Fourier Methods

We observed in separation of variables that it can be useful to expand functions by representing them as a linear combination $\phi(x) = \sum_n A_n X_n(x)$ of the eigenfunctions X_n of the differential operator $\mathcal{L}$ of the PDE. In the case that $\mathcal{L} = -d^2/dx^2$ with boundary conditions that are Dirichlet, Neumann, or Periodic, the eigenfunctions are trigonometric sine or cosine functions. This property provides useful approaches for representing functions for solving many PDEs, analysis, and other tasks. The eigenfunction expansions in this case are referred to as *Fourier Series*. We now develop some of the related results, many of which also hold more generally for eigenfunction expansions for other $\mathcal{L}$ and boundary conditions satisfying symmetry conditions.

Fourier Series (formulation I): Consider functions on $[-\ell, \ell]$ and the series expansion

$$\tilde{\phi}(x) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_n \cos\left(\frac{k\pi x}{\ell}\right) + \sum_{k=1}^{\infty} B_n \sin\left(\frac{k\pi x}{\ell}\right),$$

where the coefficients are given by

$$A_k = \frac{1}{\ell}\int_{-\ell}^{\ell} \phi(x)\cos\left(\frac{k\pi x}{\ell}\right), \qquad B_k = \frac{1}{\ell}\int_{-\ell}^{\ell} \phi(x)\sin\left(\frac{k\pi x}{\ell}\right).$$

We will have to establish the conditions under which $\tilde{\phi}$ reconstructs the function ϕ. This expansion also can be expressed using complex variables using the Euler Identity $\exp(i\theta) = \cos(\theta) + i\sin(\theta)$. This gives the following equivalent series.

Fourier Series (formulation II): Consider functions on $[-\ell, \ell]$ and the series expansion

$$\tilde{\phi}(x) = \sum_{k=-\infty}^{\infty} c_n \exp\left(ik\pi x/\ell\right),$$

where the coefficients are given by

$$c_k = \frac{1}{2\ell}\int_{-\ell}^{\ell} \phi(x)\exp\left(-ik\pi x/\ell\right).$$

Conversion Between Formulations: We can relate the coefficients between these series by using

$$A_k = c_k + \overline{c}_k, \qquad B_k = -i\left(c_k - \overline{c}_k\right),$$

and

$$c_k = \frac{1}{2}A_k - i\frac{1}{2}B_k.$$

Even, Odd, and Periodic Extensions: Consider a function f and using its evaluation on the interval $[-\ell, \ell]$. If we expand it as a Fourier series we obtain $\tilde{\phi}(x)$ which gives a *periodic extension* of f to the whole real-line $\mathbb{R}$, so $\tilde{\phi}(x + 2n\ell) = \tilde{\phi}(x)$ for any $n \in \mathbb{Z}$. We similarly can consider a function f and use its evaluations on the interval $[0, \ell]$. If we expand it as a Fourier cosine series on $[0, \ell]$ with $A_n = \frac{2}{\ell}\int_0^\ell f(x)\cos\left(\frac{m\pi x}{\ell}\right)$ and $B_n = 0$, we obtain $\tilde{\phi}$ which is an *even extension* $\tilde{\phi}(-x) = \tilde{\phi}(x)$ on the interval $x \in [-\ell, \ell]$ and this further extends to be periodic on the whole real line $\mathbb{R}$. If we expand it as a Fourier sine series on $[0, \ell]$ with $A_n = 0$ and $B_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{m\pi x}{\ell}\right)$, we obtain $\tilde{\phi}$ which is an *odd extension* $\tilde{\phi}(-x) = -\tilde{\phi}(x)$ on the interval $x \in [-\ell, \ell]$ and this further extends to be periodic on the whole real line $\mathbb{R}$.

