Classifying Second-Order PDEs

Change of Variable and Differentiation:
It is useful to investigate how the expression of differential operators is augmented when performing a change of variable. Consider the linear change of variable in $\mathbb{R}^n$ of the form

$$\boldsymbol{\xi} = Q\mathbf{x}.$$

In more detail, this can be expressed in components as

$$\underbrace{\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}}_{\boldsymbol{\xi}} = \underbrace{\begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\mathbf{x}}.$$

This can also be expressed as

$$\xi_\ell = \sum_{k'=1}^{n} Q_{\ell k'} x_{k'}.$$

We can now use the chain-rule to determine a relationship between $\partial_{\boldsymbol{\xi}} := \nabla_{\boldsymbol{\xi}}$ and $\partial_{\mathbf{x}} := \nabla_{\mathbf{x}}$. Consider

$$\frac{\partial u}{\partial x_k} = [\nabla_{\mathbf{x}} u]_k = \sum_{\ell=1}^{n} \frac{\partial u}{\partial \xi_\ell}\frac{\partial \xi_\ell}{\partial x_k} = \sum_{\ell=1}^{n} \frac{\partial u}{\partial \xi_\ell} Q_{\ell k} = \left[Q^T \nabla_{\boldsymbol{\xi}} u\right]_k,$$

where we used

$$\frac{\partial \xi_\ell}{\partial x_k} = \frac{\partial}{\partial x_k}\sum_{k'=1}^{n} Q_{\ell k'} x_{k'} = \sum_{k'=1}^{n} Q_{\ell k'}\delta_{kk'} = Q_{\ell k}.$$

The $\delta_{kk'}$ is the Kronecker δ-function which is one only when $k = k'$ and zero otherwise. This shows the linear change of variable impacts the gradients through the transpose and we have the relationships

$$\nabla_{\mathbf{x}} = Q^T \nabla_{\boldsymbol{\xi}}, \qquad \nabla_{\boldsymbol{\xi}} = Q^{-T}\nabla_{\mathbf{x}}.$$

This also can be expressed using the notation

$$\partial_{\mathbf{x}} = Q^T \partial_{\boldsymbol{\xi}}, \qquad \partial_{\boldsymbol{\xi}} = Q^{-T}\partial_{\mathbf{x}}.$$

As we shall show, this is useful in transforming PDEs and putting them into canonical forms. This is also useful later for development of solution techniques.

Classifying Second-Order PDEs
Consider the second-order constant coefficient PDE of the form

$$\mathcal{L}[u] = a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1 u_x + a_2 u_y + a_0 = 0.$$

It will be useful to also express this in the terms of coordinates $\mathbf{x} = (x_1, x_2) = (x, y)$ as

$$\mathcal{L}[u] = a_{11}u_{x_1x_1} + 2a_{12}u_{x_1x_2} + a_{22}u_{x_2x_2} + a_1 u_{x_1} + a_2 u_{x_2} + a_0 = 0.$$

The second-order terms turn out to dominate the behavior of the PDE, so we will write the differential operator as $\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_1 + \mathcal{L}_0$, where $\mathcal{L}_2$ contains the second-order terms and $\mathcal{L}_1$ the first-order terms, and $\mathcal{L}_0$ the constant. Let

$$\begin{aligned} \mathcal{L}_2[u] &= a_{11}u_{x_1x_1} + 2a_{12}u_{x_1x_2} + a_{22}u_{x_2x_2} \\ \mathcal{L}_1[u] &= a_1u_{x_1} + a_2u_{x_2} \\ \mathcal{L}_0[u] &= a_0. \end{aligned}$$

We will primarily be concerned with the behavior of the second-order terms. This operator can be expressed in components as

$$\mathcal{L}_2 = a_{11}\frac{\partial^2}{\partial x_1^2} + 2a_{12}\frac{\partial^2}{\partial x_1 x_2} + a_{22}\frac{\partial^2}{\partial x_2^2} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{bmatrix}^T}_{\partial_{\mathbf{x}}} \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{bmatrix}}_{\partial_{\mathbf{x}}} = \partial_{\mathbf{x}}^T A \partial_{\mathbf{x}}.$$

We set here $a_{21} = a_{12}$. Since A is symmetric it can be unitarily diagonalized by some invertible change of basis $\boldsymbol{\xi} = P^{-1}\mathbf{x}$ to obtain $A = P\Lambda P^{-1}$. The $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$ is the diagonal matrix of eigenvalues λ_ℓ of A. Let $Q^T = P^{-1}$, then since A is unitarily diagonalizable we have $QQ^T = I = Q^TQ$. We further have $P^{-1}AP = Q^TAQ = \Lambda$. We next use our linear change of variable relationships with $\boldsymbol{\xi} = Q\mathbf{x}$. In terms of the coordinate components, this gives

$$\mathcal{L}_2 = \partial_{\mathbf{x}}^T A \partial_{\mathbf{x}} = \partial_{\boldsymbol{\xi}}^T Q^T A Q \partial_{\boldsymbol{\xi}} = \partial_{\boldsymbol{\xi}}^T \Lambda \partial_{\boldsymbol{\xi}} = \begin{bmatrix} \frac{\partial}{\partial \xi_1} \\ \frac{\partial}{\partial \xi_2} \end{bmatrix}^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \xi_1} \\ \frac{\partial}{\partial \xi_2} \end{bmatrix} = \lambda_1 \frac{\partial^2}{\partial \xi_1^2} + \lambda_2 \frac{\partial^2}{\partial \xi_2^2}.$$

