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SUPPLEMENTS OF BOUNDED PERMUTATION GROUPS

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Abstract. Let $\lambda \leq \kappa$ be infinite cardinals and let Ω be a set of cardinality κ. The bounded permutation group $B_\lambda(\Omega)$, or simply B_λ, is the group consisting of all permutations of Ω which move fewer than λ points in Ω. We say that a permutation group G acting on Ω is a *supplement of* B_λ if $B_\lambda G$ is the full symmetric group on Ω.

In [7], Macpherson and Neumann claimed to have classified all supplements of bounded permutation groups. Specifically, they claimed to have proved that a group G acting on the set Ω is a supplement of B_λ if and only if there exists $\Delta \subset \Omega$ with $|\Delta| < \lambda$ such that the setwise stabiliser $G_{\{\Delta\}}$ acts as the full symmetric group on $\Omega \setminus \Delta$. However I have found a mistake in their proof. The aim of this paper is to examine conditions under which Macpherson and Neumann's claim holds, as well as conditions under which a counterexample can be constructed. In the process we will discover surprising links with cardinal arithmetic and Shelah's recently developed *pcf* theory.

§1. Introduction.

This paper concerns an error in Macpherson and Neumann's paper "Subgroups of Infinite Symmetric Groups" [7]. The proof they give for their Theorem 1.2 contains a tacit assumption that all cardinals are regular. In this paper I will demonstrate conditions under which the theorem is true as well as conditions under which a counterexample can be constructed.

Suppose Γ is an infinite set. We denote the full symmetric group on Γ by $\mathrm{Sym}(\Gamma)$. If $\lambda \leq |\Gamma|$ is an infinite cardinal then the *bounded group* $B_\lambda(\Gamma)$ is the group

$$\{ g \in \mathrm{Sym}(\Gamma) : |\operatorname{supp}(g)| < \lambda \}.$$

It is well known that the bounded groups $B_\lambda(\Gamma)$ and the finitary alternating group $\mathrm{Alt}(\Gamma)$ are the only non-trivial proper normal subgroups of $\mathrm{Sym}(\Gamma)$. See, for example, [9, Theorem 11.3.4].

Throughout the paper, Ω will denote an infinite set of cardinality κ and S will denote $\mathrm{Sym}(\Omega)$. If $\lambda \leq \kappa$ is an infinite cardinal then we use B_λ to denote $B_\lambda(\Omega)$. A *supplement* of B_λ in S is a group $G \leq S$ such that $B_\lambda G = S$. It can be shown that if there exists $\Delta \subset \Omega$ with cardinality less than λ such that the setwise stabiliser $G_{\{\Delta\}}$ acts as the full symmetric group on Δ^c then G is a supplement of B_λ. (See the first paragraph of the proof of [7, Theorem 1.2].) We now define two propositions, both of which imply that all supplements of B_λ are of this form.

DEFINITION 1.1. Let Semmes' Proposition, or SP for short, denote the following statement.

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> Suppose Ω is an infinite set, $\lambda \leq |\Omega|$ is an infinite cardinal, and $G \leq \mathrm{Sym}(\Omega)$ is such that $B_\lambda G = \mathrm{Sym}(\Omega)$. Then there exists $\Delta \subset \Omega$ with $|\Delta| < \lambda$ such that the pointwise stabiliser $G_{(\Delta)}$ acts as the full symmetric group on Δ^c.

Let Macpherson and Neumann's Proposition, or MNP for short, denote the following statement.

> Suppose Ω is an infinite set, $\lambda \leq |\Omega|$ is an infinite cardinal, and $G \leq \mathrm{Sym}(\Omega)$ is such that $B_\lambda G = \mathrm{Sym}(\Omega)$. Then there exists $\Delta \subset \Omega$ with $|\Delta| < \lambda$ such that the setwise stabiliser $G_{\{\Delta\}}$ acts as the full symmetric group on Δ^c.

In [10], Semmes announced that the Generalised Continuum Hypothesis implies SP. In [7], Macpherson and Neumann claimed to have proved MNP using standard set theory without the luxury of the GCH. Unfortunately, their proof was incorrect. To examine the truth or falsehood of special cases of SP and MNP we introduce the following "local" versions of these propositions.

DEFINITION 1.2. If $\lambda \leq \kappa$ are infinite cardinals then let $\mathrm{SP}(\kappa, \lambda)$ denote the following statement:

> If $|\Omega| = \kappa$ and $G \leq \mathrm{Sym}(\Omega)$ satisfies $B_\lambda G = \mathrm{Sym}(\Omega)$ then there exists $\Delta \subset \Omega$ with $|\Delta| < \lambda$ such that the pointwise stabiliser $G_{(\Delta)}$ acts as the full symmetric group on Δ^c;

and let $\mathrm{MNP}(\kappa, \lambda)$ denote the following statement:

> If $|\Omega| = \kappa$ and $G \leq \mathrm{Sym}(\Omega)$ satisfies $B_\lambda G = \mathrm{Sym}(\Omega)$ then there exists $\Delta \subset \Omega$ with $|\Delta| < \lambda$ such that the setwise stabiliser $G_{\{\Delta\}}$ acts as the full symmetric group on Δ^c.

The following concept will play a very important rôle in determining the truth or falsehood of $\mathrm{MNP}(\kappa, \lambda)$.

DEFINITION 1.3. Suppose Γ is an uncountable set and $\mu < |\Gamma|$ is an infinite cardinal. A *μ-covering of* Γ is a collection $\mathscr{C}$ of subsets of Γ, each of cardinality less than $|\Gamma|$, such that every μ-subset of Γ is a subset of some element of $\mathscr{C}$.

If $\mu < \nu$ are cardinals then let the *μ-covering number* for ν, or $\mathrm{cov}(\nu, \mu)$, be the minimum cardinality of a μ-covering of the set ν.

