Fourier Series Examples

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We give a few examples of the full Fourier series on the interval $[-\ell, \ell]$ of the form

$$\tilde{\phi}(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi x}{\ell}\right) + B_n \sin\left(\frac{n\pi x}{\ell}\right)\right)$$

where

$$A_n = \frac{1}{\ell}\int_{-\ell}^{\ell}\phi(x)\cos\left(\frac{n\pi x}{\ell}\right)$$

and

$$B_n = \frac{1}{\ell}\int_{-\ell}^{\ell}\phi(x)\sin\left(\frac{n\pi x}{\ell}\right).$$

The partial sums are given by

$$\tilde{\phi}_N(x) = \frac{1}{2}A_0 + \sum_{n=1}^{N}\left(A_n \cos\left(\frac{n\pi x}{\ell}\right) + B_n \sin\left(\frac{n\pi x}{\ell}\right)\right).$$

We also give examples of the Fourier cosine and sine series on the interval $[0, \ell]$, which correspond to even and odd extensions of the functions.

By adjusting the `num_modes=N` you can explore how the partial sums of these series converge or approximate the target function $\phi(x)$.

[1]:
```
# load needed packages
import numpy as np;
import matplotlib;
import matplotlib.pyplot as plt;

font = {'family' : 'sans-serif',
        'weight' : 'normal',
        'size'   : 10}

matplotlib.rc('font', **font)
```

Example: $\phi(x) = x$ on interval $[0, \ell]$. This has the Fourier cosine series representation

$$\tilde{\phi}(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right),$$

where

$$A_n = \begin{cases} 0, & \text{if n is even} \\ -\frac{4\ell}{n^2\pi^2}, & \text{if n is odd} \end{cases}$$

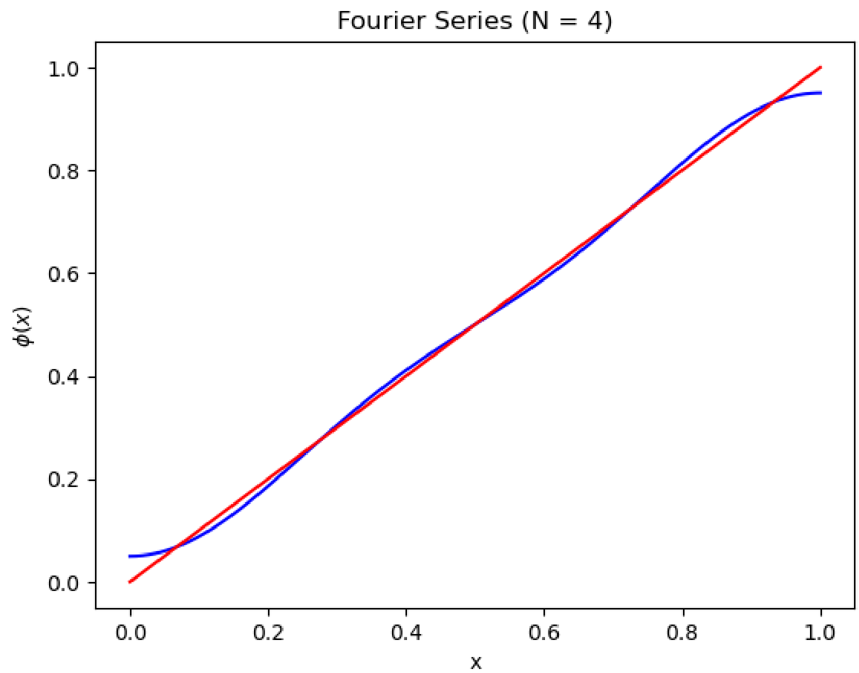
We show some partial sums below.

