FEM Approximation Properties and Convergence

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206D: Finite Element Methods
University of California Santa Barbara

Approximation by Finite Elements

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Goal: Obtain estimates of $\|v - I_h v\|_{m,h}$ in terms of $\|v\|_{t,\Omega}$ and h with $m \leq t$.

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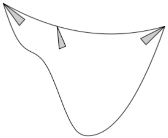

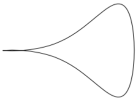

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An open domain Ω is said to satisfy the **cone condition** with angle ϕ and radius r if at every point $\mathbf{x} \in \Omega$ we have $\mathbf{x} + \mathcal{C}_{\phi,r,\mathbf{e}_\mathbf{x}} \subset \Omega$ for some orientation $\mathbf{e}_\mathbf{x}$.

Lemma

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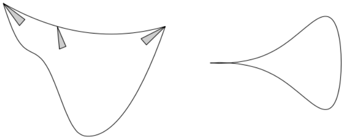

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Lemma

Consider an Ω that is bounded and star-shaped with respect to $\mathcal{B}(\mathbf{x}_c, r_c)$ and contained within $\mathcal{B}(\mathbf{x}_c, R)$. Then Ω satisfies an **interior cone condition** with radius r_c and angle $\phi = 2\arcsin\left(r_c/2R\right)$.

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Consider the interpolation operator $\mathcal{I}_s$ over $s = t(t+1)/2$ points $z_1, z_2, \ldots, z_s$ on $\overline{\Omega}$ which maps from $H^t \to \mathcal{P}_{t-1}$ well-defined for polynomials of degree $\leq t-1$, $t \geq 2$. Assume the domain $\Omega \subset \mathbb{R}^2$ has Lipschitz continuous boundary and satisfies the cone condition. Then there exists a constant $c = c(\Omega, z_1, \ldots, z_s)$ so the following bound holds

$$\|u - \mathcal{I}_s u\|_t \leq c|u|_t, \quad \forall u \in H^t(\Omega).$$

Proof: Let

$$|||v||| := |v|_t + \sum_{i=1}^{s} |v(z_i)|.$$

We show the norms $|||\cdot|||$ and $\|\cdot\|_t$ are equivalent. If this were the case, the bound would follow from

$$\|u - \mathcal{I}_s u\|_t \leq c|||u - I_s u||| = c\left(|u - \mathcal{I}_s u|_t + \sum_{i=1}^{s} |(u - \mathcal{I}_s u)(z_i)|\right) = c|u - \mathcal{I}_s u|_t = c|u|_t.$$

This makes use of $\mathcal{I}_s u(z_i) = u(z_i)$ at the interpolation points and that $D^\alpha \mathcal{I}_s u = 0$ for all $|\alpha| = t$.
We obtain one direction of equivalence, since $H^t \subset H^2 \subset C^0$ by the Sobolev Embedding Theorem, so we have

$$|v(z_i)| \leq c\|v\|_t \quad \Rightarrow \quad |||v||| \leq (1 + cs)\|v\|_t.$$

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Let $\Omega \subset \mathbb{R}^2$ be domain with Lipschitz continuous boundary. Suppose $t \geq 2$ and L is a bounded linear mapping of $H^t(\Omega)$ into a normed linear space $\mathcal{Z}$. If $\mathcal{P}_{t-1} \subset \ker(L)$, then there exists a constant $c = c(\Omega)\|L\| \geq 0$, so that

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Proof:

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Let $\Omega \subset \mathbb{R}^2$ be domain with Lipschitz continuous boundary. Suppose $t \geq 2$ and L is a bounded linear mapping of $H^t(\Omega)$ into a normed linear space $\mathcal{Z}$. If $\mathcal{P}_{t-1} \subset \ker(L)$, then there exists a constant $c = c(\Omega)\|L\| \geq 0$, so that

$$\|Lv\|_{\mathcal{Z}} \leq c|v|_t, \quad \text{for all } v \in H^t(\Omega).$$

Proof:

Let $\mathcal{I}_h : H^t(\Omega) \to \mathcal{P}_{t-1}$ be the interpolation operator. Using the first BH-Lemma and fact $\mathcal{I}_h(v) \in \ker(L)$ we have

$$\|Lv\|_{\mathcal{Z}} = \|L(v - \mathcal{I}_h v)\|_{\mathcal{Z}} \leq \|L\| \cdot \|v - \mathcal{I}_h v\|_t \leq c\|L\| \cdot |v|_t.$$