In [12], $\mathrm{cov}(\nu, \mu)$ is only defined when $\mathrm{cf}(\nu) \leq \mu < \nu$. For our purposes it is convenient to keep the definition as written when $\mu < \mathrm{cf}(\nu)$, in which case we have the following easy result.

LEMMA 1.1. *If* $\mu < \mathrm{cf}(\nu)$ *then* $\mathrm{cov}(\nu, \mu) = \mathrm{cf}(\nu)$.

On the other hand, in the case $\mu \geq \mathrm{cf}(\nu)$, we have the following, which can be proved by a fairly straightforward diagonalisation argument.

LEMMA 1.2. *If* $\mathrm{cf}(\nu) \leq \mu < \nu$ *then* $\mathrm{cov}(\nu, \mu) > \nu$.

We will also use the concept of coverings of groups by chains of proper subgroups. Macpherson and Neumann pointed out in [7, Note 3] that any group which is not finitely generated may be written as the union of an increasing chain of proper subgroups. They then went on to make the following definition.

DEFINITION 1.4. If G is not finitely generated then write $c(G)$ for the least cardinal number θ such that G can be expressed as the union of an increasing chain of θ proper subgroups of G.

Let $c(\kappa) := c(\mathrm{Sym}(\kappa))$.

The following is [7, Theorem 1.1].

LEMMA 1.3 (Macpherson and Neumann). $c(\kappa) > \kappa$.

The main results of this paper are as follows.

THEOREM 1.1. *Suppose* $\lambda \leq \kappa$. *Then* $\mathrm{SP}(\kappa, \lambda)$ *holds if and only if*

$$(\forall \mu < \lambda)(2^{\mu} < 2^{\kappa}).$$

THEOREM 1.2. *Suppose* $\lambda \leq \kappa$ *and* $\mathrm{SP}(\kappa, \lambda)$ *is false. Then* $\mathrm{MNP}(\kappa, \lambda)$ *holds if*

$$(\forall \nu \in [\lambda, \kappa])(\forall \mu < \lambda)(\mathrm{cov}(\nu, \mu) < c(\kappa)),$$

and only if

$$(\forall \nu \in [\lambda, \kappa])(\forall \mu < \lambda)(\mathrm{cov}(\nu, \mu) < 2^{\kappa}).$$

We will prove these theorems in Sections 3 and 4 respectively. In Section 5 we use Theorem 1.2, together with some results from Shelah's *pcf* theory, to prove $\mathrm{MNP}(\kappa, \lambda)$ for certain values of κ and λ. In Section 6 we find values of κ and λ for which $\neg\,\mathrm{MNP}(\kappa, \lambda)$ is consistent with ZFC. Some of these consistency results will require large cardinal axioms. However no knowledge of large cardinal axioms will be necessary if one is prepared to take on faith some results from the study of cardinal arithmetic.

To give an initial glimpse of the consequences of Theorem 1.2, we give two corollaries which follow immediately from Theorem 1.2 and the preceding Lemmas.

COROLLARY 1.1. *If* $\lambda \leq \kappa < \aleph_{\omega}$ *then* $\mathrm{MNP}(\kappa, \lambda)$ *holds.*

COROLLARY 1.2. *If* $2^{\aleph_0} = 2^{\aleph_{\omega}} = \aleph_{\omega+1}$ *then* $\neg\,\mathrm{MNP}(\aleph_{\omega}, \aleph_1)$.

§2. Preliminaries.

2.1. Notation and terminology. Throughout this paper, all functions and permutations will act on the right. We use angular brackets $\langle\,\rangle$ to denote the group generated by the enclosed list of elements or subsets of a group. If $\Gamma \subset \Omega$ then Γ^c is simply shorthand for $\Omega \setminus \Gamma$, that is, $\{\, \alpha \in \Omega : \alpha \notin \Gamma \,\}$. A *moiety* of Ω is a set $\Sigma \subset \Omega$ such that $|\Sigma| = |\Sigma^c|$. A *μ-set* is a set of cardinality μ. A *μ-subset* of a set Γ is μ-set which is a subset of Γ. If Γ is a set and λ is an infinite cardinal then $[\Gamma]^{<\lambda}$ denotes the set of all subsets of Γ which have cardinality less than λ.

If $g \in \mathrm{Sym}(\Omega)$ then the *support* of g is given by

$$\mathrm{supp}(g) := \{\, \alpha \in \Omega : \alpha g \neq \alpha \,\}.$$

If $G \leq \mathrm{Sym}(\Omega)$ then the *support* of G is given by

$$\mathrm{supp}(G) := \bigcup \{\, \mathrm{supp}(g) : g \in G \,\}.$$

Suppose $G \leq \mathrm{Sym}(\Omega)$ and $\Gamma \subset \Omega$. The *pointwise stabiliser* of Γ in G is given by

$$G_{(\Gamma)} := \{ g \in G : (\forall \gamma \in \Gamma)(\gamma g = \gamma) \}.$$

The *setwise stabiliser* of Γ in G is given by

$$G_{\{\Gamma\}} := \{ g \in G : \Gamma g = \Gamma \}.$$

This induces the following permutation group acting on Γ:

$$G^{\Gamma} := \{ g{\upharpoonright}\Gamma : g \in G_{\{\Gamma\}} \}.$$

If $G^{\Gamma} = \mathrm{Sym}(\Gamma)$ then we say that Γ is *full* for G.

All cardinals in this paper are assumed to be infinite unless otherwise stated. If $\kappa = \aleph_\alpha$ and $n < \omega$ then $\kappa^{+n} = \aleph_{\alpha+n}$ and $\kappa^{+} = \aleph_{\alpha+1}$. I use round and square brackets to indicate intervals of cardinals in the usual way: for example, if $\mu < \nu$ are cardinals then $(\mu, \nu]$ is the set of cardinals λ such that $\mu < \lambda \leq \nu$.

2.2. Preliminary results. We now list some basic background lemmas.

The following is proved in [7, Lemma 2.3].