[2]:
```
ell = 1;
x = np.linspace(0,ell,1000);
y = 0*x;

num_modes = 4;
for n in np.arange(0,num_modes):
  if (n > 0) and (n % 2 == 1):
    An = -4.0*ell/(n*n*np.pi*np.pi);
  elif n == 0:
    An = 0.5*ell;
  else:
    An = 0;

  y += An*np.cos(n*np.pi*x/ell);

plt.figure("phi(x) = x");
plt.clf();
plt.plot(x,y,'b-');
plt.plot(x,x,'r-');
plt.xlabel('x');
plt.ylabel(r'$\phi(x)$');
plt.title('Fourier Series (N = %d)'%num_modes);
plt.draw();
```

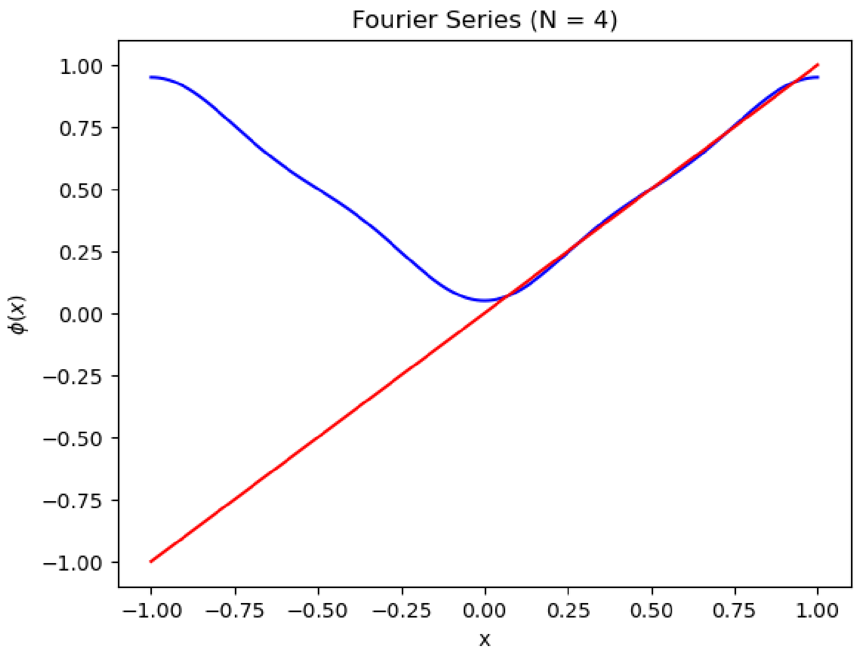


[3]:
```
ell = 1;
x = np.linspace(-ell,ell,1000);
y = 0*x;

num_modes = 4;
for n in np.arange(0,num_modes):
  if (n > 0) and (n % 2 == 1):
    An = -4.0*ell/(n*n*np.pi*np.pi);
  elif n == 0:
    An = 0.5*ell;
  else:
    An = 0;

  y += An*np.cos(n*np.pi*x/ell);

plt.figure("phi(x) = x");
plt.clf();
plt.plot(x,y,'b-');
plt.plot(x,x,'r-');
plt.xlabel('x');
plt.ylabel(r'$\phi(x)$');
plt.title('Fourier Series (N = %d)'%num_modes);
plt.draw();
```


Fourier Series (N = 4)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
$\phi(x)$
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
x

Example: $\phi(x) = 1$ on interval $[0, \ell]$. This has the Fourier sine series representation

$$\tilde{\phi}(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{\ell}\right),$$

where

$$B_n = \begin{cases} 0, & \text{if n is even} \\ \frac{4}{n\pi}, & \text{if n is odd} \end{cases}$$

