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Bowling ball representations of braid groups

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ABSTRACT

In a remark in his seminal 1987 paper, Jones describes a way to define the Burau matrix of a positive braid using a metaphor of bowling a ball down a bowling alley with braided lanes. We extend this definition to allow multiple bowling balls to be bowled simultaneously. We obtain representations of the Iwahori–Hecke algebra and a cabled version of the Temperley–Lieb representation.

Keywords: Braid group; positive braid monoid; representation; Iwahori–Hecke algebra.

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1. Introduction

The positive braid monoid B_n^+ is the monoid generated by $\sigma_1, \dots, \sigma_{n-1}$ modulo, the following relations:

- Far commutativity: $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$.
- The braid relation: $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ if $|i - j| = 1$.

Alternatively, B_n^+ is the set of n-strand geometric braids that involve only positive crossings.

In a remark in [3], Jones describes a definition of the (non-reduced) Burau representation of the positive braid monoid using a "bowling ball" metaphor. Here is the relevant passage (except we change "t" to "$1-q$" and (i, j) to (j, i), to match our conventions).

> For positive braids, there is also a mechanical interpretation of the Burau matrix: Lay the braid out flat and make it into a bowling alley with n lanes, the lanes going over each other according to the braid. If a ball traveling along a lane has probability $1 - q$ of falling off the top lane (and continuing in the lane below) at every crossing then the (j, i) entry of the

(non-reduced) Burau matrix is the probability that a ball bowled in the ith lane will end up in the jth.

This idea was generalized to string links in [7]. Subsequent papers, for example [1, 6, 8], have pursued the related idea of random walks on braids and knot diagrams. Our goal is to generalize the bowling ball definition to allow several balls to be bowled simultaneously. We obtain representations of the Iwahori–Hecke algebra and a cabled version of the Temperley–Lieb representation. I will not propose any specific applications, but I hope this definition gives a useful new way to think about these important representations.

Throughout the paper, we work over an arbitrary field containing an element q. The probability metaphor only makes literal sense when q is a real number in the range [0,1]. However, the results are true for any value of q, and even over a ring. If q is invertible then the representations extend from the braid monoid to the braid group.

2. Definition of the Representation

In this section, we give a rule for the behavior of bowling balls, and prove that it gives a representation of the braid monoid. We will describe another rule in Sec. 4.

Fix $N \geq 1$. Let β be an n-strand positive braid, thought of as a bowling alley with n lanes. Simultaneously bowl balls into the lanes so that each lane receives at most N balls.

At each crossing, some balls may fall, according to the following rule. Suppose a balls arrive on the top lane of a crossing, and b arrive on the bottom. If $a \leq b$, then no balls will fall. If $a > b$ then, with probability $1 - q$, exactly $a - b$ balls will fall from the top lane to join the b balls on the lane below. The result is that the same numbers of balls are in the exiting lanes as in the entering lanes, except possibly for a permutation.

Use this to define a matrix $\rho(\beta)$ whose rows and columns are indexed by n-tuples $\mathbf{u} = (u_1, \ldots, u_n)$ of integers such that $0 \leq u_i \leq N$. The $(\mathbf{v}, \mathbf{u})$ entry of $\rho(\beta)$ is the probability that, if u_i balls are bowled into the ith lane for all i, then v_j balls end up in the jth lane for all j.

Theorem 2.1. *ρ is a well-defined $(N+1)^n$-dimensional representation of B_n^+.*

Proof. The definition of ρ clearly respects multiplication, and the far commutativity relation. It remains to check the braid relation. This only involves three lanes, so it suffices to treat the case $n = 3$. Let

$$\beta = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2.$$

Call the three lanes the top, middle, and bottom. The top lane crosses over both other lanes, and the bottom lane crosses under both other lanes.

We will compute the entries of the matrix $\rho(\beta)$ in a way that does not depend on the specific word used to represent β. This will show that $\rho(\beta)$ is well-defined.

Case 0: No balls.

If no balls are bowled in to β then no balls will emerge.

Case 1: One ball.

Suppose one ball is bowled into β. If it is bowled into one of the lower two lanes then the top lane plays no role, so we can simply use the probabilities for a single crossing between the lower two lanes.

