Ritz-Galerkin Approximation

Paul J. Atzberger

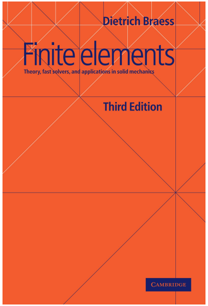
206D: Finite Element Methods
University of California Santa Barbara

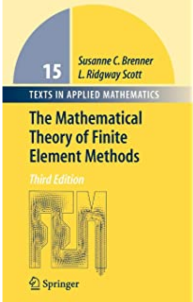
MATH 206D: Finite Element Methods

Welcome to MATH 206D: Finite Element Methods!

We will use the following books:

- *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics (third edition)*, D. Braess.
- *The Mathematical Theory of Finite Element Methods (third edition)*, S. Brenner and R. Scott.





For more information, see the course website:

http://teaching.atzberger.org/

I look forward to working with you this quarter.

Introduction to Finite Element Methods

Variational Approach

Variational Principle

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Suggests "natural boundary conditions" $\rightarrow u'(0) = u'(1) = 0$.

Variational Approach: Strong and Weak Forms

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We have "stiffness matrix" $[K]_{ij} = a(\phi_i, \phi_j)$ and "load vector" $[\mathbf{f}]_i = (f, \phi_i)$.

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Still, need theory to show $u_S \to u$ as $\mathcal{S} \to \mathcal{V}$ (i.e. we recover solution to the PDE in limit).

Linear Elements

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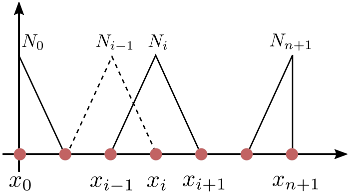
Consider space $\mathcal{S}$ generated by

$$v(x) = \sum_{i=1}^{n+1} v_i \phi_i(x)$$



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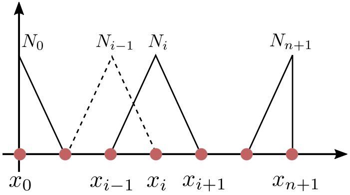

where $\phi_i(x) = N_i(x)$,

$$N_i(x) = \left\{ \begin{array}{ll} (x - x_{i-1})/h_{i-1}, & x \in [x_{i-1}, x_i] \\ (x_{i+1} - x)/h_i, & x \in [x_i, x_{i+1}] \\ 0, & \text{otherwise} \end{array} \right\} \quad \text{(Hat Functions).}$$

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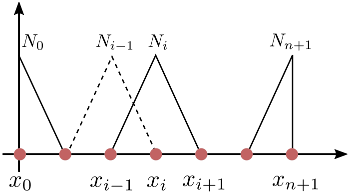

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(Hat Functions).

Here, $h_i = x_{i+1} - x_i$ and $N_i(x_j) = \delta_{ij}$.

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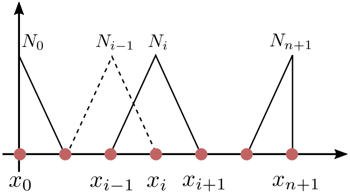

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Mesh: $x_0, x_1, \ldots, x_{n+1}$. **Elements:** $e_i = \{x | x_{i-1} \leq x \leq x_{i+1}\}$. **Shape Functions:** $N_i(x)$.

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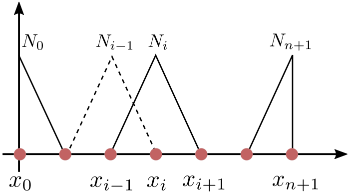

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Let $\mathcal{S} = \{v | v \in C[0, L], v(x) = \sum_{i=1}^{n} v_i N_i(x)\}$, referred to as the **shape space**.

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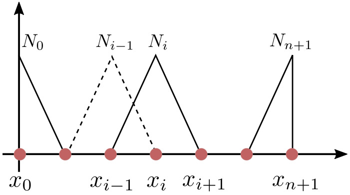

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We would like to carry-out the Ritz-Galerkin approximations over this space.

