Name: ________________________

Final Practice Problems:

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Partial Differential Equations, 124

Directions: Please answer each question carefully. Be sure to show your work. If you have questions, please feel free to ask.

Problem 1:

Consider the Fourier Series expansion

$$\tilde{\phi}(x) = \sum_{k=-\infty}^{\infty} c_n \exp\left(ik\pi x/\ell\right),$$

(i) What are the Fourier Coefficients for $\phi(x) = x$ on the interval $[-1, 1]$? Hint: Integration by parts may be useful.

(ii) What are the Fourier Coefficients for $\phi(x) = 2\sin(-6\pi x) + 3\cos(4\pi x)$ on the interval $[-1, 1]$? Hint: Euler's Identity may be useful.

(iii) What are the Fourier Coefficients for $\phi(x) = -x + 2\cos(4\pi x) + 1$ on the interval $[-1, 1]$?

Problem 2:

Consider the Fourier Series expansion

$$\tilde{\phi}(x) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_n \cos\left(\frac{k\pi x}{\ell}\right) + \sum_{k=1}^{\infty} B_n \sin\left(\frac{k\pi x}{\ell}\right),$$

(i) What are the Fourier Coefficients for $\phi(x) = x - 1$ on the interval $[-1, 1]$? Hint: Integration by parts may be useful.

(ii) What are the Fourier Coefficients for $\phi(x) = \cos(-6\pi x)$ on the interval $[-1, 1]$?

(iii) What are the Fourier Coefficients for $\phi(x) = \sin(4\pi x) + x - 1$ on the interval $[-1, 1]$?

Problem 3:

Consider the DFT/IDFT with the following discrete Fourier Expansions for values on a lattice

$$u_m = \sum_{k=0}^{n-1} \hat{u}_k e^{i2\pi km/n}, \qquad \hat{u}_k = \frac{1}{n}\sum_{m=0}^{n-1} u_m e^{-i2\pi km/n}.$$

The function is sampled using $u_m = u(x_m)$ for points $x_m = mL/n$ on a latice in the interval $[0, L]$.

(i) What are the DFT Fourier Coefficients for the function $u(x_m) = \sin(4\pi x_m/L)$ sampled on the interval $[0, 2]$ when $n = 5$? Hint: Use Euler's Identity and $\exp(i2\pi\alpha) = 1$ for $\alpha \in \mathbb{Z}$ to compute $\hat{u}_k$ for this function.

(ii) What is the Fourier Interpolation of the function on the interval $[0, 2]$ when $n = 5$? Hint: Use the centered fourier expansion and Euler's Identity.

(iii) What are the DFT Fourier Coefficients for the function $u(x_m) = \sin(-6\pi x_m/L)$ sampled on the interval $[0, 2]$ when $n = 5$? Hint: Use Euler's Identity and compute the DFT of the samples of this function.

(iv) Show that the functions $\sin(-6\pi x_m/L)$ and $\sin(4\pi x_m/L)$ are the same on the lattice x_m on the interval $[0, 2]$ when $n = 5$ and therefore have the same Fourier Coefficients. Hint: Use Euler's Identity and that $\exp(i2\pi\alpha) = 1$ with $\alpha \in \mathbb{Z}$.

(v) What is the Fourier Interpolation of the function $u(x) = \sin(-6\pi x_m/L)$ on the interval $[0, 2]$ when $n = 5$? Is this the same or different from $v(x) = \sin(4\pi x_m/L)$? Why?

(vi) Use the Aliasing Formula to show the functions $u(x) = \cos(-6\pi x_m/L)$ and $v(x) = \cos(4\pi x_m/L)$ have the same Fourier Coefficients when sampled on the interval $[0, 2]$ when $n = 5$? Hint: Use the continuous Fourier Expansions for these functions obtained from Euler's Identity.

(vii) Use the Aliasing Formula to show the functions $u(x) = 1$ and $v(x) = \cos(10\pi x_m/L)$ have the same Fourier Coefficients when sampled on the interval $[0, 2]$ when $n = 5$? Hint: Use the continuous Fourier Expansions for these functions obtained from Euler's Identity.

(viii) What values do you obtain on the lattice u_m when $\hat{u}_k = 1/n$ for all k?

(ix) What function $u(x_m)$ do you obtain on the lattice when $n = 5$ with $\hat{u}_2 = \frac{1}{2}$, $\hat{u}_3 = \frac{1}{2}$, and zero otherwise?

Problem 4:
Consider the diffusion equation on the interval with the Dirichlet Boundary Conditions:

$$\begin{cases} u_t = \kappa u_{xx}, & 0 < x < \ell, t > 0 \\ u(0,t) = j(t), \quad u(\ell,t) = h(t), & t > 0 \\ u(x,0) = \phi(x), & 0 < x < \ell, t = 0. \end{cases}$$

Determine the solution when $\kappa = 1$, $\ell = \pi$, $\phi(x) = 0$ and $j(t) = t$, $\quad h(t) = -2t$. Hint: Integration by parts may be useful.

