrating out the θ we obtain

Lemma

Let Ω be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4}\|u\|_{L^2(\Omega)}^{1/2}\|u\|_{W_2^1(\Omega)}^{1/2}.$$

Proof (sketch):

$$u(1,\theta)^2 \quad \leq \quad \int_0^1 2\left(r^2|u||\nabla u| + ru^2)\right)(r,\theta)dr \leq \int_0^1 2\left(|u||\nabla u| + u^2)\right)(r,\theta)rdr.$$

Using polar coordinates and integrating out the θ we obtain

$$\int_{\partial\Omega} u^2 d\theta \leq 2\int_\Omega \left(|u||\nabla u| + u^2\right) dxdy.$$

Lemma

Let Ω be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4}\|u\|_{L^2(\Omega)}^{1/2}\|u\|_{W_2^1(\Omega)}^{1/2}.$$

Proof (sketch):

$$u(1,\theta)^2 \leq \int_0^1 2\left(r^2|u||\nabla u| + ru^2\right)(r,\theta)dr \leq \int_0^1 2\left(|u||\nabla u| + u^2\right)(r,\theta)rdr.$$

Using polar coordinates and integrating out the θ we obtain

$$\int_{\partial\Omega} u^2 d\theta \leq 2\int_{\Omega}\left(|u||\nabla u| + u^2\right)dxdy.$$

The norm of function $u|_{\partial\Omega}$ restricted to the boundary is

Lemma

Let Ω be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4}\|u\|_{L^2(\Omega)}^{1/2}\|u\|_{W_2^1(\Omega)}^{1/2}.$$

Proof (sketch):

$$u(1,\theta)^2 \quad \leq \quad \int_0^1 2\left(r^2|u||\nabla u| + ru^2)\right)(r,\theta)dr \leq \int_0^1 2\left(|u||\nabla u| + u^2)\right)(r,\theta)rdr.$$

Using polar coordinates and integrating out the θ we obtain

$$\int_{\partial\Omega} u^2 d\theta \leq 2\int_{\Omega}\left(|u||\nabla u| + u^2\right)dxdy.$$

The norm of function $u|_{\partial\Omega}$ restricted to the boundary is

$$\|u\|_{L^2(\partial\Omega)}^2 := \int_{\partial\Omega} u^2 d\theta = \int_0^{2\pi} u(1,\theta)^2 d\theta.$$

Lemma

Let Ω be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4}\|u\|_{L^2(\Omega)}^{1/2}\|u\|_{W_2^1(\Omega)}^{1/2}.$$

Proof (sketch):

By Cauchy-Swartz we have

Lemma

Let Ω be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4}\|u\|_{L^2(\Omega)}^{1/2}\|u\|_{W_2^1(\Omega)}^{1/2}.$$

Proof (sketch):

By Cauchy-Swartz we have

$$\|u\|_{L^2(\partial\Omega)}^2 \leq 2\|u\|_{L^2(\Omega)}\left(\int_\Omega |\nabla u|^2 dxdy\right)^{1/2} + 2\int_\Omega u^2 dxdy.$$

Lemma

Let Ω be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4}\|u\|_{L^2(\Omega)}^{1/2}\|u\|_{W_2^1(\Omega)}^{1/2}.$$

Proof (sketch):
By Cauchy-Swartz we have

$$\|u\|_{L^2(\partial\Omega)}^2 \leq 2\|u\|_{L^2(\Omega)}\left(\int_\Omega |\nabla u|^2 dxdy\right)^{1/2} + 2\int_\Omega u^2 dxdy.$$

Using the arithmetic-geometric mean inequality we have

Lemma

Let Ω be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4}\|u\|_{L^2(\Omega)}^{1/2}\|u\|_{W_2^1(\Omega)}^{1/2}.$$

Proof (sketch):
By Cauchy-Swartz we have

$$\|u\|_{L^2(\partial\Omega)}^2 \leq 2\|u\|_{L^2(\Omega)}\left(\int_\Omega |\nabla u|^2 dxdy\right)^{1/2} + 2\int_\Omega u^2 dxdy.$$

Using the arithmetic-geometric mean inequality we have

$$\left(\int_\Omega |\nabla u|^2 dxdy\right)^{1/2} + \left(\int_\Omega u^2 dxdy\right)^{1/2} \leq \left(2\int_\Omega \left(|\nabla u|^2 + u^2\right) dxdy\right)^{1/2}.$$

Lemma

Let Ω be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4}\|u\|_{L^2(\Omega)}^{1/2}\|u\|_{W_2^1(\Omega)}^{1/2}.$$

Proof (sketch):
By Cauchy-Swartz we have

$$\|u\|_{L^2(\partial\Omega)}^2 \leq 2\|u\|_{L^2(\Omega)}\left(\int_\Omega |\nabla u|^2 dxdy\right)^{1/2} + 2\int_\Omega u^2 dxdy.$$

Using the arithmetic-geometric mean inequality we have

$$\left(\int_\Omega |\nabla u|^2 dxdy\right)^{1/2} + \left(\int_\Omega u^2 dxdy\right)^{1/2} \leq \left(2\int_\Omega \left(|\nabla u|^2 + u^2\right) dxdy\right)^{1/2}.$$

This implies

Lemma

Let Ω be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4}\|u\|_{L^2(\Omega)}^{1/2}\|u\|_{W_2^1(\Omega)}^{1/2}.$$

Proof (sketch):

By Cauchy-Swartz we have

$$\|u\|_{L^2(\partial\Omega)}^2 \leq 2\|u\|_{L^2(\Omega)}\left(\int_\Omega |\nabla u|^2 dxdy\right)^{1/2} + 2\int_\Omega u^2 dxdy.$$