There are three interesting cases (i) both eigenvalues are the same sign, (ii) the eigenvalues are opposite sign, (iii) one of the eigenvalues is zero. This can be characterized by consider the determinant

$$\det(A) = \lambda_1\lambda_2 = a_{11}a_{22} - a_{12}^2.$$

We see each of the cases correspond to (i) $\det(A) > 0$, (ii) $\det(A) < 0$, and (ii) $\det(A) = 0$. By making an analogy with the classification of quadratic forms which have *discriminant* $d = a_{12}^2 - a_{11}a_{22}$, we refer to the cases as a/an

(i) *elliptic PDE* when $d = a_{12}^2 - a_{11}a_{22} < 0$.

(ii) *hyperbolic PDE* when $d = a_{12}^2 - a_{11}a_{22} > 0$.

(iii) *parabolic PDE* when $d = a_{12}^2 - a_{11}a_{22} = 0$.

We can put the differential equation $\mathcal{L}[u] = 0$ above into a canoncial form using this classification. Before doing this, we note the change of variable yields for the first-order operator

$$\mathcal{L}_1 = \mathbf{a}^T\partial_{\mathbf{x}} = \mathbf{a}^TQ^T\partial_{\boldsymbol{\xi}} = \mathbf{b}^T\partial_{\boldsymbol{\xi}},$$

where we let $\mathbf{b} = Q\mathbf{a}$. The $\mathcal{L}_0$ operator does not transform under the linear change of variable. We also remark that in the parabolic PDE case it is typically assumed that if $\lambda_1 = 0$ then $b_1 \neq 0$ or if $\lambda_2 = 0$ then $b_2 \neq 0$, so that both derivatives appear at least at some order. With these considerations, we can obtain a canonical form for the PDE in each of the cases.

Elliptic Case: Let $z_\ell = \frac{1}{\sqrt{|\lambda_1|}}\xi_\ell$, then the differential operator takes on the canonical form

$$\mathcal{L}[u] = u_{z_1 z_1} + u_{z_2 z_2} + b_1 u_{z_1} + b_2 u_{z_2} + b_0 = \Delta u + \mathbf{b}^T \nabla_{\mathbf{z}} u + b_0 = 0.$$

The $\mathcal{L}_2[u] = u_{z_1 z_1} + u_{z_2 z_2} := \Delta u$ where Δ is referred to as the *Laplacian*. Let $f[u] = \mathbf{b}^T \nabla_{\mathbf{z}} u + b_0$ then we obtain

$$\Delta u = u_{z_1 z_1} + u_{z_2 z_2} = -f[u].$$

Here, the $f[u]$ acts similar to a forcing or source term which can depend on the lower-order derivatives of u. In the case that $f = 0$ we obtain the canonical form

$$\Delta u = u_{z_1 z_1} + u_{z_2 z_2} = 0.$$

Hyberbolic Case: We assume without loss of generality that the change of variable was chosen so that $\lambda_1 > 0 > \lambda_2$. We again let $z_\ell = \frac{1}{\sqrt{|\lambda_\ell|}}\xi_\ell$. The differential operator takes on the canonical form

$$\mathcal{L}[u] = u_{z_1 z_1} - u_{z_2 z_2} + b_1 u_{z_1} + b_2 u_{z_2} + b_0 = u_{z_1 z_1} - u_{z_2 z_2} + \mathbf{b}^T \nabla_{\mathbf{z}} u + b_0 = 0.$$

We remark that by making the further substitution that $t = z_1$ and $\tilde{x} = z_2$ we obtain

$$\mathcal{L}[u] = u_{tt} - u_{\tilde{x}\tilde{x}} + \mathbf{b}^T \nabla_{\mathbf{z}} u + b_0 = 0.$$