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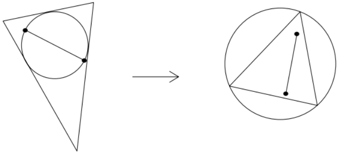
Approximation by Finite Elements

Definition



Approximation by Finite Elements

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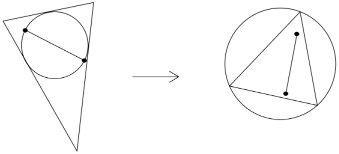

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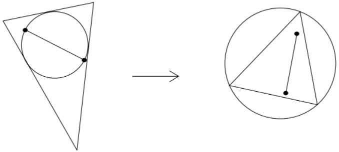

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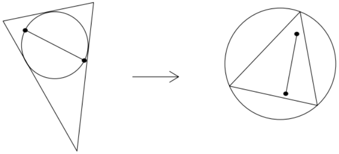

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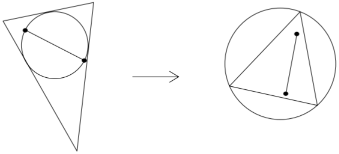

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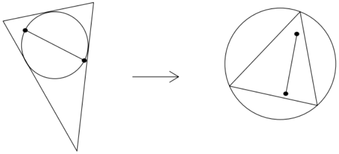

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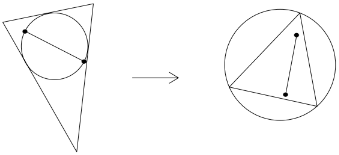

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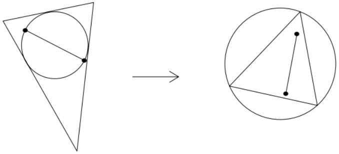

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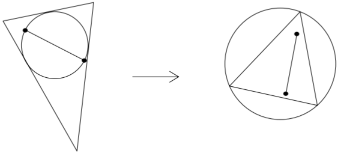

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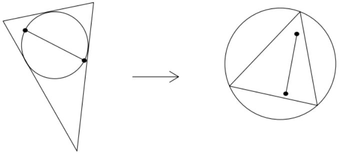

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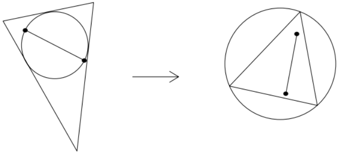

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This will become poor for triangles that are small "slivers."

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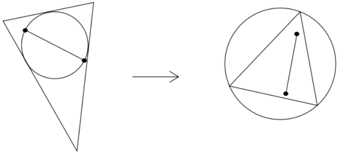

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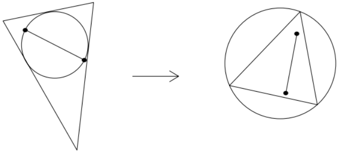

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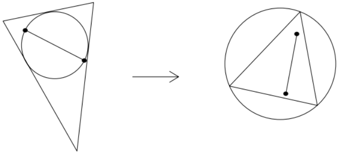

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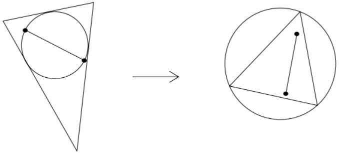

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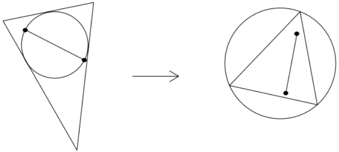

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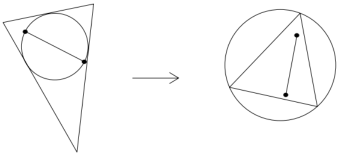

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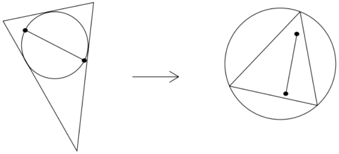

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Theorem for Quadrilateral Bilinear Elements

Consider $\mathcal{T}_h$ a quasi-uniform decomposition of Ω into parallelograms. There exists a constant $c = c(\Omega, \kappa)$ such that

