Name: ________________________

Midterm Practice Problems:

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Partial Differential Equations, 124A

Directions: Please answer each question carefully. Be sure to show your work. If you have questions, please feel free to ask.

Problem:

Classify each of the following Partial Differential Equations as either elliptic, hyperbolic, or parabolic. Give a canonical form for each PDE.

(a) $u_{tx} = 0$.

(b) $u_x + u_{tt} = 0$.

(c) $u_{xx} + u_{yy} + 4u_{xy} = 0$.

(d) $u_{xx} + u_{yy} + \frac{1}{4}u_{xy} = 0$.

Problem:

Use method of characteristics to find the solution of

$$\begin{cases} 2u_x(x,y) + 4u_y(x,y) = 0, & -\infty < x < \infty, y > 0 \\ u(x,0) = \cos(x), & -\infty < x < \infty, y = 0 \end{cases}$$

Problem:

Use method of characteristics to find the solution of

$$\begin{cases} 5u_x(x,y) + 15x^2 u_y(x,y) = -u, & -\infty < x, y < \infty, t > 0 \\ u(0,y) = e^{-|y|}, & -\infty < y < \infty, t = 0 \end{cases}$$

Problem: Consider a vibrating string with displacements $u(x,t)$ at time t. To leading order the displacement u satisfies the wave equation:

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t), & -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x), & -\infty < x < \infty, t = 0 \\ u_t(x,0) = \psi(x), & -\infty < x < \infty, t = 0. \end{cases}$$

a) Determine the solution to the wave equation when the initial configuration is $\phi(x) = \sin(x)$ and initial velocity is $\psi(x) = \cos(x)$.

b) Show that in general the total energy E of solutions of the wave equation is conserved. The total energy is given by the kinetic energy + the potential energy of the mechanical system. For the vibrating string this is

$$E(t) = \int_{-\infty}^{\infty} \frac{1}{2} |u_t(t,y)|^2 dy + \int_{-\infty}^{\infty} \frac{c^2}{2} |u_x(t,y)|^2 dy.$$

Hint: Conservation of energy corresponds to E remaining constant in time. Show this using the PDE and integration by parts to obtain $\partial E / \partial t = 0$.

c) Compute the total energy $E(0)$ at time 0 in terms of ϕ and ψ. Hint: Use that $u_x(x,0) = \phi_x(x)$.

d) Suppose that the energy is $E(t) = 0$ for all time for a twice continuously differentiable function $u(x,t)$. What must the function u be in this case? In the case that we additionally impose that $\phi(x) = 0$, what must u be? Hint: Consider what $E = 0$ implies about u_x and u_t and then integrate in x and t to obtain a form for u.

e) Suppose $u_1(x,t)$ and $u_2(x,t)$ are two solutions of the PDE with the same boundary conditions, then $v(x,t) = u_2(x,t) - u_1(x,t)$ is a solution of the PDE with $\phi(x) = \psi(x) = 0$. What is the energy $E(t)$ of v? Hint: Use linearity of the PDE and compute $E(0)$ for v at time 0. If u_1 and u_2 are both continuous, how are they related at each (x,t)? Are solutions of this PDE unique?

Problem 3:

a) Solve the diffusion equation on the real-line:

$$\begin{cases} u_t = ku_{xx}, & -\infty < x < \infty, t > 0 \\ u(x,0) = e^{-x^2}, & -\infty < x < \infty, t = 0. \end{cases}$$

b) Show that the *energy* below decays, so $E(t) \leq E(0)$ for solutions u of the diffusion equation given in part (a), $(t > 0)$

$$E\left[u\right](t) = \int_{-\infty}^{\infty} u^2(x,t)dx.$$

c) Show that if $E[w] = 0$ for a smooth function w, then w must be zero. Show this requires any two solutions u_1 and u_2 of the Diffusion Equation in part (a) be equal? This shows solutions of the Diffusion Equation are unique. Hint: Use part (b) and consider $w = u_2 - u_1$.