Wave Equation in 1D

Wave Equation:

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t), & t > 0, \ -\infty < x < \infty \\ u(x,0) = \phi(x), & t = 0 \\ u_t(x,0) = \psi(x), & t = 0. \end{cases}$$

This was motivated by a string that is subject to tension and small displacements.

Factoring Operators: To obtain representations for the solutions of the wave equation, we are going to use that the second-order differential operator can be factored into two easier to work with first-order differential operators. Since we have already seen solution techniques for first-order PDEs, we can leverage these to obtain a solution for the second-order wave equation.

The PDE can be expressed in terms of the second-order differential operator

$$\mathcal{L}[u] = 0, \quad \text{where} \quad \mathcal{L} = \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}.$$

This can be factored a few ways into a "product" of two differential operators as

$$\begin{aligned} \mathcal{L} &= \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} = \underbrace{\left(\frac{\partial}{\partial t} - c\frac{\partial^2}{\partial x}\right)}_{\mathcal{L}_1} \underbrace{\left(\frac{\partial}{\partial t} + c\frac{\partial^2}{\partial x}\right)}_{\mathcal{L}_2} = \underbrace{\left(\frac{\partial}{\partial t} + c\frac{\partial^2}{\partial x}\right)}_{\mathcal{L}_2} \underbrace{\left(\frac{\partial}{\partial t} - c\frac{\partial^2}{\partial x}\right)}_{\mathcal{L}_1} \\ &= \mathcal{L}_2[\mathcal{L}_1] = \mathcal{L}_1[\mathcal{L}_2]. \end{aligned}$$

Technically speaking, this "product" is a composition of the differential operations, $\mathcal{L}[u] = \mathcal{L}_2[\mathcal{L}_1[u]] = \mathcal{L}_1[\mathcal{L}_2[u]]$. As a consequence, if we find a function u_1 so that $\mathcal{L}_1[u_1] = 0$ then we would have $\mathcal{L}[u_1] = 0$ by using the factorization with $\mathcal{L}_1$ appearing closest to u. Similarly, if we find a function u_2 with $\mathcal{L}_2[u_2] = 0$, then $\mathcal{L}[u_1] = 0$. Such functions can be found by solving equations such as

$$\mathcal{L}_1[u] = u_t - cu_x = 0.$$

This is the first-order transport equation which has general solution $u_1 = f(x+ct)$. Similarly, we have

$$\mathcal{L}_2[u] = u_t + cu_x = 0,$$

which has general solution $u_2 = g(x - ct)$. Now we can use linearity to obtain

$$\mathcal{L}[u_1 + u_2] = \mathcal{L}[u_1] + \mathcal{L}[u_2] = \mathcal{L}_2[\mathcal{L}_1][u_1] + \mathcal{L}_1[\mathcal{L}_2][u_2] = 0.$$

This suggests that the general solutions f, g can provide representations for the wave equation, provided we can match the initial conditions.

Solution to the Initial Value Problem

We let $u = u_1 + u_2 = f + g$. To solve the initial value problem, we have to match the initial conditions at time $t = 0$. This requires

$$\begin{cases} f + g &= \phi \\ cf' - cg' &= \psi \end{cases} \Rightarrow \begin{cases} f' + g' &= \phi' \\ f' - g' &= c^{-1}\psi \end{cases} \Rightarrow \begin{cases} f' &= \frac{1}{2}\phi' + \frac{1}{2c}\psi \\ g' &= \frac{1}{2}\phi' - \frac{1}{2c}\psi. \end{cases}$$

By integrating both sides, we obtain

$$\begin{aligned} f(r) &= \frac{1}{2}\phi(r) + \frac{1}{2c}\int_0^r \psi(r')dr' + A \\ g(s) &= \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi(s')dr' + B. \end{aligned}$$

The initial condition involving ϕ requires that

$$f(x) + g(x) = \phi(x) \quad \Rightarrow \quad A + B = 0.$$

We obtain the general solution $u(x,t) = u_1(x,t) + u_2(x,t) = f(x+ct) + g(x-ct)$ which yields

$$u(x,t) = \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'.$$

Example: Solution for Hat Function: We consider the wave equation have the following initial conditions

$$\begin{cases} u_{tt}(x,t) &= c^2 u_{xx}(x,t), & t > 0,\ -\infty < x < \infty \\ u(x,0) &= \max\left(b - \frac{b|x|}{a}, 0\right), & t = 0 \\ u_t(x,0) &= 0, & t = 0. \end{cases}$$

Since $\psi = 0$ we have the solution is of the form

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] \ = \ \frac{1}{2}\max\left(b - \frac{b|x+ct|}{a}, 0\right) + \frac{1}{2}\max\left(b - \frac{b|x-ct|}{a}, 0\right).$$

As a consequence, the wave profile will always be piecewise linear. We remark that while our solution is not differentiable at all points, there are still ways to make sense of the solution by considering *weak solutions* (which we postpone discussing until a later lecture).