Fourier Transform: We define the *Fourier Transform* as the operation that gives us the coefficients from a function $A_k, B_k = \mathcal{F}_k[\phi]$ or equivalently $c_k = \mathcal{F}_k[\phi]$. Collectively, this gives $\{A_k, B_k\}_{k=0}^{\infty} = \mathcal{F}[\phi]$ and $\{c_k\}_{k=-\infty}^{\infty} = \mathcal{F}[\phi]$. Here, we take $B_0 = 0$. The *Inverse Fourier Transform* is the operation that reconstructs the function from the coefficients to yield $\tilde{\phi} = \mathcal{F}^{-1}[\{A_k, B_k\}_{k=0}^{\infty}]$ and equivalently $\tilde{\phi} = \mathcal{F}^{-1}[\{c_k\}_{k=-\infty}^{\infty}]$.

Example: Consider the function $\phi(x) = x$ on $x \in [-\ell, \ell]$. The Fourier coefficients are then given by

$$\begin{aligned} A_0 &= \frac{1}{\ell}\int_{-\ell}^{\ell} x dx = 0, \quad A_n = \frac{1}{\ell}\int_{-\ell}^{\ell} x\cos\left(\frac{m\pi x}{\ell}\right)dx = 0, \\ B_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} x\sin\left(\frac{m\pi x}{\ell}\right)dx = \frac{1}{\ell}\left[-x\left(\frac{m\pi}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right)\right]_{x=-\ell}^{x=\ell} = (-1)^{m+1}\,\frac{2\ell}{m\pi}. \end{aligned}$$

This gives the Fourier series expansion

$$\tilde{\phi}(x) = \sum_{n=1}^{\infty} (-1)^{m+1}\,\frac{2\ell}{m\pi}\sin\left(\frac{m\pi x}{\ell}\right).$$

Example: Consider the function $\phi(x) = 1$ on $x \in [0, \ell]$ and the Fourier sine series expansion. The Fourier coefficients are then given by

$$\begin{aligned} A_n &= 0 \\ B_n &= \frac{2}{\ell}\int_{-\ell}^{\ell} 1\sin\left(\frac{m\pi x}{\ell}\right)dx = \frac{2}{\ell}\left[-\left(\frac{\ell}{m\pi}\right)\cos\left(\frac{m\pi x}{\ell}\right)\right]_{x=0}^{x=\ell} = \left(\frac{2}{m\pi}\right)(1 - (-1)^m). \end{aligned}$$

The term $(1 - (-1)^m)$ evaluates to zero when m is even, so $B_{2j} = 0$, and evaluates to 2 when m is odd, so $B_{2j+1} = 4/((2j+1)\pi)$. This gives the Fourier series expansion

$$\tilde{\phi}(x) = \sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi}\sin\left(\frac{m\pi x}{\ell}\right) = \sum_{m\,:\text{odd}} \frac{4}{m\pi}\sin\left(\frac{m\pi x}{\ell}\right).$$

L^2-Analysis: It is convenient to organize our analysis using the notion of Lebesgue integrals and functions that are square integrable. We define the L^2-*inner product* of two functions f, g as the following operation

$$(f,g) = \int_a^b f(x)\overline{g(x)}dx.$$

We say that a function is *square integrable* in the L^2-sense, whenever the integral in (f, f) is defined and evaluates to a finite value. More succinctly, we say that f is L^2 and denote this by $f \in L^2[a, b]$.

The L^2-*norm* of a function f is defined as

$$\|f\|_{L^2} = \sqrt{(f,f)} = \left(\int_a^b f(x)\overline{f(x}dx\right)^{1/2}.$$

Intuitively, we can think of the L^2-inner product and L^2-norm as a way to generalize the dot-product and norm from linear algebra, where we would have $(u, v) = u\dot{v} = \sum_{k=1}^n u_k v_k$. In the L^2-inner product (f, g) the vector u, v with component index $k \in \mathbb{Z}$ is replaced with the functions f, g with parameter $x \in \mathbb{R}$. The sum over products of the discrete components is now replaced with an integral over the continuum of products of the functions.

