Practice Problems

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Problem 1:

Consider the Fourier Series expansion

$$\tilde{\phi}(x) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_n \cos\left(\frac{k\pi x}{\ell}\right) + \sum_{k=1}^{\infty} B_n \sin\left(\frac{k\pi x}{\ell}\right),$$

(a) What are the Fourier Coefficients for $\phi(x) = 3x$ on the interval $[-\pi, \pi]$?

(b) What are the Fourier Coefficients when $\phi(x) = 2$ on the interval $[0, \pi]$ and ϕ has an odd extension?

(c) What are the Fourier Coefficients when $\phi(x) = 2\sin(3x) + 3\sin(2x)$ on the interval $[0, \pi]$ and ϕ has an odd extension?

(d) What are the Fourier Coefficients when $\phi(x) = 2x$ on the interval $[0, \pi]$ and ϕ has an even extension?

(e) What are the Fourier Coefficients when $\phi(x) = 4\sin(3x)+2\cos(2x)$ on the interval $[-\pi, \pi]$ and ϕ has an periodic extension?

(f) What are the Fourier Coefficients when $\phi(x) = -2 + \sin(2x)$ on the interval $[-\pi, \pi]$ and ϕ has an periodic extension?

Problem 2:

Consider the Fourier Series expansion

$$\tilde{\phi}(x) = \sum_{k=-\infty}^{\infty} c_n \exp\left(ik\pi x/\ell\right),$$

(a) What are the Fourier Coefficients for $\phi(x) = 8\cos(4x)$ on the interval $[-\pi, \pi]$?
Hint: You can either use the conversion formula or compute them directly.

(b) What are the Fourier Coefficients when $\phi(x) = -3$ on the interval $[0, \pi]$ and ϕ has an odd extension?

(c) What are the Fourier Coefficients for $\phi(x) = 2\sin(5x) + 3\cos(2x)$ on the interval $[-\pi, \pi]$?

(d) What are the Fourier Coefficients when $\phi(x) = 2 + 4\cos(3x) + 2\sin(x)$ on the interval $[-\pi, \pi]$ and ϕ has periodic extension?

Problem 3:

Consider the diffusion equation on the interval with the Dirichlet Boundary Conditions:

$$\begin{cases} u_t = \kappa u_{xx}, & 0 < x < \ell, t > 0 \\ u(0,t) = j(t), \quad u(\ell,t) = h(t), & t > 0 \\ u(x,0) = \phi(x), & 0 < x < \ell, t = 0. \end{cases}$$

(a) Determine the solution when $\kappa = 1,\ \ell = \pi,\ \phi(x) = 2\sin(3x) + \sin(2x)$ and $j(t) = 0,\ h(t) = 0$.

(b) Determine the solution when $\kappa = 1,\ \ell = \pi,\ \phi(x) = 0$ and $j(t) = \sin(t),\ h(t) = \cos(t)$.

(c) Determine the solution when $\kappa = 1,\ \ell = \pi,\ \phi(x) = 3\sin(2x)$ and $j(t) = -2t,\ h(t) = 0$.

(d) Determine the solution when $\kappa = 1,\ \ell = \pi,\ \phi(x) = 0$ and $j(t) = 0,\ h(t) = \exp(-3t)$.

Problem 4:

Consider the wave equation on the interval with the Dirichlet Boundary Conditions:

$$\begin{cases} u_{tt} = \kappa u_{xx}, & 0 < x < \ell, t > 0 \\ u(0,t) = j(t), \;\; u(\ell,t) = h(t), & t > 0 \\ u(x,0) = \phi(x), & 0 < x < \ell, t = 0 \\ u_t(x,0) = \psi(x), & 0 < x < \ell, t = 0. \end{cases}$$

(a) Determine the solution when $\kappa = 1,\ \ell = \pi,\ \phi(x) = -3\sin(5x),\ \psi(x) = 0$, and $j(t) = 0,\ h(t) = 0$.

(b) Determine the solution when $\kappa = 1,\ \ell = \pi,\ \phi(x) = 0,\ \psi(x) = -15\cos(5x)$, and $j(t) = 0,\ h(t) = 0$.

Problem 5:

Consider the elliptic equation on the domain with the Dirichlet Boundary Conditions, $\Omega = [0, \ell] \times [0, \ell]$:

$$\begin{cases} \Delta u = f, & x \in \Omega \\ u = g, & x \in \partial\Omega. \end{cases}$$

(a) Determine the solution when $\ell = \pi$, $f(x) = 0$, $g(0, y) = g(x, \pi) = g(x, 0) = 0$, and $g(\pi, y) = q(y)$, where $q(y) = \sin(3y) + \sin(y)$.

(b) Determine the solution when $\ell = \pi$, $f(x) = -3 + 2\sin(x)$, $g(0, y) = g(x, \pi) = g(x, 0) = 0$, and $g(\pi, y) = q(y)$, where $q(y) = 0$.