Problem 5:

Consider the wave equation on the interval with the Dirichlet Boundary Conditions:

$$\begin{cases} u_{tt} = \kappa u_{xx}, & 0 < x < \ell, t > 0 \\ u(0,t) = j(t), \quad u(\ell,t) = h(t), & t > 0 \\ u(x,0) = \phi(x), & 0 < x < \ell, t = 0 \\ u_t(x,0) = \psi(x), & 0 < x < \ell, t = 0. \end{cases}$$

Determine the solution when $\kappa = 1$, $\ell = \pi$, $\phi(x) = 5\cos(2x) + 3\cos(3x)$, $\psi(x) = 0$, and $j(t) = 0$, $h(t) = 0$.

Problem 6:

Consider the elliptic equation on a domain Ω with Dirichlet Boundary Conditions and $\Omega = [0, \ell] \times [0, \ell]$:

$$\begin{cases} \Delta u = f, & x \in \Omega \\ u = g, & x \in \partial\Omega. \end{cases}$$

Determine the solution when $\ell = \pi$, $f(x, y) = 2\sin(5x)\sin(y)$, $g(0, y) = g(x, \pi) = g(x, 0) = g(\pi, y) = 0$.

Problem 7:
Consider the elliptic equation with Dirichlet Boundary Conditions on the disk $\Omega = \{x \mid \|x\| \leq a\}$:

$$\begin{cases} \Delta u = f, & x \in \Omega \\ u = h, & x \in \partial\Omega. \end{cases}$$

(i) Determine the solution when $a = 3,\ f = 0,\ h(\theta) = 2\sin(3\theta) - 2\cos(\theta)$.

(ii) Determine the solution when $a = 2,\ f = 0,\ h(\theta) = -\cos(5\theta) + 1$.

Problem 8:
Consider the elliptic equation with Dirichlet Boundary Conditions on the wedge $\Omega = \{x = (r, \theta) \mid \alpha < \theta < \beta, \ \ r < a\}$:

$$\begin{cases} \Delta u = f, & x \in \Omega \\ u = 0, & \theta = \alpha \\ u = 0, & \theta = \beta \\ u = h, & r = a. \end{cases}$$

(i) Determine the solution when $a = 3$, $\alpha = 0$, $\beta = \pi/2$, $f = 0$, $h(\theta) = 2\sin(4\theta) - 3\sin(2\theta)$.

(ii) Determine the solution when $a = 2$, $\alpha = 0$, $\beta = \pi/4$, $f = 0$, $h(\theta) = -\sin(8\theta) + 2\sin(4\theta)$.

Problem 9:
Consider approximating the Diffusion Equation using the finite difference methods below.

$$\begin{cases} u_t = \kappa u_{xx}, & t > 0, x \in \mathbb{R} \\ u(x,0) = \phi(x), & t = 0, x \in \mathbb{R}. \end{cases}$$

Perform von Neumann Analysis $v_m^n = g^n \exp(im\theta)$ to analyze stability of the finite difference method. If there are no choices possible for $\delta t, \delta x$ then state the method is unstable.

(i) Determine the stability of the finite difference method

$$\frac{v_m^{n+1} - v_m^n}{\delta t} = \kappa \frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{\delta x^2}.$$

(ii) Determine the stability of the finite difference method

$$\frac{v_m^{n+1} - v_m^n}{\delta t} = \kappa \frac{v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}}{\delta x^2}.$$

(iii) Determine the stability of the finite difference method

$$\frac{v_m^{n+1} - v_m^n}{\delta t} = \kappa \frac{1}{2}\left[\frac{v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}}{\delta x^2} + \frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{\delta x^2}\right].$$

Problem 10:
Consider approximating the following hyperbolic PDE using the finite difference method below.

$$\begin{cases} u_t + au_x = 0, & t > 0, x \in \mathbb{R} \\ u(x,0) = \phi(x), & t = 0, x \in \mathbb{R}. \end{cases}$$

Perform von Neumann Analysis $v_m^n = g^n \exp(im\theta)$ to analyze stability of the finite difference method. If there are no choices possible for $\delta t, \delta x$ then state the method is unstable.

(i) Determine the stability of the finite difference method

$$\frac{v_m^{n+1} - v_m^n}{\delta t} = -a \frac{v_m^n - v_{m-1}^n}{\delta x}.$$

(ii) Determine the stability of the finite difference method

$$\frac{v_m^{n+1} - v_m^n}{\delta t} = -a \frac{v_{m+1}^n - v_{m-1}^n}{2\delta x}.$$

(iii) Determine the stability of the finite difference method

$$\frac{v_m^{n+1} - \frac{1}{2}\left(v_{m+1}^n + v_{m-1}^n\right)}{\delta t} = -a \frac{v_{m+1}^n - v_{m-1}^n}{2\delta x}.$$