This also allows us to generalize many of the concepts from linear algebra and related geometry. We define two functions to be *orthogonal* in the L^2-sense if

$$(f,g)_{L^2} = 0.$$

We also have for L^2-inner-products the *Cauchy-Swartz Inequality*

$$(f,g) \leq \|f\|_{L^2}\|g\|_{L^2}.$$

Least-Squares Approximation: To demonstrate the utility of these concepts and approaches, consider the problem of approximating a function $f(x)$ using least-squares. For an expansion $\phi_N(x) = \sum_{n=1}^N A_n X_n(x)$, this requires finding a collection of coefficients A_n so that $\|\phi_N - f\|_{L^2}^2$ is as small as possible. This requires solving the problem

$$\min_{A_n} \|\phi_N - f\|_{L^2}^2.$$

We will consider here the case when the expansion functions $\{X_n\}$ are mutually orthogonal, $(X_n, X_k) = 0$ when $n \neq k$. We can express this as $(X_n, X_k) = (X_k, X_k)\,\delta_{n,k} = \|X_k\|^2\,\delta_{n,k}$. The $\delta_{n,k}$ denotes the Kronecker δ-function which is one when the indices agree $n = k$ and zero when $n \neq k$. We can solve the least-squares problem by differentiating in A_k and setting the derivative to zero to find the critical points. This yields

$$\begin{aligned}
\frac{\partial}{\partial A_k}\|\phi_N - f\|_{L^2}^2 &= \frac{\partial}{\partial A_k}(\phi_N - f, \phi_N - f) = \frac{\partial}{\partial A_k}(\phi_N,\phi_N) - \frac{\partial}{\partial A_k}2(f,\phi_N) + \frac{\partial}{\partial A_k}(f,f) \\
&= \sum_{n=1}^N\sum_{n'=1}^N \frac{\partial}{\partial A_k}A_n A_{n'}(X_n, X_{n'}) - 2\sum_{n=1}^N \frac{\partial}{\partial A_k}A_n(f, X_n) \\
&= \sum_{n=1}^N 2A_n(X_n, X_k) - 2(f, X_k) \\
&= 2A_k(X_k, X_k) - 2(f, X_k) = 0.
\end{aligned}$$

This gives the solution

$$A_k = \frac{(f, X_k)}{(X_k, X_k)}.$$

This contributes in the expansion ϕ_N the term

$$A_n X_n(x) = (f, \tilde{X}_n)\tilde{X}_n, \text{ where } \tilde{X}_n = \frac{X_n}{\|X_n\|}.$$

From the inner-products, we see the contribution of the n^{th} term in the expansion can be interpreted geometrically as the projection of the function f onto the unit function $\tilde{X}_n$. Note how the form of the coefficient A_n gives the normalization terms, so that we can express things in terms of $\tilde{X}_n$ with $\|\tilde{X}_n\| = 1$. Hence the least-square fit projects orthogonally the function f onto the hyper-plane spanned by linearly combining the functions $\{X_n\}$. The L^2-analysis gives a useful way to generalize a lot of the techniques and intuition from the finite dimensional setting.

Convergence of Fourier Series: We consider three different types of convergence

(i) pointwise convergence

(ii) uniform convergence

(iii) L^2-convergence.

We give below more details on each of these forms of convergence. We remark that which form of convergence to use will depend on the circumstances and what types of information is needed about the system.

Pointwise Convergence: We say a sequence of functions ϕ_N *pointwise convergences* to a function f if for each $x \in (a, b)$ we have

$$|\phi_N(x) - f(x)| \to 0, \quad \text{as } N \to \infty.$$

Uniform Convergence: A sequence of functions ϕ_N *uniformly convergences* to a function f if on $[a, b]$ we have

$$\sup_{x \in [a,b]} |\phi_N(x) - f(x)| \to 0, \quad \text{as } N \to \infty.$$

L^2-Convergence: In the case of L^2-*convergences*, we have the sequence ϕ_N converges to function f on $[a, b]$ in the L^2-sense

$$\|\phi_N - f\|_{L^2}^2 \to 0, \quad \text{as } N \to \infty.$$

We now state some results concerning when Fourier series converge in these different ways. Consider a target function $f(x)$ and the Fourier series approximation $\phi_N = \sum_{n=1}^N A_n X_n(x)$. Let $\bar{f}(x) = \frac{1}{2}[f(x-) + f(x+)]$, which gives a function having the average of the evaluations around points of discontinuity. In particular, $\bar{f}$ is the average of the left $x-$ and right $x+$ limits of f at x.

