Concise Encyclopedia of Knot Theory

Chapter 13

Algebraic and combinatorial invariants

13.6 The Temperley-Lieb algebra and planar algebras

13.6.1 Two definitions of the Temperley-Lieb algebra

The Temperley-Lieb algebra was introduced by Temperley and Lieb in 1971 [7] in connection to a problem in statistical mechanics. It has since found applications in diverse fields such as knot theory, representation theory, and von Neumann algebras. It is arguably the first and simplest example of a *diagrammatic algebra*.

When we speak of "the" Temperley-Lieb algebra, we really refer to a family of algebras $\mathrm{TL_n}(\delta)$, indexed by a non-negative integer n and a complex number δ. The easiest way to define $\mathrm{TL_n}(\delta)$ is by generators and relations.

Definition 1 $\mathrm{TL_n}(\delta)$ *is the unital associative* $\mathbb{C}$*-algebra generated by*

$$e_1, \ldots, e_{n-1}$$

subject to the relations:

- $e_i^2 = \delta e_i$.
- $e_i e_j e_i = e_i$ *if* $|i-j| = 1$.
- $e_i e_j = e_j e_i$ *if* $|i-j| > 1$.

(We could use an arbitrary ring of scalars, but we will stick with $\mathbb{C}$*.)*

What makes $\mathrm{TL_n}(\delta)$ a diagrammatic algebra is an alternative definition in terms of *Temperley-Lieb diagrams.* Fix a rectangle with n marked points on the bottom edge and n marked points on the top edge. An n-strand Temperley-Lieb diagram is a collection of n disjoint arcs in the rectangle, each connecting a pair of marked points. The arcs are called *strands.*

Two n-strand Temperley-Lieb diagrams are considered to be the same if there is an isotopy that goes from one diagram to the other, while remaining a Temperley-Lieb diagram at all times. Thus, all that matters is which pairs of marked points are connected.

The set of n-strand Temperley-Lieb diagrams is a basis for $\mathrm{TL_n}(\delta)$, as a vector space. There is no diagrammatic interpretation of addition or scalar multiplication: a typical element of $\mathrm{TL_n}(\delta)$ is a purely formal linear combination of diagrams.

The product of Temperley-Lieb diagrams is defined by the following procedure of "stacking and bursting bubbles". Given two n-strand Temperley-Lieb diagrams a and b, place a on top of b, and connect the marked points on the top of a to the corresponding marked points on the bottom of b. The resulting diagram might contain closed loops made up of arcs from a and b. Let ℓ be the number of closed loops, and let c be the Temperley-Lieb diagram obtained by removing these closed loops. The product of a and b is then defined to be $ab = \delta^{\ell} c$.

Extend the above multiplication of diagrams to a bilinear operation on $\mathrm{TL_n}(\delta)$. Thus, to multiply two linear combinations of diagrams, use the distributive law, and then the above rule for multiplication of diagrams. In summary, the diagrammatic definition of $\mathrm{TL_n}(\delta)$ is as follows.

Definition 2 $\mathrm{TL_n}(\delta)$ *is the algebra of formal linear combinations of n-strand Temperley-Lieb diagrams. Multiplication is the unique bilinear operation that extends the above "stacking and bursting bubbles" operation on diagrams.*

It is not immediately obvious, but Definitions 1 and 2 do in fact give the same algebra. This was proved by Kauffman [5].

The identity element $\mathbf{1}$ of $\mathrm{TL_n}(\delta)$ is the diagram in which every strand is vertical. The generator e_i is the diagram in which every strand is vertical except for a strand connecting marked points number i and $i+1$ on the bottom, and a strand connecting marked points number i and $i+1$ on the top. See Figure 13.1.

See Figure 13.2 for an example of a Temperley-Lieb diagram rewritten as a product of diagrams e_i. Kauffman proved that this is always possible [5]. See also his survey article [6], which fills in some of the details of the proof

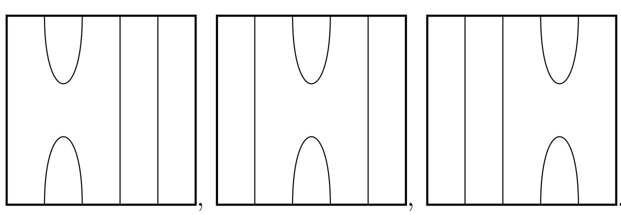

FIGURE 13.1: Generators e_1, e_2, e_3 of $\mathrm{TL}_4(\delta)$.

that were previously left to the reader. Ernst, Hastings and Salmon [1] used a similar procedure to go from a diagram to a word of shortest possible length.

The diagrams e_i satisfy the relations in Definition 1, as can be verified by simply drawing the diagrams representing each side of each relation. The fact that no additional relations are needed can be proved by a dimension count. In the diagrammatic definition, the dimension of $\mathrm{TL_n}(\delta)$ is the number of Temperley-Lieb diagrams, which is known to equal the *Catalan number*

$$\dim \mathrm{TL_n}(\delta) = \mathrm{C_n} = \frac{1}{\mathrm{n}+1}\binom{2\mathrm{n}}{\mathrm{n}}.$$

Jones [2] showed that the relations given in the algebraic definition of $\mathrm{TL_n}(\delta)$ can be used to put any word in the generators into a certain normal form, and the number of words in normal form is also C_n.

