Consider transforms for functions $u(x)$ sampled as $u_m = u(x_m)$ at lattice locations $x_m = mL/n$ on the interval $[0, L]$.

Discrete Fourier Transform (DFT)
The function is transformed to a frequency space representation using

$$\hat{u}_k = \mathcal{F}_k[\{u_m\}] = \frac{1}{n}\sum_{m=0}^{n-1} u_m e^{-i2\pi km/n}.$$

Inverse Discrete Fourier Transform (IDFT)
The function is reconstructed at the lattice sites x_m using the inverse transform (IDFT) given by

$$u_m = \mathcal{F}_m^{-1}[\{\hat{u}_k\}] = \sum_{k=0}^{n-1} \hat{u}_k e^{i2\pi km/n}.$$

We can also express this concisely by using vector notation $\hat{\mathbf{u}} = \{\hat{u}_k\}_{k=0}^{n-1}$ and $\mathbf{u} = \{\hat{u}_m\}_{m=0}^{n-1}$. The DFT and IDFT are then seen to be linear transforms with

$$\hat{\mathbf{u}} = \mathcal{F}[\mathbf{u}], \quad \mathbf{u} = \mathcal{F}^{-1}[\hat{\mathbf{u}}].$$

1. Consider the continuous Fourier transform for periodic functions on $[0, L]$.

$$u(x) = \sum_{k=-\infty}^{\infty} \mathbb{U}_k e^{i2\pi kx/L}, \quad \mathbb{U}_k = \frac{1}{L}\int_0^L u(x)e^{-i2\pi kx/L}dx.$$

Derive the aliasing formula

$$\hat{u}_k = \sum_{\alpha=-\infty}^{\infty} \mathbb{U}_{k+\alpha n}.$$

Hint: Use that when restricted to the lattice $x_m = mL/n$, we can not distinguish between the fourier modes $e^{i2\pi kx_m/L} = e^{i2\pi(k+\alpha n)x_m/L}$ for any $\alpha \in \mathbb{Z}$.

2. For approximating the action of the differential operators we would like to have representations for interpolating u between the lattice sites.

(a) Show that we can obtain a real-valued interpolation $\tilde{u}$ when n is odd by rewriting the IDFT and extending it off-lattice as

$$\tilde{u}(x) = \sum_{k=-N/2}^{N/2} \hat{u}_k e^{i2\pi kx/L}.$$

Here, $N = n-1$. Give a counter-example for a real-valued function u showing that we would obtain complex values at say location $\frac{1}{2}(x_1 + x_0)$ if we used simply

$$\tilde{u}(x) = \sum_{k=0}^{n-1} \hat{u}_k e^{i2\pi kx/L}.$$

(b) What happens when n is even? By using equivalence between the complex-valued and real-valued Fourier series what is the real-valued interpolation when n is even in terms of $\hat{u}_k$?

(c) Derive the fourier symbol for the Laplacian $\hat{\mathcal{L}}_k$ using the interpolation $\tilde{u}$ and $\Delta\tilde{u}$. The fourier symbol gives $\mathcal{F}_k[\mathcal{L}u] = \mathcal{F}_k[\Delta u] = \hat{\mathcal{L}}_k u_k$.

3. Consider the Poisson Partial Differential Equation (PDE) on the interval $[0, L]$,

$$\Delta u = -f, \;\; u(0) = u(L), \;\; \int_0^L u(x)dx = 0.$$

(a) Give the steps for obtaining a spectral approximation of the solution u to the Poisson PDE using the DFT and IDFT. In particular, how do you represent the solution in terms of the Fourier coefficients of $\hat{f}_k$ to obtain $\hat{u}_k$? How do you process f_m and obtain the solution u_m?

(b) (extra) Implement this spectral approximation method in python or your preferred programming language. Perform a convergence study by fixing a choice for a non-trivial function f and plotting on a log-scale how the error decreases as you increase n. Note a known solution can be manufactured by letting $f = \Delta u$ for some chosen function u satisfying the boundary and other conditions. Compare your findings with how an error model that scales like C/n^p for $p = 1, 2, 4$ by plotting it on the same graph.