Theorem: Uniform Convergence. The Fourier series has **uniform convergence** on $[a, b]$ when the following holds

(i) $f(x), f'(x), f''(x)$ exist and are continuous on $[a, b]$.

(ii) $f(x)$ satisfies the same boundary conditions (BCs) as $X_n(x)$.

Theorem: Pointwise Convergence.

- If $\bar{f}(x)$ is continuous and $\bar{f}'(x)$ exists then the classical Fourier series (sine,cosine) has $\phi_N \to \bar{f}$ with **pointwise convergence**.
- If $f(x)$ is only piecewise continuous, then if $f'(x)$ exists and is piecewise continuous the Fourier series has $\phi_N \to \bar{f}$ with **pointwise convergence**.

Theorem: L^2-Convergence. The Fourier series has **L^2-convergence** for any $f \in L^2[a, b]$. In other words, provided that f is a Lebesgue measurable function and has finite L^2-norm

$$\|f\|_2^2 = \int_a^b |f(x)|^2 dx < \infty.$$

We can see that the conditions become progressively less stringent as we move from asking for uniform convergence to only pointwise convergence to L^2-convergence. The L^2 theory is also closely related to the concept of *weak convergence*.

Weak Convergence and Approximation: It can be shown that convergence in L^2 implies that $(\phi_N, w)_{L^2} \to (f, w)_{L^2}$ for all $w \in L^2$. This follows from the Cauchy-Swartz Inequality, since $(\phi_N - f, w)_{L^2} \leq \|\phi_N - f\| \, \|w\|$ and $\|\phi_N - f\| \to 0$. We can think about w serving the role of a *test function* that probes the properties of a function g and reports its results as a scalar value r as $r = (g, w)_{L^2}$. With this in mind, the L^2-convergence $(\phi_N, w)_{L^2} \to (f, w)_{L^2}$ can be viewed as saying that for each test function w the function properties characterized for ϕ_N by $r_N = (\phi_N, w)_{L^2}$ converge to the value $r = (f, w)_{L^2}$. In this way, we obtain in the limit from ϕ_N the same properties as when probing the function f, so $r_N \to r$. In this sense, we have a weak notion of convergence of functions which turns out to be very useful for analysis and in many practical settings, including numerical approximation.

Green's Identities: In many of the derivations we will use integration by parts which follow a similar pattern. These also have analogues in higher dimensions. To organize our calculations along these lines and to help in identifying these commonalities, we will make use of the following two identities.

Green's First Identity:

$$\int_a^b uv_{xx} - u_x v_x dx = [uv_x]_{x=a}^{x=b}.$$

Green's Second Identity:

$$\int_a^b uv_{xx} - u_{xx} v dx = [uv_x - u_x v]_{x=a}^{x=b}.$$

Solution of Parabolic PDEs with Homogeneous Dirichlet Boundary Conditions
Consider

$$\begin{cases} u_t & = & \kappa u_{xx}, & t > 0,\ 0 < x < \ell \\ u(0,t) & = & u(\ell,t) = 0, & t > 0 \\ u(x,0) & = & \phi(x), & t = 0. \end{cases} \tag{1}$$

We aim to construct a solution by seeking at each time t a representation in terms of the Fourier series expansion

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{\ell}\right).$$

Formally, differentiating through the series using $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial x^2}$ yields the ODE

$$u_n'(t) = -\left(\frac{n\pi}{\ell}\right)^2 \kappa u_n = -\kappa\lambda_n u_n.$$

In general, as we shall see in other examples, we do need to be cautious about differentiating series, since this may not always yield a valid result, especially when the series expansion for the derivative does not converge uniformly. In the ODE, we are using that the differential operator is $\mathcal{L} = -\frac{\partial^2}{\partial x^2}$ and $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$ is the eigenvalue of $\mathcal{L}X_n = \lambda_n X_n$. The eigenfunctions are $X_n = \sin\left(\sqrt{\lambda_n} x\right)$. The solution to the ODE is given by

$$u_n(t) = u_n(0) \exp\left(-\kappa\lambda_n t\right).$$

To obtain the solution, we need to choose $u_n(0)$ so that we match the initial condition at time $t = 0$

$$u(x,0) = \sum_{n=1}^{\infty} u_n(0) \sin\left(\frac{n\pi x}{\ell}\right) = \phi(x).$$