Now, suppose the ball is bowled into the top lane. The probability that it will end up in the top lane is q^2, since it must pass over two empty lanes. The probability that it will end up among the top two lanes is q, since it must pass over the bottom lane exactly once, regardless of whether or not it falls to the middle lane. By subtraction, the probability that it will end up in the middle lane is $q - q^2$, and the probability that it will end up in the bottom lane is $1 - q$.

Case 2: Two balls.

Suppose two balls are bowled into β. If they are bowled into the same lane then they behave as a single ball, which was covered in the Case 1. If one of them is bowled into the bottom lane then the bottom lane plays no role, so we can simply use the probabilities for a single crossing between the upper two lanes.

Now, suppose the two balls are bowled into the top and the middle lanes. The probability that the bottom lane ends up empty is q^2, since the two balls must pass over the bottom lane. The probability that one of the lower two lanes ends up empty (that is, that the ball remains in the top lane) is q, since the top ball must pass over the empty lane. By subtraction, the probability that the middle lane will end up empty is $q - q^2$, and the probability that the top lane will end up empty is $1 - q$.

Case 3: Three balls.

Suppose three balls are bowled into β. If they are bowled into the same lane then they behave as a single ball, which was covered in Case 1. If they are bowled one into each lane then no balls will fall.

Now, suppose one ball is bowled into one lane and two balls are bowled into another. We use a triple (x, y, z) to denote the outcome where x balls emerge from the left (lowest) lane, y from the middle, and z from the right (highest). We will compute the probabilities of all six possible outcomes in the following order:

$$(0, 1, 2), (0, 2, 1), (1, 2, 0), (2, 1, 0), (2, 0, 1), (1, 0, 2).$$

First, we compute the probability of the outcome $(0, 1, 2)$. If two balls were bowled into the top lane and one into the middle, then the probability of $(0, 1, 2)$ is q^3, because there are three crossings at which a larger number of balls must pass over a smaller number without falling. For any other input, the probability of $(0, 1, 2)$ is 0, since balls cannot fall up.

Next, we compute the probability that the outcome is either $(0,1,2)$ or $(0,2,1)$. To do this, ignore the distinction between having one or two balls in a lane, and proceed as if one ball had been bowled in to each of two lanes. In other words, for a given input of a total of three balls into two lanes, the probability that the outcome will have the form $(0,y,z)$ is the same as the probability of an outcome $(0,1,1)$ from a modified input of a total of two balls into the same two lanes. This probability was computed in Case 2. Take the probability of $(0,y,z)$ minus the probability of $(0,1,2)$ to obtain the probability of $(0,2,1)$.

Next, we compute the probability that the outcome is either $(0,2,1)$ or $(1,2,0)$. To do this, ignore the distinction between having zero or one ball in a lane and proceed as if only one ball had been bowled in. In other words, for a given input of two balls into a lane and one into another, the probability of an outcome of the form $(x,2,z)$ is the same as the probability of an outcome $(0,1,0)$ from a modified input of one ball where originally there was two, and no ball where originally there was one. This probability was computed in Case 1. Take the probability of $(x,2,z)$ minus the probability of $(0,2,1)$ to obtain the probability of $(1,2,0)$.

Continue in a similar fashion. Compute the probability that the outcome has the form $(x,y,0)$ similarly to $(0,y,z)$ above, and then subtract the outcome $(1,2,0)$ to get $(2,1,0)$. Compute the probability that the outcome has the form $(2,y,z)$ similarly to $(x,2,z)$ above, and subtract the outcome $(2,1,0)$ to get $(2,0,1)$. Finally, compute the probability of $(x,0,z)$ and subtract $(2,0,1)$ to get $(1,0,2)$.

In the end, for any given input, we have computed the probabilities of all six outputs, independently of which of the two bowling lane configurations was used for β.

Case 4: The general case.

Suppose a, b and c balls are bowled into the lanes. The only thing that matters about the numbers a, b and c is which equalities and inequalities hold between them. Thus, we can reduce to one of the cases we have already covered.