Lemma 2.1. *Suppose Σ_1 and Σ_2 are subsets of Ω such that $|\Sigma_1 \cap \Sigma_2| = \kappa$ and $\Sigma_1 \cup \Sigma_2 = \Omega$. If $G \leq S$ is such that Σ_1 and Σ_2 are full for G then $G = S$.*

The following is salvaged from Macpherson and Neumann's incorrect proof of MNP. It is proved in the second paragraph of the proof of [7, Theorem 1.2].

Lemma 2.2 (Macpherson and Neumann). *If $B_\lambda G = S$ then there is a moiety Σ of Ω and a group $H \leq G_{\{\Sigma\}}$ such that Σ is full for H and $H^{\Omega \setminus \Sigma} \leq B_\lambda(\Omega \setminus \Sigma)$.*

Finally, we will use the following result.

Lemma 2.3. *If λ is uncountable and $B_\lambda G = S$ then there exists $\mu < \lambda$ such that $B_{\mu^+} G = S$.*

Proof. Suppose $\lambda = \aleph_\alpha \leq \kappa$ and $B_\lambda G = S$. Note that $B_\lambda = \bigcup\{ B_{\aleph_{\beta+1}} : \beta < \alpha \}$. It follows that $(B_{\aleph_{\beta+1}} G)_{\beta<\alpha}$ is an increasing chain of subgroups of S whose union is S. But the length of this chain is no greater than κ. Thus, by Lemma 1.3, not all of the groups in the chain are proper subgroups of S, and we are done. ⊣

§3. Semmes' proposition. The aim of this section is to prove Theorem 1.1. Macpherson and Neumann state this in [7, Theorem 4.4] as a corollary to MNP. The proof given here is completely different and does not assume MNP. We will use the following lemmas.

Lemma 3.1. *Suppose Σ is a moiety of Ω and $K \leq S$ is such that Σ is full for K and $|K^{\Omega\setminus\Sigma}| < 2^\kappa$. Then $K_{(\Omega\setminus\Sigma)} = S_{(\Omega\setminus\Sigma)}$.*

Proof. Let $\mathscr{C}$ be a collection of pairwise disjoint κ-subsets of Σ such that $|\mathscr{C}| = \kappa$. For every subset $\mathscr{A}$ of $\mathscr{C}$, choose $k_{\mathscr{A}} \in K$ such that $\mathrm{supp}(k_{\mathscr{A}}) \cap \Sigma = \bigcup \mathscr{A}$. There must exist distinct subsets $\mathscr{A}$ and $\mathscr{B}$ of $\mathscr{C}$ such that $k_{\mathscr{A}}{\upharpoonright}\Sigma^c = k_{\mathscr{B}}{\upharpoonright}\Sigma^c$. Let $k := k_{\mathscr{A}} k_{\mathscr{B}}{}^{-1}$. Then $k \in K_{(\Omega\setminus\Sigma)}$. Also, $\mathrm{supp}(k)$ must include the symmetric difference of $\bigcup\mathscr{A}$ and $\bigcup\mathscr{B}$, so $|\,\mathrm{supp}(k) \cap \Sigma| = \kappa$. Thus $K_{(\Omega\setminus\Sigma)}$ contains a permutation of degree κ. But $K_{(\Omega\setminus\Sigma)}$ acts on Σ as a normal subgroup of $\mathrm{Sym}(\Sigma)$. Thus $K_{(\Omega\setminus\Sigma)} = S_{(\Omega\setminus\Sigma)}$. ⊣

LEMMA 3.2. *Suppose $B_\lambda G = S$ and there exists a moiety Σ of Ω such that $G_{(\Omega\setminus\Sigma)} = S_{(\Omega\setminus\Sigma)}$. Then there exists a set $\Delta \subset \Omega$ with $|\Delta| < \lambda$ such that $G_{(\Delta)} = S_{(\Delta)}$.*

PROOF. Since $B_\lambda G = S$, there exists $g \in G$ such that $|\Sigma \cap \Sigma g| = \kappa$ and $|\Omega \setminus (\Sigma \cup \Sigma g)| < \lambda$. Let $\Delta := \Omega \setminus (\Sigma \cup \Sigma g)$. Then Σ and Σg are both full for $G_{(\Delta)}$. By Lemma 2.1, $G_{(\Delta)} = S_{(\Delta)}$. ⊣

PROOF OF THEOREM 1.1. The "only if" half was correctly proved in [7, Lemma 4.2]. Conversely, suppose $G \leq S$ satsisfies $B_\lambda G = S$, where $\lambda \leq \kappa$ are such that $2^\mu < 2^\kappa$ for all $\mu < \lambda$. We are required to show that there exists a set $\Delta \subset \Omega$ such that $|\Delta| < \lambda$ and $G_{(\Delta)} = S_{(\Delta)}$. By Lemmas 3.1 and 3.2, it suffices to show that there exists a moiety Σ of Ω and a group $K \leq G$ such that Σ is full for K and $|K^{\Omega\setminus\Sigma}| < 2^\kappa$.

Suppose $\lambda = \aleph_0$. By Lemma 2.2, there is a moiety Σ of Ω and a subgroup $H \leq G_{\{\Sigma\}}$ such that Σ is full for H and $H^{\Omega\setminus\Sigma} \leq B_{\aleph_0}(\Omega \setminus \Sigma)$. But $|B_{\aleph_0}(\Omega \setminus \Sigma)| \leq \kappa$, so we are done.

From now on, we will assume that $\lambda > \aleph_0$.

By Lemma 2.3, there exists $\mu < \lambda$ such that $B_{\mu^+} G = S$. By Lemma 2.2, there is a moiety Σ of Ω and a subgroup $H \leq G_{\{\Sigma\}}$ such that Σ is full for H and $H^{\Omega\setminus\Sigma} \leq B_{\mu^+}(\Omega \setminus \Sigma)$.