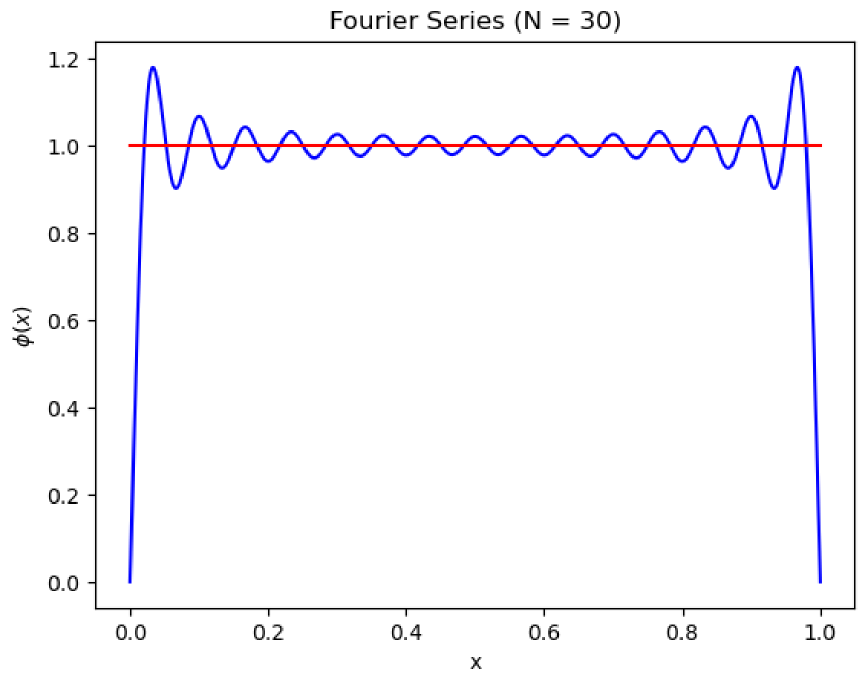
We show some partial sums below.

[4]:
```
ell = 1;
x = np.linspace(0,ell,1000);
y = 0*x;

num_modes = 30;
for n in np.arange(1,num_modes):
  if (n > 0) and (n % 2 == 1):
    Bn = 4.0/(n*np.pi);
  elif n == 0:
    Bn = 0;
  else:
    Bn = 0;

  y += Bn*np.sin(n*np.pi*x/ell);

plt.figure("phi(x) = x");
plt.clf();
plt.plot(x,y,'b-');
plt.plot(x,0*x + 1.0,'r-');
plt.xlabel('x');
plt.ylabel(r'$\phi(x)$');
plt.title('Fourier Series (N = %d)'%num_modes);
plt.draw();
```