In every case, for any given input, we can compute the probability of any given output. The computation is the same for $\sigma_1\sigma_2\sigma_1$ and $\sigma_2\sigma_1\sigma_2$, so these have the same matrix. □

3. The Iwahori–Hecke Algebra

Let ρ be the representation of B_n^+ defined in the previous section. The Iwahori–Hecke algebra $H_n(q)$ is the monoid algebra of formal linear combinations of positive braids modulo the two-sided ideal generated by the quadratics

$$(q+\sigma_i)(1-\sigma_i).$$

Theorem 3.1. *ρ factors through $H_n(q)$.*

Proof. Only two lanes are involved, so it suffices to treat the case $n=2$. Let $\mathbf{v}$ be the vector corresponding a balls in the left lane and b in the right. We must show that $\mathbf{v}$ is in the kernel of

$$(q + \rho(\sigma_1))(1 - \rho(\sigma_1)).$$

If $a = b$ then $\mathbf{v}$ is fixed by $\rho(\sigma_1)$, and we are done.

If $a \neq b$ then the only thing that matters is which of a and b is larger. Thus, we can reduce to the case they are equal to 0 and 1. The action of $\rho(\sigma_1)$ is then the same as the Burau representation. It is well known, and easily checked, that this satisfies the required quadratic relation. □

Note that if q is invertible then the σ_i are invertible in $H_n(q)$, with

$$\sigma_i^{-1} = q^{-1}(\sigma_i + q - 1).$$

In this case, ρ is a representation of the braid group B_n and not just B_n^+.

There is another important element of the kernel of ρ in the case $n = N + 2$. Suppose w is a permutation of $\{1, \ldots, N + 2\}$. Let $(-1)^w = \pm 1$ denote the sign of the permutation. Let β_w denote the unique positive braid with a minimal number of crossings such that the lane at position i goes to position $w(i)$ for all $i = 1, \ldots, N+2$. Let x be the following element of $H_n(q)$.

$$x = \sum_w (-1)^w \beta_w.$$

A generalization of x appears in the definition of the Specht modules in [2].

For every $i = 1, \ldots, N + 1$, we have

$$x = \left(\sum_{w(i)<w(i+1)} (-1)^w \beta_w \right) (1 - \sigma_i), \tag{3.1}$$

and thus

$$x\sigma_i = -qx. \tag{3.2}$$

Theorem 3.2. *For $n = N + 2$, $\rho(x) = 0$.*

Proof. Consider a basis vector $\mathbf{v}$ corresponding to bowling v_i balls into the ith lane, where

$$0 \leq v_1 \leq \cdots \leq v_{N+2} \leq N.$$

Then $v_i = v_{i+1}$ for some i. The action of σ_i then fixes $\mathbf{v}$, so by Eq. (3.1), $\rho(x)\mathbf{v} = 0$.

Now, consider an arbitrary basis vector $\mathbf{v}'$ corresponding to bowling v_i' balls into the ith lane. Let $v_1, \ldots, v_{N+2}$ be the numbers v_i' arranged into increasing order. Then $v_i' = v_{w(i)}$ for some permutation w. If we bowl v_i balls into the ith lane of β_w for all i then no balls will fall, since a smaller number passes over a larger number at every crossing. Thus,

$$\mathbf{v}' = \rho(\beta_w)\mathbf{v}.$$

By repeatedly applying Eq. (3.2), $x\beta_w$ is a scalar multiple of x. Thus,

$$\rho(x)\mathbf{v}' = \rho(x\beta_w)\mathbf{v} = 0,$$

so every basis vector is in the kernel of the action of x. $\square$

For $n \geq N + 2$ and $1 \leq k \leq n - N - 1$, let $x_k \in H_n(q)$ be the result of placing $k - 1$ straight lanes on the left-hand side of x and $n - k - N - 1$ straight lanes on the right-hand side. Obviously, x_k is also in the kernel of ρ.

If $N = 1$, and our field is $\mathbb{C}$, and $q \neq 0$, then the quotient of $H_n(q)$ by the two-sided ideal generated by the elements x_k is isomorphic to the Temperley–Lieb algebra. See [4, Theorem 5.29], although the conventions there are slightly different. Thus, in this case, the representation ρ factors through the Temperley–Lieb algebra.

Remark 3.3. I have not been able to find any reference in the literature to the quotient of $H_n(q)$ by the two-sided ideal generated by the elements x_k when $N > 1$. I suspect it is related to the action of $H_n(q)$ on the nth tensor power of the standard representation of $U_q(\mathfrak{sl}_{N+1})$, as analyzed in [5].

