Elasticity Theory

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206D: Finite Element Methods
University of California Santa Barbara

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The symbol $\wedge$ denotes the vector cross-product in $\mathbb{R}^3$.

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Elasticity Theory

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Elasticity Theory

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Valid for small deformations, but if valid all deformations, called *St. Venant-Kirkhhoff material*.

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From definition of $\iota_j(A)$ we have $\iota_j(E) = O(E^j)$. From smoothness of γ_i, we have only $\iota_1(E)$ contributes in the expressions. From $C = I + 2E$, $C^2 = I + 4E + o(E)$, to obtain leading order we expand as $\gamma_0(E) = a_0 + b_1\iota_1(E) + o(E)$, $\gamma_1(E) = a_1 + O(E)$, $\gamma_2(E) = a_2 + O(E)$. This yields

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Significance: Gives general constitutive relation expressed in terms of strain E when deformations are small.

Remark: Typically, $C = I$ with unstressed conditions so that $\pi = 0$. The λ and μ are called *Lame' constants*.

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Valid for small deformations, but if valid all deformations, called *St. Venant-Kirkhhoff material*.

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Remark: For small deformations, if we replace linearization in E with linearization in ϵ approach is called **geometrically linear theory**.

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Remark: Results in saddle-point problems.

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By introducing L^2 inner-product notation, we can express as

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Useful in establishing variational problems are elliptic and for coercivity.

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There are results establishing conditions for well-posedness for the other formulations.

Displacement Formulation

Existence Theorem (Displacement Formulation)

Let $\Omega \subset \mathbb{R}^3$ be a domain with piecewise smooth boundary, and Γ_0 has positive two-dimensional measure. Then the variational problem of linear elasticity has exactly one solution.

This follows by establishing the coercivity condition for the bilinear form in the variational problem. For the weak displacement formulation this is done using the Korn Inequalities.

The Lax-Milgram Theorem then gives the well-posedness of the variational problem.

There are results establishing conditions for well-posedness for the other formulations. These typically involve analysis establishing the Babuska-Brezzi inf-sup conditions hold (discussed with mixed method theory).

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In the nearly incompressible regime, referred to as **volume locking** or **Poisson locking**.

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$$\begin{aligned} 2\mu(\epsilon(u),\epsilon(v))_0 + (\operatorname{div} u, p)_0 &= \langle \ell, v\rangle, \quad && \forall v \in H^1_\Gamma(\Omega), \\ (\operatorname{div} u, q)_0 - \lambda^{-1}(p,q)_0 &= 0, && \forall q \in L_2(\Omega). \end{aligned}$$

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In the displacement formulation on $v \in H_\Gamma^1$, we have

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Discretization: Need to choose appropriate finite element spaces for mixed methods (upcoming lectures).