This gives that

$$u_n(0) = \frac{2}{\ell}\int_0^{\ell} \phi(x) \sin\left(\frac{n\pi x}{\ell}\right).$$

This gives the Fourier series representation for the solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(0) \exp\left(-\kappa\lambda_n t\right) \sin\left(\frac{n\pi x}{\ell}\right).$$

Alternative Derivation without Series Differentiation: We now show how we can avoid differentiation of the series and obtain more reliable solution techniques. Consider the Fourier expansions of the function u and its derivatives u_t, u_x, u_{xx},

$$\begin{aligned} u_n(t) &= \frac{2}{\ell}\int_{-\ell}^{\ell} u(x,t) \sin\left(\frac{n\pi x}{\ell}\right) dx, \quad v_n(t) = \frac{2}{\ell}\int_{-\ell}^{\ell} \frac{\partial u}{\partial t} \sin\left(\frac{n\pi x}{\ell}\right) dx, \\ q_n(t) &= \frac{2}{\ell}\int_{-\ell}^{\ell} \frac{\partial u}{\partial x} \sin\left(\frac{n\pi x}{\ell}\right) dx, \quad w_n(t) = \frac{2}{\ell}\int_{-\ell}^{\ell} \frac{\partial^2 u}{\partial x^2} \sin\left(\frac{n\pi x}{\ell}\right) dx. \end{aligned}$$

Since $u_t = \kappa u_{xx}$ and the coefficients of Fourier series are unique we have $v_n = \kappa w_n$. We now use that we can expression u_n' as

$$u_n'(t) = \frac{\partial}{\partial t}\frac{2}{\ell}\int_{-\ell}^{\ell} u(x,t)\sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{2}{\ell}\int_{-\ell}^{\ell}\frac{\partial u}{\partial t}\sin\left(\frac{n\pi x}{\ell}\right)dx = v_n(t).$$

This yields that $u_n'(t) = v_n(t)$. Using that $v_n(t) = \kappa w_n(t)$ we obtain $u_n'(t) = \kappa w_n(t)$. Now we use integration by parts twice to relate $w_n(t)$ to $u_n(t)$, which is equivalent to Green's second identity. Integration by parts yields

$$\begin{aligned} w_n &= \frac{2}{\ell}\int_{-\ell}^{\ell}\frac{\partial^2 u}{\partial x^2}\sin\left(\frac{n\pi x}{\ell}\right)dx \\ &= \frac{2}{\ell}\int_{-\ell}^{\ell} u\frac{\partial^2}{\partial x^2}\sin\left(\frac{n\pi x}{\ell}\right)dx + \underbrace{\frac{2}{\ell}\left[u\frac{\partial}{\partial x}\sin\left(\frac{n\pi x}{\ell}\right) - u_x\sin\left(\frac{n\pi x}{\ell}\right)\right]_{x=0}^{x=\ell}}_{=0} \\ &= -\left(\frac{n\pi}{\ell}\right)^2\frac{2}{\ell}\int_{-\ell}^{\ell} u\sin\left(\frac{n\pi x}{\ell}\right)dx = -\left(\frac{n\pi}{\ell}\right)^2 u_n = -\lambda_n u_n. \end{aligned}$$

We used here in the integration by parts that the boundary terms $u(0,t) = u(\ell,t) = 0$ and similarly the sine terms vanish. This yields $w_n = -\lambda_n u_n$ and combining this with $v_n = \kappa w_n$ we obtain the ODE

$$u_n'(t) = -\kappa\lambda_n u_n,$$

with solution $u_n(t) = u_n(0)\exp(-\kappa\lambda_n t)$. This yields the series representation for the solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(0)\exp(-\kappa\lambda_n t)\sin\left(\frac{n\pi x}{\ell}\right).$$

We emphasize that while the solution is the same as we obtained before, it was derived without differentiating through the series. As we will see for many PDEs, this alternative approach will be crucial for obtaining viable solution techniques.

