Elliptic PDEs
We give some brief discussions on solution techniques for elliptic PDEs using Fourier Methods.

Poisson Problem
Consider the general elliptic poisson problem

$$\begin{cases} \Delta u & = & f, & \mathbf{x} \in \Omega \\ u(\mathbf{x}) & = & g(\mathbf{x}), & \mathbf{x} \in \partial\Omega. \end{cases}$$

Case of 2D Square: One Inhomogeneous Dirichlet Boundary and $f = 0$.
We consider the square in $\mathbb{R}^2$ with side-lengths $[0, \pi]$, giving the domain $\Omega = [0, \pi] \times [0, \pi]$. We also consider the case when all boundaries have $g = 0$ except for the right-most edge of the square. We take this to have the Dirichlet boundary condition $u(\pi, y) = q(y)$.

From separation of variables derivations in lecture, this has the following series expansion

$$u(x, y) = \sum_{m=1}^{\infty} A_m \sinh\left(\sqrt{m^2}x\right) \sin(my).$$

We obtain from integration

$$\begin{aligned} \int_0^\pi u(x, y) \sin(m^* y) dy &= \sum_{m=1}^{\infty} A_m \sinh\left(\sqrt{m^2}x\right) \cdot \\ & \quad \int_0^\pi \sin(my) \sin(m^* y) dy \\ &= \frac{\pi}{2} A_m \sinh\left(\sqrt{m^2}x\right). \end{aligned}$$

By applying this when $\mathbf{x} = (\pi, y)$, substituting the boundary condition for u, and solving for A_m, we obtain for the coefficients the solution

$$A_m = \frac{2}{\pi \sinh\left(\sqrt{m^2}\pi\right)} \int_0^\pi q(y) \sin(my) dy. \tag{1}$$

As a summary, we obtain our series solution to the PDE by performing for the boundary condition $q(y)$ the Fourier Transform given in equation 1. We also remark that a similar expression can be obtained when the inhomogeneous boundary term is on any of the other edges of the square. One way to readily obtain these expressions is to do a change of variable. For example, suppose we had $u(x, \pi) = q(x)$, then doing the change of variable to $x' = y, y' = x$ in the series above would yield the solution. A similar approach can be used for the faces when $x = 0$. Also, if we had a combination of inhomoegeneous boundary conditions these also can be obtained by sovling for each separately u_1, u_2 and then successively applying the superposition principle to obtain the solution $u = u_1 + u_2$. These are a few ways the invariance and linear properties of the Laplacian Δ can be useful in obtaining practical solutions.

Case of 2D Square: Homogeneous Dirichlet Boundary Conditions and $f \neq 0$. We consider the square in $\mathbb{R}^2$ with side-lengths $[0, \pi]$, giving the domain $\Omega = [0, \pi] \times [0, \pi]$. We also consider the case when all boundaries have $g = 0$. We address the case when there is a non-zero f source term on the right-hand side.

From our eigenfunction derivations in lecture, this leads to the following fourier series expansion

$$u(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{m,n} \sin(mx) \sin(ny).$$

We can similarly obtain an expansion for f as

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{m,n} \sin(mx) \sin(ny).$$

By direct differentiation we formally obtain

$$\begin{aligned} f &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{m,n} \sin(mx) \sin(ny) \\ &= \Delta u(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -\lambda_{m,n} A_{m,n} \sin(mx) \sin(ny), \end{aligned}$$

were $\lambda_{m,n} = m^2 + n^2$. We obtain the solution by using the uniqueness of the Fourier expansions. This gives the solution for $A_{m,n}$ in terms of $F_{m,n}$ as

$$A_{m,n} = \frac{-F_{m,n}}{\lambda_{m,n}}. \tag{2}$$

This provides the solution to the Poisson problem using the series expansion for u.

We remark we can also derive the expression for obtaining $F_{m,n}$ as follows from the expansion

$$\begin{aligned} \int_0^{\pi} \int_0^{\pi} f(x, y) \sin(m^* y) \sin(n^* y) dx dy &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{m,n} \sin(mx) \sin(ny) \sin(m^* x) \sin(n^* y) dx dy \\ &= \frac{\pi^2}{4} F_{m^*, n^*}. \end{aligned}$$

This gives

$$F_{m,n} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} f(x, y) \sin(mx) \sin(ny) dx dy.$$

By applying this to a given function $f(x, t)$ we obtain the solution to the poisson problem with Homogeneous Dirichlet Boundary conditions. This can be combined with our other solution techniques by using the superposition principle.

Case of 3D Cube: One Inhomogeneous Dirichlet Boundary and $f = 0$.
We consider the cube in $\mathbb{R}^3$ with side-lengths $[0,\pi]$, giving the domain $\Omega = [0,\pi]\times[0,\pi]\times[0,\pi]$. We also consider the case when all boundaries have $g = 0$ except for the right-most face of the cube. We take this to have the Dirichlet boundary condition $u(\pi,y,z) = q(y,z)$.