Linear Elements

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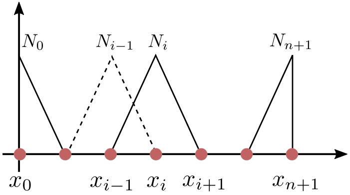

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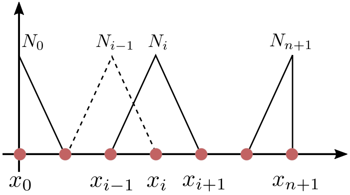

Consider the heat equation in 1D on $[0, L]$

$$\left\{ \begin{array}{ll} \frac{d^2u}{dx^2} = f(x), & x \in [0, L] \\ u(0) = T_1, u(L) = T_2, & x \text{ on boundary} \end{array} \right.$$

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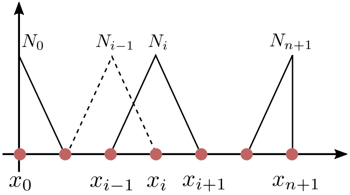

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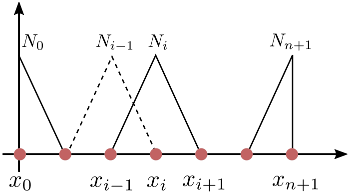

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$$a(u, v) = (-f, v), \ \forall v \in \mathcal{V}, \qquad \text{WLOG } \mathcal{V} = \{v | v \in C[0, L], v(0) = v(L) = 0\}$$

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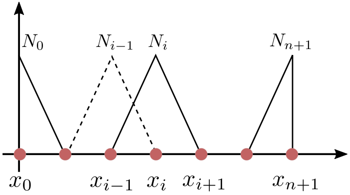

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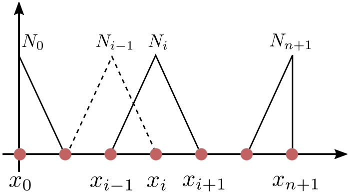

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To obtain stiffness matrix K and load vector $\mathbf{f}$, we need to compute the inner-products.

Linear Elements

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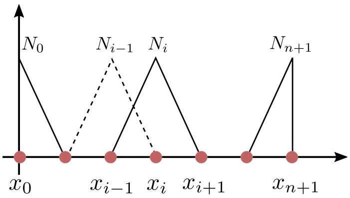

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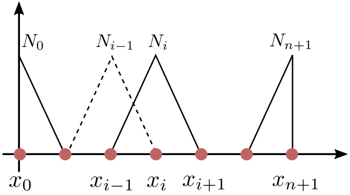

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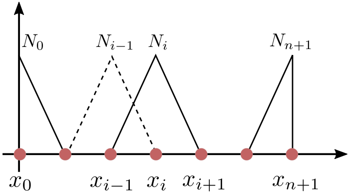
Linear Elements



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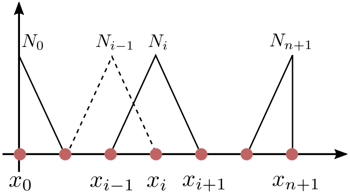

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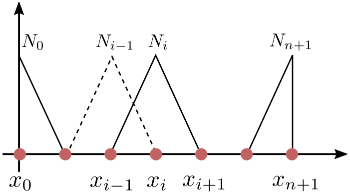

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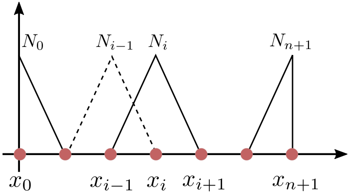

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$$(-f, N_i) = -\int_{x_{i-1}}^{x_{i+1}} f(x) N_i(x) dx.$$

When $f = \sum_{i=0}^{n+1} f_i N_i(x)$, compute via "mass matrix" $M_{ij} = (N_i, N_j)$, and $[\mathbf{f}]_i = M_{ij} f_j$.