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$\|u - I_h u\|_{m,h} \leq ch^{t-m}\|u\|_{t,\Omega}$	$0 \leq m \leq t$	
C^0 elements		
linear triangle	$t = 2$	
quadratic triangle	$2 \leq t \leq 3$	
cubic triangle	$2 \leq t \leq 4$	
bilinear quadrilateral	$t = 2$	
serendipity element	$2 \leq t \leq 3$	
9 node quadrilateral	$2 \leq t \leq 3$	
C^1 elements		
Argyris element	$3 \leq t \leq 6$	
Bell element	$3 \leq t \leq 5$	
Hsieh–Clough–Tocher element	$3 \leq t \leq 4$	$(m \leq 2)$
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Approximation by Finite Elements

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Consider $\mathcal{T}_h$ a quasi-uniform decomposition of Ω into parallelograms. There exists a constant $c = c(\Omega, \kappa)$ such that

$$\|u - \mathcal{I}_h u\|_{m,\Omega} \leq ch^{2-m}|u|_{2,\Omega}, \quad \forall u \in H^2(\Omega).$$

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Remark: For Serendipity Elements a similar proof technique can be used to obtain $\|u - \mathcal{I}_h u\|_{m,\Omega} \leq ch^{t-m}|u|_{t,\Omega}, \ \forall u \in H^t(\Omega), \ m = 0, 1, \ t = 2, 3.$

Approximation by Finite Elements

Theorem (Inverse Estimate)

Consider affine family of elements $\{\mathcal{S}_h\}$ with piecewise polynomials of degree k having uniform partitions. There exists a constant $c = c(\kappa, k, t)$ so that

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We use the equivalence of the norms $\|\cdot\|_{t,T}$ and $\|\cdot\|_{m,T}$ on the finite dimensional space $\mathcal{Q} = \mathcal{P} \bigoplus \mathcal{P}_{m-1}$.

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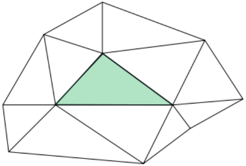

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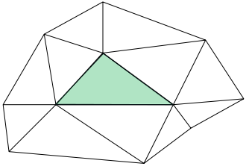

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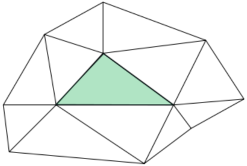

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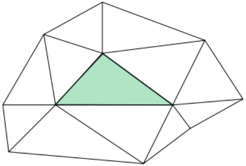

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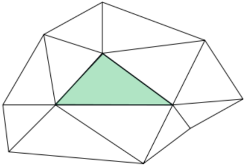

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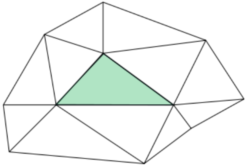

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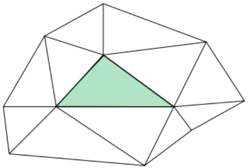

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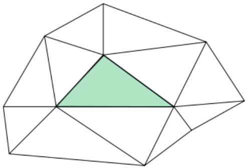

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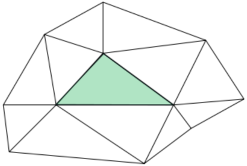

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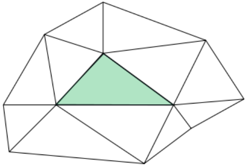

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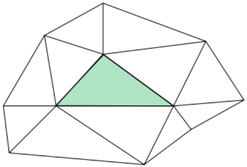

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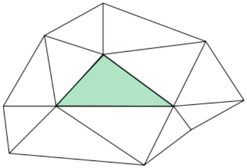

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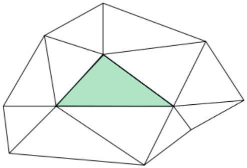

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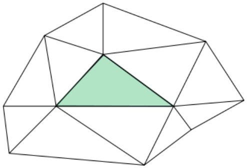

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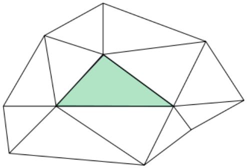

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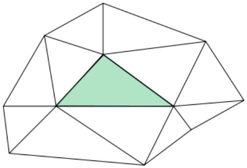

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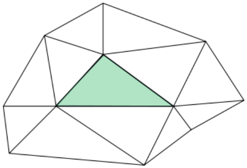

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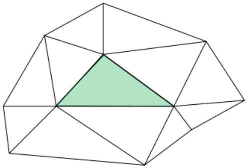

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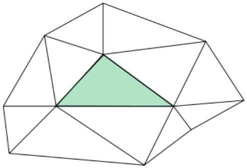

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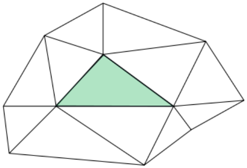

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