Variational Formulation of Elliptic PDEs

Paul J. Atzberger

206D: Finite Element Methods
University of California Santa Barbara

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Since $a(\cdot,\cdot)$ is coercive we have $a(v,v) = 0 \rightarrow v \equiv 0$, so a is an inner-product and $\|v\|_E = \sqrt{a(v,v)}$ is a norm.

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i Given $F \in \mathcal{V}'$, find u satisfying

$$a(u, v) = F[v], \quad \forall v \in \mathcal{V}, \quad (*)$$

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We show uniqueness first, then existence. Suppose $Tv_1 = v_1$ and $Tv_2 = v_2$, then by the contraction principle

$$\|Tv_1 - Tv_2\| \leq M\|v_1 - v_2\|$$

where $0 \leq M < 1$. By the fix-point property

$$\|v_1 - v_2\| \leq M\|v_1 - v_2\|.$$

This implies $\|v_1 - v_2\| = 0 \;\rightarrow\; v_1 = v_2$.

We next show existence.

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Lemma (Fix Point Theorem)

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Proof (continued):

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Theorem (Lax-Milgram)

Given a Hilbert space $(V, (\cdot,\cdot))$, a continuous, coercive bilinear form $a(\cdot,\cdot)$ (not necessarily symmetric), and $F \in \mathcal{V}'$, there exists a unique $u \in \mathcal{V}$ so that

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Implications: Also shows for the Galerkin approximations for the finite-dimensional problems the existence and uniqueness of solution u_h.

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We show $\tau Au = \tau F$ in $\mathcal{V}$, which implies solution to the variational problem holds.
We do this using a contraction mapping principle for $T[v] := v - \rho(\tau Av - \tau F)$. The fixed point theorem yields $Tu = u - \rho(\tau Au - \tau F) = u$. This implies $\tau Au - \tau F = 0$.
We now show that such a $\rho \neq 0$ exists making T a contraction map.

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For any $v_1, v_2 \in \mathcal{V}$, let $v = v_1 - v_2$, then

$$\|Tv_1 - Tv_2\|^2 \quad = \quad \|v_1 - v_2 - \rho(\tau A v_1 - \tau A v_2)\|^2$$

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&= \|v\|^2 - 2\rho(\tau A v, v) + \rho^2\|\tau A v\|^2 \\
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&= \|v\|^2 - 2\rho a(v, v) + \rho^2 a(v, \tau A v), \ \ (\text{definition of } A)\ , \\
&\leq \|v\|^2 - 2\rho \|v\|^2 + \rho^2 C \|v\| \|\tau A v\|, \ \ (\text{cocercivity and continuity of } A)\ ,
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&\leq \left(1 - 2\rho\alpha + \rho^2 C^2\right)\|v\|^2 \ (A \text{ bounded}, \ \tau \text{ isometric})
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