From separation of variables derivations in lecture, this has the following series expansion

$$u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n}\sinh\left(\sqrt{m^2+n^2}x\right)\sin(my)\sin(nz).$$

We obtain from integration

$$\begin{aligned}\int_0^{\pi}\int_0^{\pi} u(x,y,z)\sin(m^*y)\sin(n^*z)dydz &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n}\sinh\left(\sqrt{m^2+n^2}x\right)\cdot \\ &\quad \int_0^{\pi}\int_0^{\pi}\sin(my)\sin(nz).\sin(m^*y)\sin(n^*z)dydz \\ &= \frac{\pi^2}{4}A_{m^*,n^*}\sinh\left(\sqrt{m^2+n^2}x\right).\end{aligned}$$

By applying this when $\mathbf{x} = (\pi,y,z)$, substituting the boundary condition for u, and solving for $A_{m,n}$ we obtain for the coefficients the solution

$$A_{m,n} = \frac{4}{\pi^2\sinh\left(\sqrt{m^2+n^2}\pi\right)}\int_0^{\pi}\int_0^{\pi} q(y,z)\sin(my)\sin(nz)dydz. \quad (3)$$

As a summary, we obtain our series solution to the PDE by performing for the boundary condition $q(y,x)$ the 2D Fourier Transform given in equation 3. We also remark that a similar expression can be obtained when the inhomogeneous boundary term is on any of the other faces of the cube. One way to readily obtain these expressions is to do a change of variable. For example, suppose we had $u(x,\pi,z) = q(x,z)$, then doing the change of variable to $x' = y, y' = x, z' = z$ in the series above would yield the solution. A similar approach can be used for the faces when $x = 0$. These are a few ways the invariance properties of the Laplacian Δ can be useful in obtaining practical solutions.

Case of 3D Cube: Homogeneous Dirichlet Boundary Conditions and $f \neq 0$.
We consider the cube in $\mathbb{R}^3$ with side-lengths $[0,\pi]$, giving the domain $\Omega = [0,\pi]\times[0,\pi]\times[0,\pi]$. We also consider the case when all boundaries have $g = 0$. We address the case when there is a non-zero f source term on the right-hand side.

From eigenfunction derivations in lecture, this has the following series expansion

$$u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} A_{m,n,p}\sin(mx)\sin(ny)\sin(pz).$$

We can similarly obtain an expansion for f as

$$f(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} F_{m,n,p}\sin(mx)\sin(ny)\sin(pz).$$

By direct differentiation we formally obtain

$$\begin{aligned} f(x,y,z) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} F_{m,n,p}\sin(mx)\sin(ny)\sin(pz) \\ &= \Delta u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} -\lambda_{m,n,p}A_{m,n,p}\sin(mx)\sin(ny)\sin(pz), \end{aligned}$$

were $\lambda_{m,n,p} = m^2 + n^2 + p^2$. We can use uniqueness of the Fourier expansions to obtain the solution for $A_{m,n,p}$ given by

$$A_{m,n,p} = \frac{-F_{m,n,p}}{\lambda_{m,n,p}}. \tag{4}$$

This provides the solution to the Poisson problem using the series expansion for u.

We remark we can also derive the expression for obtaining $F_{m,n,p}$ as follows from the expansion

$$\begin{aligned} &\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} f(x,y,z)\sin(m^*y)\sin(n^*y)\sin(p^*z)dxdydz \\ &\quad = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} F_{m,n,p}\cdot \\ &\qquad \int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\sin(mx)\sin(ny)\sin(pz)\sin(m^*x)\sin(n^*y)\sin(p^*z)dxdydz \\ &\quad = \frac{\pi^3}{2^3}F_{m^*,n^*,p^*}. \end{aligned}$$

This gives

$$F_{m,n,p} = \frac{2^3}{\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} f(x,y,z)\sin(mx)\sin(ny)\sin(pz)dxdydz.$$

Summary. These techniques provide a few methods for using fourier approaches for solving elliptic PDEs on squares and cubes. These methods also extend in a similar manner to higher dimensions. The results can also be used for numerical approximation of solutions to these PDEs. One approach would be to truncate the series expansions to a finite number of terms N and then replacing the fourier transforms by performing approximations to the integrals, such as using quadratures like the trapezoidal method we discussed in previous lectures. This would yield approaches closely related to *discrete fourier transforms* and *spectral numerical methods.* The solution techniques mentioned here can be extended in several ways, including performing additional analysis and derivations to obtain similar series expansion solution in other geometries, such as disks, wedges, and annuli. We will discuss these and other related topics in upcoming lectures.