4. A Cabling of the Temperley–Lieb Algebra

Fix $K \geq 1$. Let β be a positive braid, thought of as a bowling alley with n lanes. Create a *cabling* of β by replacing every lane with K parallel lanes. Suppose $\mathbf{a} = (a_1, \ldots, a_n)$ is an n-tuple of integers such that $0 \leq a_i \leq K$. Bowl balls into the lanes so that each lane gets at most one ball and, for all i, a_i balls go into the ith collection of K parallel lanes. Whenever a ball passes over an empty lane, it falls with probability $1 - q$.

Use this to define a matrix $\rho_K(\beta)$ whose rows and columns are indexed by n-tuples $(a_1, \ldots, a_n)$ of integers such that $0 \leq a_i \leq K$. The $(\mathbf{b}, \mathbf{a})$ entry of $\rho_K(\beta)$ is the probability that, if a_i balls are bowled into the ith set of K parallel lanes for all i, then b_i balls end up in the ith set of K parallel lanes for all i.

Theorem 4.1. *ρ_K is a well-defined representation of B_n^+.*

Proof. Our definition only keeps track of the number of balls in each collection of K parallel lanes. We must check that it is not necessary to know precisely which lanes they are in.

Consider the cabling of a single positive crossing. Suppose we bowl a balls into the upper K lanes of a cabled crossing, and b into the lower K lanes. The empty lanes in the upper lanes will remain empty, and the balls in the lower lanes will remain in their lane. The probability that exactly c balls will fall depends only on the number a of occupied upper lanes and the number $K - b$ of empty lower lanes.

The representation ρ_K clearly respects multiplication. It therefore assigns a well-defined matrix to any positive braid β that is written as a product of crossings. By Theorem 2.1, the probability of any specific outcome is invariant under applying braid relations to the cabling of β. Therefore, our matrix $\rho_K(\beta)$ is also invariant under braid relations. $\square$

A trace function of ρ_K can be used to compute the colored Jones polynomial of a knot. Other apparently similar approaches to the colored Jones polynomial have appeared in [8, 1]. It would be interesting to know something about the limiting behavior of ρ_K if we set $q = e^{2i\pi/K}$ and let K go to infinity. This may have some connection to the Kashaev conjecture.

For all $0 \le c \le a \le N$ and $0 \le b \le N$, let $f_b^a(c)$ denote the probability, when a balls enter the top lane of a crossing and b balls enter the bottom, that c balls fall from the upper lane to the lower lane. We will compute a formula for $f_b^a(c)$. First, we define some notation.

If k is a non-negative integer, we define the quantum integer

$$[k] = \frac{1-q^k}{1-q}.$$

Note that we are using the definition that involves only positive exponents of q, not the definition that is symmetric under mapping q to q^{-1}.

The q-factorial is $[k]! = [k][k-1]\cdots[1]$. If $0 \le r \le k$, the Gaussian binomial is

$$\binom{k}{r}_q = \frac{[k]!}{[r]![k-r]!}.$$

These have a combinatorial interpretation as follows.

An *inversion of a permutation* w of $\{1, \dots, k\}$ is a pair $i < j$ such that $w(i) > w(j)$. The quantum factorial $[k]!$ is the sum over w of q to the power of the number of inversions of w.

An *inversion of a sequence* $(\epsilon_1, \dots, \epsilon_k)$ of ones and zeros is a pair $i < j$ such that $\epsilon_i = 1$ and $\epsilon_j = 0$. The Gaussian binomial $\binom{k}{r}_q$ is the sum, taken over all such sequences that have r ones, of q to the power of the number of inversions.

We now give a formula for $f_b^a(c)$.

Theorem 4.2. *If a, b, c are integers and $0 \le c \le a$ then*

$$f_b^a(c) = q^{(a-c)(K-b-c)} \binom{a}{c}_q \binom{K-b}{c}_q (1-q)^c [c]!$$

Proof. Consider a crossing where K parallel lanes pass over K parallel lanes. Now, bowl a balls into the upper collection of lanes and b into the lower. We must compute the probability that exactly c balls will fall.

Fix a choice of c of the upper a balls, a choice of c of the lower $K-b$ initially empty lanes, and a bijection w from these balls to these empty lanes. We compute the probability that our chosen balls fall into our chosen lanes according to w, and then sum over w and these choices to obtain $f_b^a(c)$.