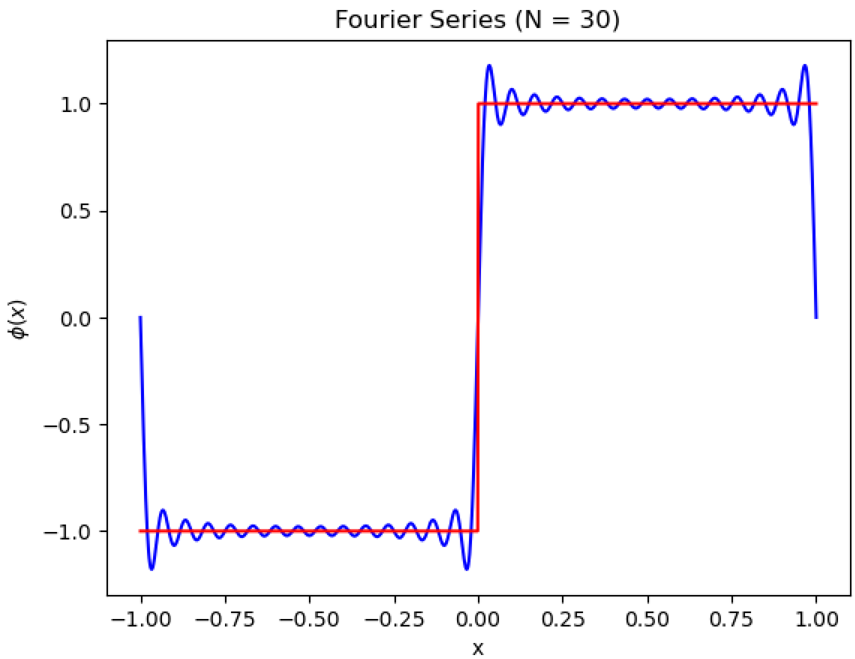


[5]:
```
ell = 1;
x = np.linspace(-ell,ell,1000);
y = 0*x;

num_modes = 30;
for n in np.arange(1,num_modes):
  if (n > 0) and (n % 2 == 1):
    Bn = 4.0/(n*np.pi);
  elif n == 0:
    Bn = 0;
  else:
    Bn = 0;

  y += Bn*np.sin(n*np.pi*x/ell);

plt.figure("phi(x) = x");
plt.clf();
plt.plot(x,y,'b-');
plt.plot(x,np.sign(x),'r-');
plt.xlabel('x');
plt.ylabel(r'$\phi(x)$');
plt.title('Fourier Series (N = %d)'%num_modes);
plt.draw();
```


Fourier Series (N = 30)
1.0
0.5
0.0
−0.5
−1.0
$\phi(x)$
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
x

Example Fourier transform of Dirac δ-function $\phi(x) = \delta(x - x_0)$, on interval $[-\ell, \ell]$ where $x_0 = 0$.

The Fourier series is given by

$$\tilde{\phi}(x) = \frac{1}{2\ell} + \frac{1}{\ell}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{\ell}\right),$$

where

$$A_n = \frac{1}{\ell}, \quad B_n = 0.$$

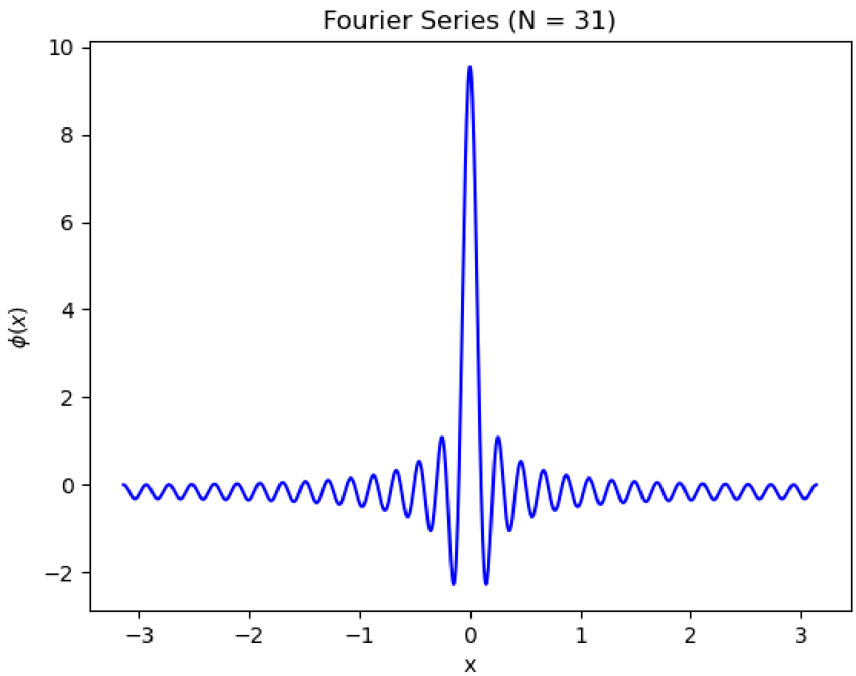
We show some partial sums below.

[6]:
```
ell = np.pi;
x = np.linspace(-ell,ell,1000);
y = 0*x;

num_modes = 31;
for n in np.arange(1,num_modes):
  An = 1.0/np.pi;
  if n == 0:
    An = 0.5*An;

  #y += An*np.cos(n*np.pi*x/ell) + Bn*np.sin(n*np.pi*x/ell);
  y += An*np.cos(n*np.pi*x/ell);

plt.figure("phi(x) = x");
plt.clf();
plt.plot(x,y,'b-');
#plt.plot(x,np.sign(x),'r-');
plt.xlabel('x');
plt.ylabel(r'$\phi(x)$');
plt.title('Fourier Series (N = %d)'%num_modes);
plt.draw();
```


Fourier Series (N = 31)
10
8
6
4
2
0
−2
$\phi(x)$
−3
−2
−1
0
1
2
3
x

```
[ ]:
```