Using the arithmetic-geometric mean inequality we have

$$\left(\int_\Omega |\nabla u|^2 dxdy\right)^{1/2} + \left(\int_\Omega u^2 dxdy\right)^{1/2} \leq \left(2\int_\Omega \left(|\nabla u|^2 + u^2\right) dxdy\right)^{1/2}.$$

This implies

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4}\|u\|_{L^2(\Omega)}^{1/2}\|u\|_{W_2^1(\Omega)}^{1/2}.$$

Lemma

Let Ω be the unit disk. For all $u \in W_2^1(\Omega)$ the restriction of $u|_{\partial\Omega}$ can interpreted as a function in $L^2(\partial\Omega)$. Furthermore, it satisfies the bound

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4}\|u\|_{L^2(\Omega)}^{1/2}\|u\|_{W_2^1(\Omega)}^{1/2}.$$

Proof (sketch):
By Cauchy-Swartz we have

$$\|u\|_{L^2(\partial\Omega)}^2 \leq 2\|u\|_{L^2(\Omega)}\left(\int_\Omega |\nabla u|^2 dxdy\right)^{1/2} + 2\int_\Omega u^2 dxdy.$$

Using the arithmetic-geometric mean inequality we have

$$\left(\int_\Omega |\nabla u|^2 dxdy\right)^{1/2} + \left(\int_\Omega u^2 dxdy\right)^{1/2} \leq \left(2\int_\Omega \left(|\nabla u|^2 + u^2\right) dxdy\right)^{1/2}.$$

This implies

$$\|u\|_{L^2(\partial\Omega)} \leq 8^{1/4}\|u\|_{L^2(\Omega)}^{1/2}\|u\|_{W_2^1(\Omega)}^{1/2}.$$

■

Trace Theorems (boundary conditions)

Theorem

Trace Theorem: Consider Ω with a Lipschitz boundary and p real number with $1 \leq p \leq \infty$. We then have there exists a constant C so that

$$\|v\|_{L^p(\partial\Omega)} \leq C\|v\|_{L^p(\Omega)}^{1-1/p}\|v\|_{W_p^1(\Omega)}^{1/p}, \ \forall v \in W_p^1(\Omega).$$

Trace Theorems (boundary conditions)

Theorem

Trace Theorem: Consider Ω with a Lipschitz boundary and p real number with $1 \leq p \leq \infty$. We then have there exists a constant C so that

$$\|v\|_{L^p(\partial\Omega)} \leq C\|v\|_{L^p(\Omega)}^{1-1/p}\|v\|_{W_p^1(\Omega)}^{1/p}, \ \forall v \in W_p^1(\Omega).$$

Definition

Trace-Free Sobolev Spaces: We denote by $\mathring{W}_p^1(\Omega)$ the subset of $W_p^1(\Omega)$ consisting of the functions whose trace on the boundary $v|_{\partial\Omega}$ is zero. In particular,

$$\mathring{W}_p^1(\Omega) = \{v \in W_p^1(\Omega) \mid v|_{\partial\Omega} = 0 \text{ in } L^2(\partial\Omega)\}.$$

Trace Theorems (boundary conditions)

Theorem

Trace Theorem: Consider Ω with a Lipschitz boundary and p real number with $1 \leq p \leq \infty$. We then have there exists a constant C so that

$$\|v\|_{L^p(\partial\Omega)} \leq C\|v\|_{L^p(\Omega)}^{1-1/p}\|v\|_{W_p^1(\Omega)}^{1/p}, \ \forall v \in W_p^1(\Omega).$$

Definition

Trace-Free Sobolev Spaces: We denote by $\mathring{W}_p^1(\Omega)$ the subset of $W_p^1(\Omega)$ consisting of the functions whose trace on the boundary $v|_{\partial\Omega}$ is zero. In particular,

$$\mathring{W}_p^1(\Omega) = \{v \in W_p^1(\Omega) \mid v|_{\partial\Omega} = 0 \text{ in } L^2(\partial\Omega)\}.$$

Similarly, $\mathring{W}_p^k(\Omega)$ consists of functions whose derivatives of order $k-1$ are in $\mathring{W}_p^1(\Omega)$, so that

Trace Theorems (boundary conditions)

Theorem

Trace Theorem: Consider Ω with a Lipschitz boundary and p real number with $1 \leq p \leq \infty$. We then have there exists a constant C so that

$$\|v\|_{L^p(\partial\Omega)} \leq C\|v\|_{L^p(\Omega)}^{1-1/p}\|v\|_{W_p^1(\Omega)}^{1/p}, \ \forall v \in W_p^1(\Omega).$$

Definition

Trace-Free Sobolev Spaces: We denote by $\mathring{W}_p^1(\Omega)$ the subset of $W_p^1(\Omega)$ consisting of the functions whose trace on the boundary $v|_{\partial\Omega}$ is zero. In particular,

$$\mathring{W}_p^1(\Omega) = \{v \in W_p^1(\Omega) \mid v|_{\partial\Omega} = 0 \text{ in } L^2(\partial\Omega)\}.$$

Similarly, $\mathring{W}_p^k(\Omega)$ consists of functions whose derivatives of order $k-1$ are in $\mathring{W}_p^1(\Omega)$, so that

$$\mathring{W}_p^k(\Omega) = \{v \in W_p^k(\Omega) \mid v^{(\alpha)}|_{\partial\Omega} = 0 \text{ in } L^2(\partial\Omega), \ \ \forall|\alpha| < k\}.$$