Linear Elements

Shape functions:

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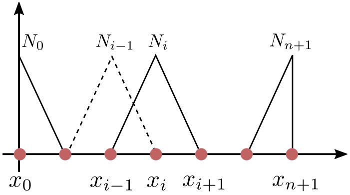

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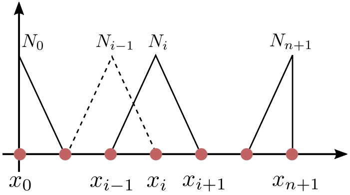

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Stiffness matrix when $h_i = h_0 = h$ and load vector when $f(x) = f_0$,

$$K = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ 0 & 0 & -1 & \ddots & -1 \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix},$$

Linear Elements

Shape functions:

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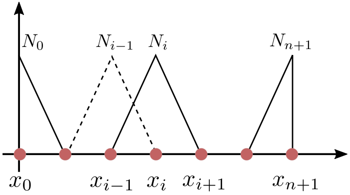

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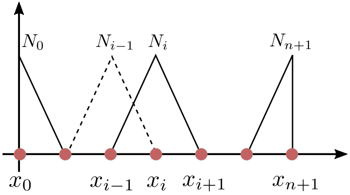
Linear Elements

Shape functions:

$$N_i(x) = \left\{ \begin{array}{ll} (x - x_{i-1})/h_{i-1}, & x \in [x_{i-1}, x_i] \\ (x_{i+1} - x)/h_i, & x \in [x_i, x_{i+1}] \\ 0, & \text{otherwise} \end{array} \right\}$$



Ritz-Galerkin Approximation

$$\left\{ \begin{array}{l} a(u_S, v) = (-f, v), \ \forall v \in \mathcal{S} \\ \mathcal{S} = \{v | v = \sum_{i=1}^{n} v_i N_i(x)\} \end{array} \right.$$

Stiffness matrix when $h_i = h_0 = h$ and load vector when $f(x) = f_0$,

$$K = \frac{1}{h} \left[\begin{array}{ccccc} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ 0 & 0 & -1 & \ddots & -1 \\ 0 & 0 & 0 & \cdots & 2 \end{array} \right], \mathbf{f} = h \left[\begin{array}{c} f_0 \\ f_0 \\ f_0 \\ \vdots \\ f_0 \end{array} \right]$$

In this case, the $K\mathbf{u} = -\mathbf{f}$ has similarities to Finite Difference Method for the heat equation.

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$$a(u - u_S, w) = 0, \forall w \in \mathcal{S}$$

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$$\begin{aligned} \|u - u_S\|_E^2 &= a(u - u_S, u - u_S) = a(u - u_S, u - v) + a(u - u_S, v - u_S) \\ &= a(u - u_S, u - v) \leq \|u - u_S\|_E \|u - v\|_E \qquad \text{(swartz)} \end{aligned}$$

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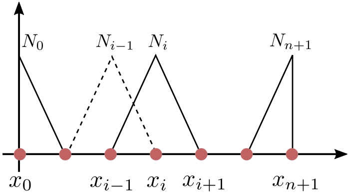

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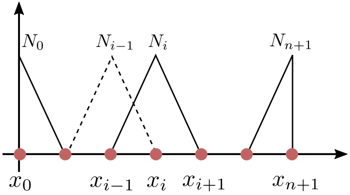

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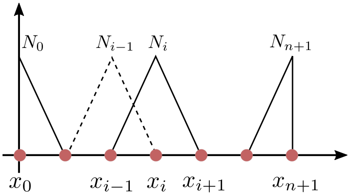

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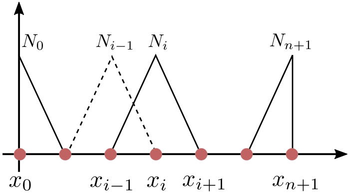

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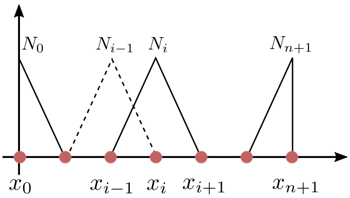