If $2^\mu \geq \kappa$ then

$$|H^{\Omega\setminus\Sigma}| \leq \kappa^\mu \leq (2^\mu)^\mu = 2^\mu < 2^\kappa,$$

so we are done.

From now on, we will assume that $2^\mu < \kappa$.

CLAIM. *There is a μ-set Φ such that every μ-subset of $\Omega \setminus \Phi$ is full for $G \cap B_{\mu^+}$.*

PROOF OF CLAIM. Suppose the contrary: that for every μ-set Φ there is a μ-subset of $\Omega \setminus \Phi$ which is not full for $G \cap B_{\mu^+}$. We can inductively construct a sequence $(\Phi_\xi)_{\xi<\mu^+}$ of pairwise disjoint μ-sets which are not full for $G \cap B_{\mu^+}$. Let $\Psi := \bigcup_{\xi<\mu^+} \Phi_\xi$. We can assume that $|\Omega \setminus \Psi| = \kappa$ (by omitting every second term in the sequence $(\Phi_\xi)_{\xi<\mu^+}$ if necessary). Since $B_{\mu^+} G = S$, there exists $g \in G$ such that $|\Psi g \setminus \Sigma| \leq \mu$. There must exist $\xi < \mu^+$ such that $\Phi_\xi g \subset \Sigma$. Then $\Phi_\xi g$ is full for $G \cap B_{\mu^+}$. Hence Φ_ξ is also full for $G \cap B_{\mu^+}$. This contradiction completes the proof of the claim. ⊣

CLAIM. *There exists a set Ψ with the following properties*:
- $\Phi \subset \Psi$;
- $|\Psi| = 2^\mu$;
- *For every μ-subset Γ of Ψ we have $(G \cap B_{\mu^+})^\Gamma \leq (G_{(\Omega\setminus\Psi)})^\Gamma$.*

Note that the second and third of these properties will also hold for Ψg for any $g \in G$.

PROOF OF CLAIM. To construct Ψ, we first construct an increasing chain $(\Psi_\xi)_{\xi<\mu^+}$ such that for every $\xi < \mu^+$ we have:
- $\Phi \subset \Psi_\xi$;
- $|\Psi_\xi| = 2^\mu$;

- If $\eta < \xi$ then for every μ-subset Γ of Ψ_η we have $(G \cap B_{\mu^+})^\Gamma \leq (G_{(\Omega\setminus\Psi_\xi)})^\Gamma$.

Let Ψ_0 be an arbitrary 2^μ-set such that $\Phi \subset \Psi_0$.

If ξ is a non-zero limit ordinal then let $\Psi_\xi := \bigcup_{\eta<\xi} \Psi_\eta$.

Now suppose $\xi = \eta + 1$. There are only 2^μ distinct μ-subsets of Ψ_η, each of which can be permuted in only 2^μ distinct ways. Thus there is a group $K \leq G \cap B_{\mu^+}$ such that $|K| = 2^\mu$ and $K^\Gamma = (G \cap B_{\mu^+})^\Gamma$ for every μ-subset Γ of Ψ_η. Let $\Psi_\xi := \operatorname{supp}(K)$. Then $K \leq G_{(\Omega\setminus\Psi_\xi)}$. Thus $(G \cap B_{\mu^+})^\Gamma \leq (G_{(\Omega\setminus\Psi_\xi)})^\Gamma$ for every μ-subset Γ of Ψ_η. Also $\Phi \subset \Psi_\xi$ and $|\Psi_\xi| = 2^\mu$, so Ψ_ξ satisfies all of our requirements.

Having constructed the chain, we now let

$$\Psi := \bigcup_{\xi<\mu^+} \Psi_\xi.$$

Clearly $\Phi \subset \Psi$ and $|\Psi| = 2^\mu$. If Γ is a μ-subset of Ψ then $\Gamma \subset \Psi_\eta$ for some $\eta < \mu^+$, so

$$(G \cap B_{\mu^+})^\Gamma \leq (G_{(\Omega\setminus\Psi_{\eta+1})})^\Gamma \leq (G_{(\Omega\setminus\Psi)})^\Gamma.$$

Thus Ψ satisfies all of our requirements. $\dashv$

Having constructed Ψ, we now replace H by $H_{(\Sigma\cap\Psi)}$ and replace Σ by $\Sigma \setminus \Psi$. Thus we may assume that Σ is disjoint from Ψ and that the conclusions of Lemma 2.2 still hold, namely that $H \leq G_{\{\Sigma\}}$ is such that Σ is full for H and $H^{\Omega\setminus\Sigma} \leq B_{\mu^+}(\Omega \setminus \Sigma)$.

Claim. *If Γ is a μ-subset of $\Omega \setminus \Sigma$ then there exists $g \in G_{(\Sigma)}$ such that $\Gamma g \subset \Psi$.*

Proof of claim. Let Γ be a μ-subset of $\Omega \setminus \Sigma$. Let $\Xi \subset \Omega$ be a μ-set which includes $\Gamma \setminus \Psi$, meets Ψ at a μ-set, and is disjoint from Φ. Since Ξ is disjoint from Φ, it is full for $G \cap B_{\mu^+}$. It follows that there exists $h \in G \cap B_{\mu^+}$ such that $\Gamma \subset \Psi \cup \Psi h$.

Now let $\Theta \subset \Psi h$ be a μ-set which includes $\Psi h \setminus \Psi$, meets $(\Psi \cap \Psi h) \setminus \Gamma$ at a μ-set, and is disjoint from Φ. Since Θ is disjoint from Φ, it is full for $G \cap B_{\mu^+}$. Since Θ is a μ-subset of Ψh, it is also full for $G_{(\Omega\setminus\Psi h)}$. It follows that there exists a permutation $g \in G_{(\Omega\setminus\Psi h)}$ which fixes $\Psi h \cap \Sigma$ pointwise and sends $\Gamma \cap \Psi h$ into Ψ.