Solution of Parabolic PDEs with Inhomogeneous Dirichlet Boundary Conditions
Consider

$$\begin{cases} u_t = \kappa u_{xx}, & t > 0,\ 0 < x < \ell \\ u(0,t) = h(t),\ u(\ell,t) = j(t), & t > 0 \\ u(x,0) = \phi(x), & t = 0. \end{cases} \tag{2}$$

We aim to construct a solution using a representation in terms of a Fourier series expansion of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\left(\frac{n\pi x}{\ell}\right).$$

In this case, we can already see such an expansion would not technically satisfy pointwise the inhomogeneous boundary conditions, since when evaluated at $x = 0$ and $x = \ell$ we always get the sine term is zero. However, we do know that the Fourier series can still converge to approximate functions in the weaker sense of L^2-convergence (which by design does not rely on pointwise evaluations of the functions). In this case, we are seeking a *weak solution* u of the inhomogeneous PDE in the sense

$$\begin{cases} (u_t, w) & = & (\kappa u_{xx}, w), \quad t > 0,\ 0 < x < \ell,\ \forall w \in \mathcal{W} \\ (u(x,0), w) & = & (\phi(x), w), \quad t = 0,\ \forall w \in \mathcal{W}. \end{cases} \tag{3}$$

In the notation, we are using the L^2-inner product $(f,g) = (f,g)_{L^2} = \int f(x)g(x)dx$. The $\mathcal{W}$ refers to a space of *test functions*, such as C_0^∞. The C_0^∞ here denotes the collection of all infinitely continuously differentiable functions f with compact support in $[-\ell, \ell]$. Compact support means for a given function $f(x)$ there exists some compact set $\mathcal{K} \subset [-\ell, \ell]$ outside of which the function vanishes, that is $x \notin \mathcal{K}$ then $f(x) = 0$. For the boundary conditions, we also will only require the weaker conditions $\lim_{x\to 0} u(x,t) = h(t)$ and $\lim_{x\to\ell} u(x,t) = j(t)$ for $t > 0$. The boundary conditions also can be formulated more abstractly to obtain even weaker conditions, but we will use these limit conditions for now.

Now with this background and motivations in mind, we will derive a solution representation for the PDE using Fourier series. Since we are seeking a weak solution, we will use our techniques based on integration instead of trying to differentiate the series, which we already see would be problematic for this PDE given the inhomogeneous boundary conditions.

Consider the Fourier expansions of the function u and its derivatives u_t, u_x, u_{xx},

$$\begin{aligned} u_n(t) &= \frac{2}{\ell}\int_{-\ell}^{\ell} u(x,t)\sin\left(\frac{n\pi x}{\ell}\right)dx, \quad v_n(t) = \frac{2}{\ell}\int_{-\ell}^{\ell} \frac{\partial u}{\partial t}\sin\left(\frac{n\pi x}{\ell}\right)dx, \\ q_n(t) &= \frac{2}{\ell}\int_{-\ell}^{\ell} \frac{\partial u}{\partial x}\sin\left(\frac{n\pi x}{\ell}\right)dx, \quad w_n(t) = \frac{2}{\ell}\int_{-\ell}^{\ell} \frac{\partial^2 u}{\partial x^2}\sin\left(\frac{n\pi x}{\ell}\right)dx. \end{aligned}$$

Since $u_t = \kappa u_{xx}$ and the coefficients of Fourier series are unique, we have $v_n = \kappa w_n$. We now use that we can expression u_n' as

$$u_n'(t) = \frac{\partial}{\partial t}\frac{2}{\ell}\int_{-\ell}^{\ell} u(x,t)\sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{2}{\ell}\int_{-\ell}^{\ell}\frac{\partial u}{\partial t}\sin\left(\frac{n\pi x}{\ell}\right)dx = v_n(t).$$