Example (linear elements):
The Green's function for $-d^2u/dx^2 = f$ is given by

$$G(x, x_0) = \left\{ \begin{array}{ll} x, & x < x_0 \\ x_0, & x_0, \text{ otherwise} \end{array} \right\}, \quad \frac{dG}{dx} = \left\{ \begin{array}{ll} 1, & x < x_0 \\ 0, & x_0, \text{ otherwise} \end{array} \right\}, \quad \frac{d^2G}{dx^2} = -\delta(x - x_0).$$

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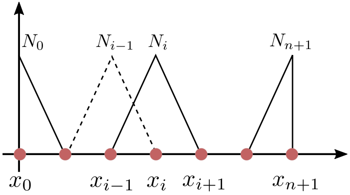

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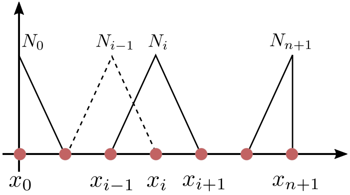
The solution u above can be expressed as

$$u(x) = \int G(x, y) f(y) dy.$$

Error Estimates

Example (linear elements) (continued)



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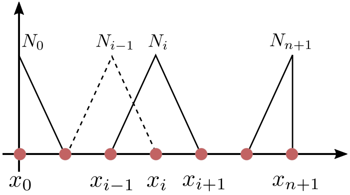

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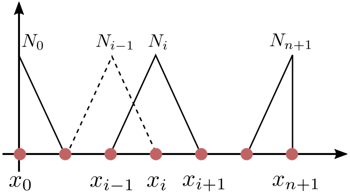

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Putting this together we have the error can be expressed as

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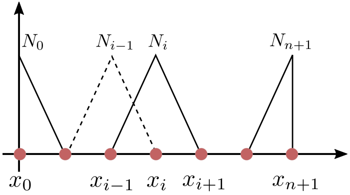

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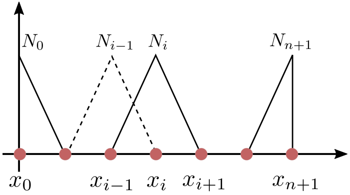

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This means u_S is piece-wise linear with $u_S(x_i) = u(x_i)$. We denote $u_S = u_I$ where u_I is the linear interpolation of the solution.

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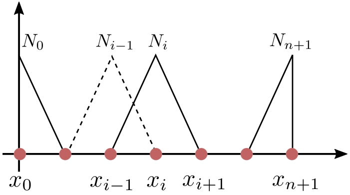
Lemma: The error of linear interpolation satisfies
$\|u - u_I\|_\infty \leq Ch^2 \|u''\|_\infty$.



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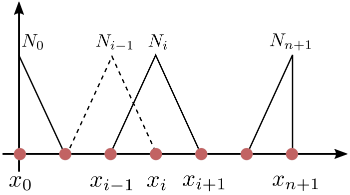

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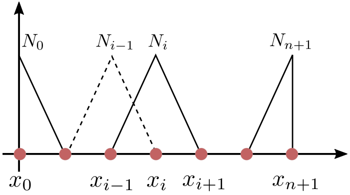

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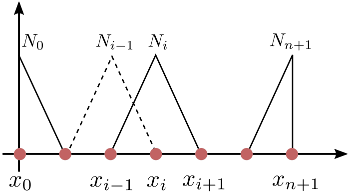

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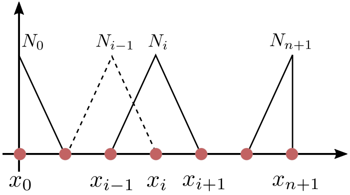

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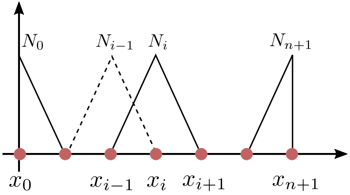

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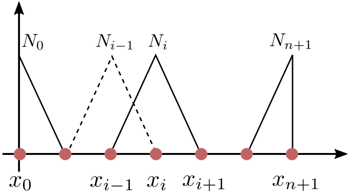

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Key is to design function spaces and study their interpolation theory, since this indicates FEM errors.