Now $\Gamma \setminus \Psi h$ is a subset of Ψ and is fixed pointwise by g, so g in fact sends all of Γ into Ψ. Also, both $\Omega \setminus \Psi h$ and $\Psi h \cap \Sigma$ are fixed pointwise by g, so g fixes Σ pointwise. Thus g satisfies the requirements of the claim. $\dashv$

Suppose $h \in H$. Let $\Gamma := \operatorname{supp}(h) \setminus \Sigma$. By the previous claim, there exists $g \in G_{(\Sigma)}$ such that $\Gamma g \in \Psi$. Let $k_h := g^{-1}hg$. Then $(k_h \upharpoonright \Sigma) = (h \upharpoonright \Sigma)$ and $(k_h \upharpoonright \Sigma^c) \in B_{\mu^+}(\Psi)$. (I am abusing notation slightly by considering $B_{\mu^+}(\Psi)$ to be a subgroup of $\operatorname{Sym}(\Sigma^c)$ in the obvious way.)

Now let $K := \langle k_h : h \in H \rangle$. Then Σ is full for K and $K^{\Omega\setminus\Sigma} \leq B_{\mu^+}(\Psi)$. But

$$|B_{\mu^+}(\Psi)| \leq (2^\mu)^\mu = 2^\mu < 2^\kappa.$$

This completes the proof of the theorem.

§4. Macpherson and Neumann's proposition.

The aim of this section is to prove Theorem 1.2.

4.1. The "if" half. We now prove the "if" half of Theorem 1.2.

The proof is based on Macpherson and Neumann's incorrect proof of [7, Theorem 1.2]. I have overcome the mistake in their proof by using the added assumption that certain covering numbers are sufficiently small. We will use the following simple lemma, whose proof is almost identical to that of [7, Corollary 3.1].

LEMMA 4.1. *Suppose $\mathscr{C}$ is a collection of subsets of S such that $|\mathscr{C}| < c(\kappa)$ and $S = \langle \bigcup \mathscr{C} \rangle$. Then there exists a finite subcollection $\mathscr{B} \subset \mathscr{C}$ such that $S = \langle \bigcup \mathscr{B} \rangle$.*

PROOF OF THE "IF" HALF OF THEOREM 1.2. Fix $\lambda \leq \kappa$. Suppose $\mathrm{SP}(\kappa, \lambda)$ is false and $\mathrm{cov}(\nu, \mu) < c(\kappa)$ for all $\nu \in [\lambda, \kappa]$ and $\mu < \lambda$. Let G be such that $B_\lambda G = S$. We are required to find a set $\Delta \subset \Omega$ such that $|\Delta| < \lambda$ and Δ^c is full for G.

Since $\mathrm{SP}(\kappa, \lambda)$ is false, Theorem 1.1 implies that λ must be uncountable. By Lemma 2.3, there exists $\mu < \lambda$ such that $B_{\mu^+} G = S$. By Lemma 2.2, there exists a moiety Σ_0 of Ω and a group $H \leq G_{\{\Sigma_0\}}$ such that Σ_0 is full for H and $H^{\Omega \setminus \Sigma_0} \leq B_{\mu^+}(\Omega \setminus \Sigma_0)$.

If $\Psi \subset \Omega \setminus \Sigma_0$ then let H_Ψ denote the group $\{ h \in H : \mathrm{supp}(h) \subset \Sigma_0 \cup \Psi \}$. Let $\Psi_0 \subset \Omega \setminus \Sigma_0$ be a set of minimal cardinality such that Σ_0 is full for H_{Ψ_0}. I claim that $|\Psi_0| < \lambda$. Suppose not. Let $\nu := |\Psi_0|$. Then $\mathrm{cov}(\nu, \mu) < c(\kappa)$, so we can choose a μ-covering $\mathscr{C}$ of Ψ_0 such that $|\mathscr{C}| < c(\kappa)$. Then $H_{\Psi_0} = \langle H_\Gamma : \Gamma \in \mathscr{C} \rangle$. Thus Σ_0 is full for $\langle H_\Gamma : \Gamma \in \mathscr{C} \rangle$. By Lemma 4.1, there exists a finite subcollection $\mathscr{B}$ of $\mathscr{C}$ such that Σ_0 is full for $\langle H_\Gamma : \Gamma \in \mathscr{B} \rangle$. Let $\Psi := \bigcup \mathscr{B}$. Then Σ_0 is full for H_Ψ. But $|\Psi| < \nu$, thus contradicting the minimality of $|\Psi_0|$. Thus $|\Psi_0| < \lambda$.

Let $H_0 := H_{\Psi_0}$. Then Σ_0 is full for H_0 and $\mathrm{supp}(H_0) \subset \Sigma_0 \cup \Psi_0$. Since $B_\lambda G = S$, there exists $g \in G$ such that $|\Sigma_0 \cap \Sigma_0 g| = \kappa$ and $|\Omega \setminus (\Sigma_0 \cup \Sigma_0 g)| < \lambda$. Let $\Sigma_1 := \Sigma_0 g$, let $\Psi_1 := \Psi_0 g$, and let $H_1 := g^{-1} H_0 g$. Then Σ_1 is full for H_1 and $\mathrm{supp}(H_1) \subset \Sigma_1 \cup \Psi_1$.

