Name: Solutions

Midterm Exam:

Professor: Paul J. Atzberger

Partial Differential Equations, 124A

Scoring:

Problem 1: __________

Problem 2: __________

Problem 3: __________

Problem 4: __________

Directions: Please answer each question carefully. Be sure to show your work. If you have questions, please feel free to ask.

Problem 1:
Use the method of characteristics to find the solution of

$$\begin{cases} 2u_x + 6u_y = 0, & -\infty < x < \infty, y \geq 0 \\ u(x,0) = \sin(x), & -\infty < x < \infty, y = 0. \end{cases}$$

$$2u_x + 6u_y = 0 \Rightarrow u_x + 3u_y = 0$$

$$\frac{dy}{dx} = 3 \Rightarrow y(x) = 3x + C$$

$$(x_1, y_1): \quad y_1 = 3x_1 + C \Rightarrow C = y_1 - 3x_1$$

$$y(x) = 3(x - x_1) + y_1$$

$$(x, 0): \quad 0 = 3(x - x_1) + y_1 \Rightarrow x = \frac{-y_1 + 3x_1}{3} = x_1 - \frac{1}{3}y_1$$

Solution

$$u(x_1, y_1) = u\left(x_1 - \frac{1}{3}y_1, 0\right) = \sin\left(x_1 - \frac{1}{3}y_1\right).$$

Problem 2:
Classify each of the following Partial Differential Equations as either *elliptic*, *hyperbolic*, or *parabolic*.

(a) $u_{xx} + u_{yy} + 6u_{xy} + u_y = 0$, $\mathcal{L}[u] = \mathcal{L}_2[u] + \mathcal{L}_1[u]$,

$\mathcal{L}_2[u] = u_{xx} + 6u_{xy} + u_{yy} = a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy}$

$a_{11} = 1,\ a_{12} = 3,\ a_{22} = 1$

$d = a_{12}^2 - a_{11}a_{22} = 9 - 1 = 8 > 0$

hyperbolic

(b) $u_{xx} + u_{yy} + u_x + u_y = 0$.

$a_{11} = 1,\ a_{12} = 0,\ a_{22} = 1$

$d = 0 - 1 = -1 < 0$

elliptic

(c) $u_{xx} + u_{yy} + 2u_{xy} + u_x = 0$, $\mathcal{L}[u] = \mathcal{L}_2[u] + \mathcal{L}_1[u]$,

$a_{11} = 1,\ a_{12} = 1,\ a_{22} = 1$

$d = 1^2 - 1 \cdot 1 = 0$

$\mathcal{L}_1[u] = a_1u_x + a_2u_y$

$a_1 \neq 0$,

parabolic.

Problem 3:
Use the method of characteristics to find the solution of

$$\begin{cases} 2u_x - 2yu_y = 0, & -\infty < y < \infty, x > 0 \\ u(0,y) = y^2, & -\infty < y < \infty, x = 0 \end{cases}$$

$2u_x - 2yu_y = 0 \Rightarrow u_x - yu_y = 0$

$\frac{dy}{dx} = -y \Rightarrow y(x) = Ce^{-x}$

(x_1, y_1): $y_1 = Ce^{-x_1} \Rightarrow C = y_1 e^{x_1}$

$(0, y)$: $y = Ce^{-0} \Rightarrow y = C = y_1 e^{x_1}$

Solution:

$u(x_1, y_1) = u(0, C) = C^2 = y_1^2 e^{2x_1}$.

Problem 4:

Consider a vibrating string with displacements $u(x,t)$ at time t satisfying the wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx}, & -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x), & -\infty < x < \infty, t = 0 \\ u_t(x,0) = \psi(x), & -\infty < x < \infty, t = 0. \end{cases}$$

a) Determine the solution to the wave equation when the initial configuration is $\phi(x) = \exp(-x^2)$ and initial velocity is $\psi(x) = -2x\exp(-x^2)$.

$$u(x,t) = \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\tilde{x})\, d\tilde{x}$$

$$= \frac{1}{2}\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} -2\tilde{x}e^{-\tilde{x}^2}\, d\tilde{x}$$

$$= \frac{1}{2}\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right] + \frac{1}{2c}\left[e^{-x^2}\right]_{x-ct}^{x+ct}$$

$$= \left[\frac{1}{2} + \frac{1}{2c}\right]\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right] = \left(\frac{c+1}{2c}\right)\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right]$$

b) Compute the total energy $E(0)$ at time 0 in terms of ϕ and ψ.
Hint: Use $\int x^2 \exp(-x^2)dx = \sqrt{\pi}/2$.

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2 dx + \frac{c^2}{2}\int_{-\infty}^{\infty} u_x^2 dx.$$

$$E[0] = \frac{1}{2}\int_{-\infty}^{\infty} \psi^2(x)\, dx + \frac{c^2}{2}\int_{-\infty}^{\infty} (\phi_x(x))^2 dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} 4x^2 e^{-2x^2} dx + \frac{c^2}{2}\int_{-\infty}^{\infty} 4x^2 e^{-2x^2} dx$$

let $z = \sqrt{2}x$, $dx = \frac{1}{\sqrt{2}}dz$

$$= \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} z^2 e^{-z^2} dz + \frac{c^2}{\sqrt{2}}\int_{-\infty}^{\infty} z^2 e^{-z^2} dz = \frac{\sqrt{\pi}}{2\sqrt{2}}(1+c^2).$$

c) Show that the total energy E of solutions of the wave equation is conserved by showing $dE/dt = 0$. Hint: Use the PDE and integration by parts.

$$\frac{dE}{dt} = \frac{1}{2}\int_{-\infty}^{\infty} 2u_t u_{tt}\, dx + \frac{c^2}{2}\int_{-\infty}^{\infty} 2u_x u_{xt}\, dx$$

$0 \leftarrow$ since $u_x \to 0$ $x \to \pm\infty$.

$$= \cancel{\frac{1}{2}\int_{-\infty}^{\infty} 2c^2 u_t u_{xx}\, dx} + \frac{c^2}{2}\cancel{\left[2u_x u_t\right]_{-\infty}^{\infty}} - \cancel{\frac{c^2}{2}\int_{-\infty}^{\infty} 2u_{xx} u_t\, dx}$$

$$= 0.$$