The solution goes through a few phases. For times with $t > a/c$, there is no overlap in the two wave profiles and at any given point we have

$$u(x,t) = \begin{cases} 0, & x < -a - ct \\ \frac{1}{2}\max\left(b - \frac{b|x+ct|}{a}, 0\right), & -a - ct < x < a - ct \\ 0, & a - ct < x < -a + ct \\ \frac{1}{2}\max\left(b - \frac{b|x-ct|}{a}, 0\right), & -a + ct < x < a + ct \\ 0, & x > a + ct \end{cases}$$

At intermediate times the solution is more complicated. For example, at $t = a/2c < a/c$ we have

$$u\left(x, t = \frac{a}{2c}\right) = \begin{cases} 0, & x < \frac{-3a}{2} \\ \frac{1}{2}\left(b + \frac{b(x+\frac{1}{2}a)}{a}\right) = \frac{3b}{4} + \frac{bx}{2a}, & \frac{-3a}{2} < x < -\frac{a}{2} \\ \frac{1}{2}\left(b + \frac{b(x+\frac{1}{2}a)}{a}\right) + \frac{1}{2}\left(b - \frac{b(x-\frac{1}{2}a)}{a}\right) = \frac{b}{2}, & \frac{-a}{2} < x < -\frac{a}{2} \\ \frac{1}{2}\left(b - \frac{b(x-\frac{1}{2}a)}{a}\right) = \frac{3b}{4} - \frac{bx}{2a}, & \frac{a}{2} < x < \frac{3a}{2} \\ 0, & x > \frac{3a}{2}. \end{cases}$$

We notice from this example a few interesting behaviors. The form of the wave at time zero has an amplitude of b. This form then splits into two copies which each propagate in opposite directions. The form of the wave that emerges has an amplitude of $\frac{1}{2}b$ but retains the initial width $2a$. Other general insights can be obtained into the behaviors of the wave equation by studying this and other examples.

Domain of Dependence and Domain of Influence: We make a few related comments. As we saw in the hat-function example, the initial conditions play an important role in the form of the solution. Importantly in the case of the wave equation, a given point (x, t) only depends on a subset of the data in the initial conditions. We can see from the form of the general solution $u(x, t)$ can only depend on ϕ and ψ over the interval $[x - ct, x + ct]$. This is referred to as the *Domain of Dependence* for the solution point (x, t). Similarly, we see the initial conditions can only influence a rather limit set of solution points. For the given initial conditions at $(x_0, t_0 = 0)$, we see this can only influence the solution at points (x, t) where $x \in [x_0 - c(t - t_0), x_0 + c(t - t_0)]$. This interval is referred to as the *Domain of Influence* for the data. These are important in understanding the role what specified data in the PDE play in the solution of u at (x, t). The wave equation provides a clear illustration given that we are able to solve for its general solution explicitly.

Numerical Methods for Simulations of 1D Waves: This representation for the solutions obtained is also readily amenable for performing numerical simulations to investigate behaviors of 1D waves. For example, suppose one wants to obtain a numerical approximation of the solutions. We can model initial conditions specified over a finite interval $[-L, L]$ by the values of ϕ and ψ at a collection of n equally spaced points. This can be obtained by using grid points $x_m = -L + m\Delta x$, where $\Delta x = 2L/(n-1)$ with $m \in [0, 1, \ldots, n-1]$. To keep things simple, we will also consider representing the solution at these sample grid points with the restriction that times to be of the form $t_k = k\Delta t$ with k an integer and $\Delta t = \Delta x/c$. This ensures that $x \pm ct$ will be a grid point x_m for some integer m. To compute a numerical approximation of the solution at $u(x_m, t_k)$ we use

$$\begin{aligned} u(x_m, t_k) &= \frac{1}{2}\left[\phi(x_m - ct_k) + \phi(x_m + ct_k)\right] + \frac{1}{2c}\int_{x_m - ct_k}^{x_m + ct_k} \psi(x')dx' \\ &\approx \frac{1}{2}\left[\phi(x_m - ct_k) + \phi(x_m + ct_k)\right] + \frac{1}{2c}\sum_{\ell = m-k}^{m+k} q_\ell \psi(x_\ell). \end{aligned}$$

The main approximation is in replacing the integration with a finite sum using the function evaluations of $\psi(x_\ell)$ at the grid points x_ℓ. This is referred to as a *quadrature* approximation of the integral and almost any method could be used here. For simplicity, we will use the trapezoid method where the integral is approximated by the sum

$$\int_{x_m - ct_k}^{x_m + ct_k} \psi(x')dx' \approx \frac{\Delta x}{2}\left(f(x_{m-k}) + 2\sum_{m'=m-k+1}^{m+k-1} f(x'_m) + f(x_{m+k})\right).$$

This would be the same as approximating the area under the curve of the integrand $\psi(x)$ by the Riemann sum approximation where trapezoids are used to approximate the areas on each sub-interval $[x_{m-1}, x_m)$. Our restriction of the times to t_k ensures that $x_m - ct_k = x_{m-k}$ and $x_m + ct_k = x_{m+k}$ so that the calculations stay "on lattice." In summary, we obtain the numerical method

$$\tilde{u}(x_m, t_k) \quad = \quad \frac{1}{2}\left[\phi(x_{m-k}) + \phi(x_{m+k})\right] + \frac{\Delta x}{2}\left(f(x_{m-k}) + 2\sum_{m'=m-k+1}^{m+k-1} f(x_{m'}) + f(x_{m+k})\right).$$

For a fixed time $t = t_k$ an approximate solution $\tilde{u}(x,t)$ can be obtained by sweeping over the grid points x_m. By looking at these solutions at successive times by increasing k, the evolution of solutions of the wave equation can be studied numerically. I encourage everyone to implement this in your preferred programming language, such as python, and to give this a try. In python, this can be done using the packages: *numpy* and *matplotlib*. A good study would be to investigate empirically the different roles that ϕ and ψ play in the solution of the resulting wave and how it evolves over time. The ϕ gives the initial configuration and ψ the initial momentum (velocity). Looking at a variety of choices for these functions is a good way to help build intuition for the behaviors that can be exhibited by solutions of the wave equation.