By replacing H_0 with $H_{0(\Psi_1 \cap \Sigma_0)}$ and Σ_0 with $\Sigma_0 \setminus \Psi_1$, we can assume that Σ_0 is disjoint from Ψ_1. Similarly, we can assume that Σ_1 is disjoint from Ψ_0. We still have that Σ_i is full for H_i and $\mathrm{supp}(H_i) \subset \Sigma_i \cup \Psi_i$. Let $\Delta := \Omega \setminus (\Sigma_0 \cup \Sigma_1)$. Now H_0 fixes Σ_0 setwise and $\mathrm{supp}(H_0) \subset \Sigma_0 \cup \Delta$, so H_0 must fix Δ setwise. Thus Σ_0 is full for $G_{\{\Delta\}}$. Similarly, Σ_1 is full for $G_{\{\Delta\}}$. Thus Δ^c is full for G, by Lemma 2.1. But $|\Delta| < \lambda$, so we are done. ⊣

4.2. The "only if" half. We now prove the "only if" half of Theorem 1.2. We will use the following lemmas.

LEMMA 4.2. *There exists a group $F \leq \mathrm{Sym}(\omega)$ such that F is a free group of rank $\aleph_0$ and $\mathrm{supp}(g) = \omega$ for every non-identity element g of F.*

PROOF. Use the right-regular representation of the free group of rank $\aleph_0$. ⊣

DEFINITION 4.1. Let X and Y be arbitrary sets. We say a collection $\mathscr{F}$ of functions from X to Y is *independent* if, for every finite collection $f_1, \ldots, f_n$ of distinct elements of $\mathscr{F}$, there exists $x \in X$ such that $xf_1, \ldots, xf_n$ are distinct.

The following is proved in [2].

LEMMA 4.3. *For every cardinal μ there exists an independent collection $\mathscr{F}$ of functions from μ to ω such that $|\mathscr{F}| = 2^\mu$.*

PROOF OF THE "ONLY IF" HALF OF THEOREM 1.2. Fix $\lambda \leq \kappa$. Suppose $\mathrm{SP}(\kappa, \lambda)$ is false, so $2^{\mu_0} = 2^\kappa$ for some $\mu_0 < \lambda$. Suppose further that $\nu \in [\lambda, \kappa]$ and $\mu_1 < \lambda$ are such that $\mathrm{cov}(\nu, \mu_1) = 2^\kappa$. Let $\mu := \max(\mu_0, \mu_1)$. Then $2^\mu = 2^\kappa$ and $\mathrm{cov}(\nu, \mu) = 2^\kappa$. We will now construct a group G such that $B_{\mu^+}G = S$ but for all $\Delta \subset \Omega$ such that $|\Delta| < \nu$ we have that Δ^c is not full for G.

Let Ψ be a ν-subset of Ω such that $|\Omega \setminus \Psi| = \kappa$. Let Φ be a μ-subset of Ψ.

Let $(\Phi_{\eta,n})_{\eta<\mu, 1\leq n<\omega}$ be a collection of pairwise disjoint μ-subsets of Φ. Let $(\Psi_\eta)_{\eta<\mu}$ be a collection of pairwise disjoint ν-subsets of $\Psi \setminus \Phi$.

Let $F \leq \mathrm{Sym}(\omega)$ be a free group of rank $\aleph_0$ such that $\mathrm{supp}(g) = \omega$ for every non-identity element g of F. Let $X \subset F$ freely generate F.

Let $(f_\xi)_{\xi<2^\kappa}$ be an independent sequence of functions from μ to X.

Let $(s_\xi)_{1\leq\xi<2^\kappa}$ enumerate $\mathrm{Sym}(\Psi^c)$.

Let $(\Delta_\xi)_{\xi<2^\kappa}$ enumerate the collection of all sets Δ such that $\Phi \subset \Delta \subset \Omega$ and $|\Delta| < \nu$.

We now construct sequences $(g_\xi)_{\xi<2^\kappa}$ and $(\Gamma_{\eta,\xi})_{\eta<\mu,\xi<2^\kappa}$ such that for all $\xi < 2^\kappa$ and $\eta < \mu$, the following are all satisfied:

- $\Gamma_{\eta,\xi}$ is a μ-subset of Ψ_η and $g_\xi \in S$;
- If $\alpha < \xi$ and $g \in \langle g_\beta : \beta < \xi \rangle$ then $\Gamma_{\eta,\xi} \not\subset \Delta_\alpha g$;
- If $m \in \omega$ and $n = m(\eta f_\xi)$ then:
  - if $m, n > 0$ then $\Phi_{\eta,m} g_\xi = \Phi_{\eta,n}$,
  - if $m = 0$ then $\Gamma_{\eta,\xi} g_\xi = \Phi_{\eta,n}$, and
  - if $n = 0$ then $\Phi_{\eta,m} g_\xi = \Gamma_{\eta,\xi}$;
- If $\xi > 0$ then $g_\xi \restriction \Psi^c = s_\xi$;
- If $\xi = 0$ then $|\Psi \cap \Psi g_\xi| = \mu$ and $|\Omega \setminus (\Psi \cup \Psi g_\xi)| = \kappa$.

We construct these sequences simultaneously by induction on ξ. Suppose we are up to stage ξ in the construction. Note that there are fewer than 2^κ sets of the form $\Delta_\alpha g$, where $\alpha < \xi$ and $g \in \langle g_\beta : \beta < \xi \rangle$. But $\mathrm{cov}(\nu, \mu) = 2^\kappa$. Thus, for every $\eta < \mu$, we can choose a μ-subset $\Gamma_{\eta,\xi}$ of Ψ_η such that $\Gamma_{\eta,\xi} \not\subset \Delta_\alpha g$ for all $\alpha < \xi$ and $g \in \langle g_\beta : \beta < \xi \rangle$. It is now trivial to construct a permutation g_ξ which satisfies all of the other requirements.

Having constructed the chain, let $G := \langle g_\alpha : \alpha < 2^\kappa \rangle$. I claim that G satisfies our requirements.

Choose $x \in B_{\mu^+}$ such that $\Psi g_0 x$ is disjoint from Ψ. Then both $\Omega \setminus \Psi$ and $\Omega \setminus \Psi g_0 x$ are full for $B_{\mu^+}G$. Thus $B_{\mu^+}G = S$, by Lemma 2.1. Now suppose $\Delta \subset \Omega$ and $|\Delta| < \nu$. Then $\Delta \cup \Phi = \Delta_\alpha$ for some $\alpha < 2^\kappa$. To show that Δ^c is not full for G, it suffices to show that Δ_α^c is not full for G. This follows from the following claim, by simple cardinality considerations.