Some terminology will help us to stay organized. The K upper lanes consist of our chosen c *briefly-full* lanes, $a-c$ *always-full* lanes, and $K-a$ *irrelevant* lanes. The

K lower lanes consist of our chosen c *briefly-empty* lanes, $K-b-c$ *always-empty* lanes, and b *irrelevant* lanes. The irrelevant lanes are either upper lanes that start and remain empty, or lower lanes that start and remain full. These have no effect on the probability.

Consider the crossings where an always-full lane passes over an always-empty lane. At each such crossing, a ball will pass over an empty lane, contributing a factor of q. Taken together, these crossings contribute the term

$$q^{(a-c)(K-b-c)}.$$

Consider the crossings where an always-full lane passes over a briefly-empty lane. Such a crossing will contribute a factor of q if and only if the briefly-empty lane is still empty, having not yet met its corresponding briefly-full lane. The number of times this happens is the number of pairs of upper lanes consisting of a briefly-full lane to the left of an always-full lane. This is independent of w and of the choice of c briefly-empty lower lanes. Ranging over the choice of c briefly-full upper lanes contributes the powers of q in the combinatorial definition of

$$\binom{a}{c}_q.$$

Consider the crossings where a briefly-full lane passes over an always-empty lane. Such a crossing will contribute a factor of q if and only if the briefly-full lane is still full, having not yet met its corresponding briefly-empty lane. The number of times this happens is the number of pairs of lower lanes consisting of an always-empty lane to the left of a briefly-empty lane. This is independent of w and of the choice of c briefly-full upper lanes. Ranging over the choice of c briefly-empty lower lanes contributes the powers of q in the combinatorial definition of

$$\binom{K-b}{c}_q.$$

Consider the crossings where a ball falls from a briefly-full lane to a briefly-empty lane. Each such crossing contributes a factor of $(1-q)$. Taken together, these contribute the term

$$(1-q)^c.$$

Finally, consider the crossings where a briefly-full lane passes over a briefly-empty lane but no ball falls there. This will contribute a factor of q if and only if the briefly-full lane is still full and the briefly-empty lane is still empty. The number of times this happens is the number of pairs of briefly-full upper lanes i and j such that i is to the left of j and $w(i)$ is to the left of $w(j)$. This is independent of the choices of c briefly-empty lower lanes and of c briefly-full upper lanes. Ranging over the choice of w contributes the powers of q in the combinatorial definition of

$$[c]!$$

Multiply the above contributions to get the desired formula for $f_b^a(c)$. $\square$

References

[1] C. W. Armond, Walks along braids and the colored Jones polynomial, *J. Knot Theory Ramifications* **23**(2) (2014) 1450007.

[2] R. Dipper and G. James, Representations of Hecke algebras of general linear groups, *Proc. London Math. Soc.* (3) **52**(1) (1986) 20–52.

[3] V. F. R. Jones, Hecke algebra representations of braid groups and link polynomials, *Ann. of Math.* (2) **126**(2) (1987) 335–388.

[4] C. Kassel and V. Turaev, *Braid Groups*, Graduate Texts in Mathematics, Vol. 247 (Springer, New York, 2008). With the graphical assistance of Olivier Dodane.

[5] D. Kazhdan and H. Wenzl, Reconstructing monoidal categories, in *I. M. Gelfand Seminar*, Advances in Soviet Mathematics, Vol. 16 (American Mathematical Society, Providence, RI, 1993), pp. 111–136.

[6] P. Kirk, C. Livingston and Z. Wang, The Gassner representation for string links, *Commun. Contemp. Math.* **3**(1) (2001) 87–136.

[7] X.-S. Lin, F. Tian, and Z. Wang, Burau representation and random walks on string links, *Pacific J. Math.* **182**(2) (1998) 289–302.

[8] X.-S. Lin and Z. Wang, Random walk on knot diagrams, colored Jones polynomial and Ihara-Selberg zeta function, in *Knots, Braids, and Mapping Class Groups — Papers dedicated to Joan S. Birman*, AMS/IP Studies in Advanced Mathematics, Vol. 24 (American Mathematical Society, Providence, RI, 2001), pp. 107–121.