This yields that $u_n'(t) = v_n(t)$. Using that $v_n(t) = \kappa w_n(t)$ we obtain $u_n'(t) = \kappa w_n(t)$. Now we use integration by parts twice to relate $w_n(t)$ to $u_n(t)$, which is equivalent to Green's second identity. Integration by parts yields

$$\begin{aligned} w_n &= \frac{2}{\ell}\int_{-\ell}^{\ell} \frac{\partial^2 u}{\partial x^2} \sin\left(\frac{n\pi x}{\ell}\right) dx \\ &= \frac{2}{\ell}\int_{-\ell}^{\ell} u\frac{\partial^2}{\partial x^2} \sin\left(\frac{n\pi x}{\ell}\right) dx + \underbrace{\frac{2}{\ell}\left[u\frac{\partial}{\partial x}\sin\left(\frac{n\pi x}{\ell}\right) - u_x \sin\left(\frac{n\pi x}{\ell}\right)\right]_{x=0}^{x=\ell}}_{=-2n\kappa\pi\ell^{-2}((-1)^n j(t) - h(t))} \\ &= -\left(\frac{n\pi}{\ell}\right)^2 \frac{2}{\ell}\int_{-\ell}^{\ell} u \sin\left(\frac{n\pi x}{\ell}\right) dx - 2n\kappa\pi\ell^{-2}\left((-1)^n j(t) - h(t)\right) \\ &= -\left(\frac{n\pi}{\ell}\right)^2 u_n - 2n\kappa\pi\ell^{-2}\left((-1)^n j(t) - h(t)\right) \\ &= -\lambda_n u_n - 2n\kappa\pi\ell^{-2}\left((-1)^n j(t) - h(t)\right). \end{aligned}$$

We used here in the integration by parts that the boundary terms $u(0,t) = u(\ell,t) = 0$ and similarly the sine terms vanish. This yields $w_n = -\lambda_n u_n$ and combining this with $v_n = \kappa w_n$ we obtain the ODE

$$u_n'(t) = -\kappa\lambda_n u_n - 2n\kappa\pi\ell^{-2}\left((-1)^n j(t) - h(t)\right).$$

We find that the boundary terms now contribute in the ODE a source term in addition to the previous decay term $-\kappa\lambda_n u_n$. We can express the solution to the ODE using Duhamel's principle to obtain

$$\begin{aligned} u_n(t) &= u_n(0)\exp\left(-\kappa\lambda_n t\right) + \int_0^t \exp\left(-\kappa\lambda_n(t-s)\right) g_n(s) ds \\ g_n(s) &= -2n\kappa\pi\ell^{-2}\left((-1)^n j(t) - h(t)\right). \end{aligned} \tag{4}$$

This yields the series representation for the solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{\ell}\right),$$

where $u_n(t)$ is given in equation 4. We emphasize this general solution was derived without differentiating through the series. We remark that such series expansion approaches will typically reduce a more challenging PDE problem down to the simpler problem of solving a collection of ODEs. This demonstrates some of the ways Fourier methods can be used to obtain representations of functions and solutions. In future lectures we will further develop these techniques for analysis and solving other pdes.

Summary:
The *Fourier Methods* and related approaches provide powerful methods for obtaining solutions of pdes and for performing analysis. We see that Fourier series can approximate quite general functions only requiring that they be Lebesgue measurable and have finite L^2-norm $\|f\|_{L^2} < \infty$. We also found that the L^2 theory leads to a weak form of convergence for functions with many useful properties. We also should mention we discussed here only a small

subset of results and techniques using Fourier methods. For those interested in exploring these topics more, Fourier methods and their generalizations are part of a larger field referred to as *Harmonic Analysis.* We also remark that these techniques also play a prominent role in numerical approximation of solutions of PDEs as part of the fields of *Spectral Numerical Methods* and *Finite Element Methods.* The Fourier methods and related techniques we discuss here are also used in many other fields as part of obtaining representations of operations on functions, approximating functions, data analysis and compression, and solving other pdes. As we shall discuss in future lectures, many of the methods also can be extended beyond 1d to higher dimensional spaces.