13.6.2 The Temperley-Lieb planar algebra

The Temperley-Lieb planar algebra takes the idea of multiplying by stacking diagrams, and extends it to allow for infinitely many other ways of connecting diagrams. In a sense, this uses the two-dimensional nature of the diagrams to give extra structure, whereas a traditional algebra only uses the one-dimensional operation of concatenating strings of letters. We use *planar tangles* to specify operations; see, for example, Figure 13.3.

A planar tangle consists of the following data:

- an *output disk* D,
- *input disks* $D_1, \ldots, D_k$ in the interior of D,
- an even number of *marked points* on the boundary of each input and output disk,
- finitely many *strands*, and
- a *marked interval* touching the boundary of each input or output disk.

The marked points are connected in pairs by strands that are disjoint from each other and from the interior of the input disks. We also allow strands that

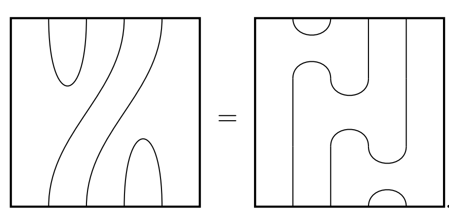

FIGURE 13.2: The element $e_1e_2e_3$ of $\mathrm{TL_4}(\delta)$.

are closed loops. Each input and output disk has one marked interval between adjacent marked points, usually indicated by a star near the boundary. If a disk has no marked points on the boundary then the marked interval is the entire boundary. Two planar tangles are considered to be the same if there is an isotopy that goes from one to the other, while remaining a planar tangle at all times.

For each planar tangle T, there is a corresponding *partition function* Z_T. Suppose T has k input disks with $2n_1, \ldots, 2n_k$ marked points, and $2n$ marked points on the boundary of the output disk. Then Z_T will be a linear map

$$Z_T\colon \mathrm{TL}_{\mathrm{n}_1} \otimes \cdots \otimes \mathrm{TL}_{\mathrm{n}_\mathrm{k}} \to \mathrm{TL}_\mathrm{n}.$$

To define Z_T, it suffices to define its action on

$$v_1 \otimes \cdots \otimes v_k,$$

where v_i is an n_i-strand Temperley-Lieb diagram.

The idea is to insert the diagrams v_i into the input disks of T. It helps to think of them as diagrams in a disk, by "rounding off the corners" of the rectangles, but keeping a marked interval at what was the left edge of the rectangle. Insert each diagram v_i into the input disk D_i, rotated so that their marked intervals align, and then join each marked point on the boundary of v_i to the corresponding marked point on the boundary of D_i.

After inserting a diagram into each input disk, we obtain a diagram in D. Let ℓ be the number of closed loops in this diagram, and let v be the Temperley-Lieb diagram obtained by deleting these closed loops. Then

$$Z_T(v_1 \otimes \cdots \otimes v_k) = \delta^\ell v.$$

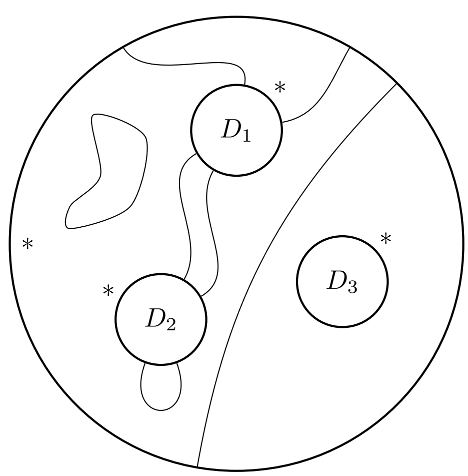

FIGURE 13.3: A planar tangle with three input disks.

Extend this to a linear function on all of $\mathrm{TL}_{\mathrm{n}_1} \otimes \cdots \otimes \mathrm{TL}_{\mathrm{n}_\mathrm{k}}$.

Definition 3 *The Temperley-Lieb planar algebra is the collection of vector spaces*

$$\mathrm{TL}_0, \mathrm{TL}_1, \mathrm{TL}_2, \ldots$$

together with the above rule that assigns a partition function to each planar tangle.

A general planar algebra is defined similarly.

Definition 4 *A planar algebra is a collection of vector spaces*

$$V_0, V_1, V_2, \ldots$$

together with a rule that assigns a multilinear operation Z_T to each planar tangle T, and satisfies a certain naturality condition.

The naturality condition is cumbersome to state precisely, but comes for free in most examples where the vector spaces V_n are spanned by some kind of diagrams. Here, the diagrams include Temperley-Lieb diagrams, but may also allow extra features like crossings, orientations, or vertices. The function Z_T is defined by inserting diagrams into input disks of T, and perhaps performing local operations that simplify the resulting diagram. The naturality condition then says that if you insert diagrams into planar tangles, and insert those planar tangles into other planar tangles, then the order of those two operations should not matter.

Planar algebras were originally defined by Jones [3] as a way to axiomatize the standard invariant of a subfactor. This original definition, now called a *subfactor planar algebra*, has additional features and axioms that we have left out. (The relevant adjectives are: shaded, evaluable, unital, involutive, spherical, and positive.) Conversely, a less restrictive definition than ours might allow odd number of marked points on boundaries of a disk, or might not require Z_T to be invariant under isotopy of T. A survey of variations on the notion of planar algebra can be found in Jones's lecture notes from 2011 [4].

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