CLAIM. $G_{\{\Delta_\alpha\}} \leq \langle g_\xi : \xi < \alpha \rangle$.

PROOF OF CLAIM. Suppose $g \in G \setminus \langle g_\xi : \xi < \alpha \rangle$. We will show that $g \notin G_{\{\Delta_\alpha\}}$.

We can express g as a word in $\{ g_\xi : \xi < 2^\kappa \}$. This word must involve at least one element g_ξ for which $\xi > \alpha$. Thus we can write g in the form $g = hw(\bar{g})$, where $h \in \langle g_\xi : \xi < \alpha \rangle$ and

$$w(\bar{g}) = g_{\beta_0}^{a_0} g_{\beta_1}^{a_1} \cdots g_{\beta_r}^{a_r}$$

is a reduced word in $\{ g_\beta : \beta < 2^\kappa \}$ with $a_i = \pm 1$ and $\beta_0 > \alpha$. By the independence of $(f_\xi)_{\xi<2^\kappa}$, we can choose $\eta < \mu$ such that if $\beta_i \neq \beta_j$ then $\eta f_{\beta_i} \neq \eta f_{\beta_j}$. For all

$i \leq r$, let $x_i := \eta f_{\beta_i}$. Then

$$w(\bar{x}) := x_0^{a_0} \dots x_r^{a_r}$$

is a reduced word in X. Thus every non-empty subword of $w(\bar{x})$ is a non-identity element of F, and hence has no fixed points in ω. In particular, for every $i \leq r$ we have

$$0 \neq 0(x_0^{a_0} \dots x_i^{a_k}).$$

Thus it can be shown by induction on r that

$$\Gamma_{\eta,\beta_0} w(\bar{g}) = \Phi_{\eta,n}$$

where $n = 0w(\bar{x})$. Let $\Gamma := \Gamma_{\eta,\beta_0} h^{-1}$. By our choice of Γ_{η,β_0} we have that $\Gamma \not\subset \Delta_\alpha$. But $\Gamma g = \Phi_{\eta,n} \subset \Delta_\alpha$. It follows that $g \notin G_{\{\Delta_\alpha\}}$, thus proving the claim. ⊣

Thus G is a counterexample to $\mathrm{MNP}(\kappa, \lambda)$. ⊣

§5. **Proving cases.** In this section we will use Theorem 1.2 to prove $\mathrm{MNP}(\kappa, \lambda)$ for certain values of κ and λ. The following two cases are easy corollaries of Theorems 1.1 and 1.2.

COROLLARY 5.1. $\mathrm{MNP}(\kappa, \aleph_0)$ *holds for any infinite cardinal* κ.

PROOF. Indeed Theorem 1.1 yields the stronger result $\mathrm{SP}(\kappa, \aleph_0)$. ⊣

COROLLARY 5.2. *If λ is a regular cardinal and $\kappa = \lambda^{+n}$ for some $n < \omega$ then* $\mathrm{MNP}(\kappa, \lambda)$ *holds.*

PROOF. Let $\kappa = \lambda^{+n}$ where λ is a regular cardinal and $n < \omega$. Suppose ν and μ are cardinals such that $\nu \in [\lambda, \kappa]$ and $\mu < \lambda$. Either $\nu = \lambda$ or ν is a successor cardinal. In either case, ν is regular, so $\mathrm{cov}(\nu, \mu) = \nu$ by Lemma 1.1. Thus $\mathrm{cov}(\nu, \mu) < c(\kappa)$, by Lemma 1.3. Thus $\mathrm{MNP}(\kappa, \lambda)$ holds, by Theorem 1.2. ⊣

To prove other cases of $\mathrm{MNP}(\kappa, \lambda)$, we will require upper bounds on values of $\mathrm{cov}(\nu, \mu)$ in cases where $\mu \geq \mathrm{cf}(\nu)$. We will use results from Shelah's recently developed *pcf* theory to obtain some such bounds in the case $\mu = \mathrm{cf}(\nu)$. We now proceed to show that the case $\mu > \mathrm{cf}(\nu)$ will take care of itself.

LEMMA 5.1. *Suppose $\mu < \lambda \leq \nu \leq \kappa$ are cardinals such that* $\mathrm{cov}(\theta, \mu) \leq \kappa$ *for all $\theta \in [\lambda, \nu]$. Then there exists a collection $\mathscr{C} \subset [\nu]^{<\lambda}$ such that $\mathscr{C}$ forms a μ-covering of ν and $|\mathscr{C}| \leq \kappa$.*

PROOF. We proceed by induction on ν. If $\nu = \lambda$ then we can simply let $\mathscr{C}$ be a μ-covering of ν such that $|\mathscr{C}| = \mathrm{cov}(\nu, \mu)$. Now suppose $\nu > \lambda$. Let $\mathscr{A}$ be a μ-covering of ν such that $|\mathscr{A}| \leq \kappa$. We can assume that every element of $\mathscr{A}$ has cardinality at least λ. By the induction hypothesis, for each $A \in \mathscr{A}$ there exists a collection $\mathscr{B}_A \subset [A]^{<\lambda}$ which forms a μ-covering of A and satisfies $|\mathscr{B}_A| \leq \kappa$. Let $\mathscr{C} := \bigcup\{\mathscr{B}_A : A \in \mathscr{A}\}$. Then it is not hard to check that $\mathscr{C}$ satisfies our requirements, so we are done. ⊣

LEMMA 5.2. *Let $\lambda \leq \kappa$ be uncountable cardinals such that λ is regular. Suppose we have that* $\mathrm{cov}(\nu, \mathrm{cf}(\nu)) \leq \kappa$ *for all $\nu \in [\lambda, \kappa]$ such that* $\mathrm{cf}(\nu) < \lambda$. *Then* $\mathrm{cov}(\nu, \mu) \leq \kappa$ *for all $\nu \in [\lambda, \kappa]$ and $\mu < \lambda$, and hence* $\mathrm{MNP}(\kappa, \lambda)$ *holds.*

PROOF. Proof is by double induction on μ and ν. Suppose $\mu < \lambda$ is such that:

$$(\forall \nu \in [\lambda, \kappa])(\forall \theta < \mu)(\mathrm{cov}(\nu, \theta) \leq \kappa).$$

Note that this is vacuously true in the base case $\mu = \aleph_0$.