We further let $f[u] = f(t, \tilde{x}; u) = -\mathbf{b}^T \nabla_{\mathbf{z}} u - b_0$. This gives

$$u_{tt} = u_{\tilde{x}\tilde{x}} + f[u].$$

This shows the canonical form above for the hyperbolic PDE is similar to the wave equation of a string under tension with an additional forcing term. Here, the forcing term can depend on lower-order derivatives of u. For example, if $f[u] = -u_t$ this would act like friction dampening the wave. As another example, if $f[u] = -g$ this would act similar to gravity pulling downward on the string. In the case that $\mathbf{b} = 0$ we obtain the canonical form

$$u_{tt} = u_{\tilde{x}\tilde{x}}.$$

Parabolic Case: We assume without loss of generality that the change of variable was chosen so that $\lambda_1 = 0$ and $\lambda_2 > 0$, and $b_1 \neq 0$. We then let $z_1 = -b_1^{-1}\xi_1$ and $z_2 = \frac{1}{\sqrt{|\lambda_2|}}\xi_2$. The differential operator takes on the canonical form

$$\mathcal{L}[u] = u_{z_2 z_2} - u_{z_1} + b_2 u_{z_2} + b_0 = 0.$$

We remark that by making the further substitution that $t = z_1$ and $\tilde{x} = z_2$ we obtain

$$\mathcal{L}[u] = u_{\tilde{x}\tilde{x}} - u_t + b_2 u_{\tilde{x}} + b_0 = 0.$$

We further let $f[u] = f(t, \tilde{x}; u) = b_2 u_{\tilde{x}} + b_0$. This gives

$$u_t = u_{\tilde{x}\tilde{x}} + f[u].$$

The f plays a role similar to a forcing or source term that can depend on lower-order derivatives of u in $\tilde{x}$. Note in the case that $u_t = 0$ we obtain a similar form as in the forced elliptic case (but now in one variable). This suggests the elliptic case can be viewed in some circumstances as the steady-state of the parabolic case. In the case that $f = 0$ we have the canonical form

$$u_t = u_{\tilde{x}\tilde{x}}.$$

Summary These are the cases for constant coefficient second-order partial differential equations (PDEs) in two dimensions. Some similar behaviors are also found in higher dimensions. We briefly discuss a generalization of the elliptic case which arises in many theoretical studies and applications of PDEs.

Elliptic Operators in $\mathbb{R}^n$
We now consider linear second-order differential operators in $\mathbb{R}^n$ of the form

$$\mathcal{L}_2 = \sum_{i=1}^n \sum_{j=1}^n a_{ij} \frac{\partial^2}{\partial x_i x_j} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix}^T}_{\partial_{\mathbf{x}}^T} \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix}}_{\partial_{\mathbf{x}}} = \partial_{\mathbf{x}}^T A \partial_{\mathbf{x}}.$$

This operator is referred as being an *elliptic operator* if A is symmetric positive definite. In this case, it can be diagonalized by a unity change of basis, so that $\Lambda = QAQ^T$, where $QQ^T = I = Q^TQ$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the diagonal matrix of eigenvalues λ_ℓ of A. We will also use that $Q^{-T} = Q$ in this case. Now let $\boldsymbol{\xi} = Q\mathbf{x}$, then we have $\partial_{\boldsymbol{\xi}} = Q\partial_{\mathbf{x}}$, $\partial_{\mathbf{x}} = Q^T\partial_{\boldsymbol{\xi}}$, and $\partial_{\mathbf{x}}^T = \partial_{\boldsymbol{\xi}}^T Q$. This gives

$$\mathcal{L}_2 = \partial_{\mathbf{x}}^T A \partial_{\mathbf{x}} = \partial_{\boldsymbol{\xi}}^T QAQ^T \partial_{\boldsymbol{\xi}} = \partial_{\boldsymbol{\xi}}^T \Lambda \partial_{\boldsymbol{\xi}}.$$

Expressing this in terms of the components, this gives the following canonical form

$$\mathcal{L}_2 = \sum_{i=1}^n \sum_{j=1}^n a_{ij} \frac{\partial^2}{\partial x_i x_j} = \begin{bmatrix} \frac{\partial}{\partial \xi_1} \\ \frac{\partial}{\partial \xi_2} \\ \vdots \\ \frac{\partial}{\partial \xi_n} \end{bmatrix}^T \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \xi_1} \\ \frac{\partial}{\partial \xi_2} \\ \vdots \\ \frac{\partial}{\partial \xi_n} \end{bmatrix} = \sum_{\ell=1}^n \lambda_\ell \frac{\partial^2}{\partial \xi_\ell^2} = \sum_{\ell=1}^n \frac{\partial^2}{\partial z_\ell^2} =: \Delta.$$

In the last step, we used that $\lambda_\ell > 0$ and define $z_\ell = \frac{1}{\sqrt{\lambda_\ell}}\xi_\ell$. The operator $\Delta u := \sum_{\ell=1}^n \frac{\partial^2}{\partial z_\ell^2} u = \sum_{\ell=1}^n u_{z_\ell z_\ell}$ is referred to as the *Laplacian.*

By use of these types of changes of variable, we can take general PDEs and transform many of them into canonical forms for which we have solution techniques. This allows us to greatly extend the range of applicability of our solution techniques and gain insights into the behaviors of diverse PDEs that arise in practice.