Now suppose $\nu \in [\lambda, \kappa]$ is such that

$$(\forall \theta \in [\lambda, \nu))(\mathrm{cov}(\theta, \mu) \leq \kappa).$$

Note that this is vacuously true in the base case $\nu = \lambda$.

We will show that $\mathrm{cov}(\nu, \mu) \leq \kappa$. If $\mu < \mathrm{cf}(\nu)$ then this holds by Lemma 1.1, while if $\mathrm{cf}(\nu) = \mu$ then this holds by our assumptions. Thus we assume $\mu > \mathrm{cf}(\nu)$.

By applying Lemma 5.1 to the induction hypothesis on ν, we have that for each ordinal ξ such that $\lambda \leq \xi < \nu$, there exists a collection $\mathscr{A}_\xi \subset [\xi]^{<\lambda}$ such that $\mathscr{A}_\xi$ is a μ-covering of ξ and $|\mathscr{A}_\xi| \leq \kappa$. Let

$$\mathscr{A} := \bigcup \{ \mathscr{A}_\xi : \lambda \leq \xi < \nu \}.$$

Now $\mathrm{cf}(\nu) < \mu$ and $|\mathscr{A}| \leq \kappa$. Thus, by applying Lemma 5.1 to the inductive hypothesis on μ we have that there exists a collection $\mathscr{B} \subset [\mathscr{A}]^{<\lambda}$ such that $\mathscr{B}$ is a $\mathrm{cf}(\nu)$-covering of $\mathscr{A}$ and $|\mathscr{B}| \leq \kappa$. Let

$$\mathscr{C} := \left\{ \bigcup B : B \in \mathscr{B} \right\}.$$

Then $|\mathscr{C}| \leq \kappa$. Also $\mathscr{C} \subset [\nu]^{<\lambda}$, since λ is regular. Finally, $\mathscr{C}$ is a μ-covering of ν, since any μ-subset of ν can be expressed as the union of at most $\mathrm{cf}(\nu)$ bounded μ-subsets of ν. Thus $\mathrm{cov}(\nu, \mu) \leq \kappa$ and we are done. ⊣

We can prove cases of $\mathrm{MNP}(\kappa, \lambda)$ where λ is singular by using the following lemma, which is an easy consequence of Lemma 2.3.

LEMMA 5.3. *Suppose $\theta < \lambda \leq \kappa$ are cardinals such that* $\mathrm{MNP}(\kappa, \mu^+)$ *holds for all* $\mu \in [\theta, \lambda)$. *Then* $\mathrm{MNP}(\kappa, \lambda)$ *holds.*

We are now ready to use Shelah's results concerning a quantity called the *pseudopower* of ν, or $\mathrm{pp}(\nu)$. For a definition of $\mathrm{pp}(\nu)$, see [12]. Pseudopowers are related to covering numbers by [12, Theorem 5.7], which is as follows.

LEMMA 5.4 (Shelah). *If ν is a singular cardinal satisfying $\nu < \aleph_\nu$ then* $\mathrm{pp}(\nu) = \mathrm{cov}(\nu, \mathrm{cf}(\nu))$.

Note that results on pseudopowers will tell us nothing about the values $\mathrm{cov}(\nu, \mathrm{cf}(\nu))$ can take when ν is a cardinal fixed point, that is, when $\nu = \aleph_\nu$. We will avoid this problem by only attempting to prove cases of $\mathrm{MNP}(\kappa, \lambda)$ for which the intervals $[\lambda, \kappa]$ contains no cardinal fixed points. Theorem 6.4 and Problem 6.1 suggest that this may indeed be prudent.

The following is given as the last theorem in [12].

THEOREM 5.1 (Shelah). *If $\nu < \aleph_{\omega_4}$ has countable cofinality then* $\mathrm{pp}(\nu) < \aleph_{\omega_4}$.

COROLLARY 5.3. $\mathrm{MNP}(\aleph_{\omega_4}, \aleph_1)$ *holds.*

The following is [12, Theorem 6.3].

THEOREM 5.2 (Shelah). *If α is an ordinal, δ is a limit ordinal and $\delta < \aleph_{\alpha+\delta}$ then* $\mathrm{pp}(\aleph_{\alpha+\delta}) < \aleph_{\alpha+|\delta|^{+4}}$.

COROLLARY 5.4. *Suppose λ and $\kappa = \aleph_\delta$ are such that $[\lambda, \kappa]$ contains no cardinal fixed points, and δ can be expressed as a finite ordinal sum:*

$$\delta = \delta_1 + \cdots + \delta_r + n$$

where r, $n < \omega$ and for each $i < r$ we have that δ_i is a limit cardinal and $\mathrm{cf}(\delta_i) \geq \lambda$. *Then* $\mathrm{MNP}(\kappa, \lambda)$ *holds.*

As a special case, if δ is a limit cardinal which is smaller than the first cardinal fixed point then $\mathrm{MNP}(\aleph_\delta, \lambda)$ holds for all $\lambda \leq \mathrm{cf}(\delta)$. Thus, for example, we have the curious fact that $\mathrm{MNP}(\aleph_{\omega_{\omega_1}}, \aleph_1)$ is a theorem of ZFC.

It is not known whether "+4" can be replaced by "+" in Theorem 5.2. If it could, then the word "limit" could be removed from the corollary.

§6. Forcing failures. In this section we will use Theorem 1.2 to find values of κ and λ for which $\neg\,\mathrm{MNP}(\kappa, \lambda)$ is consistent with ZFC. Throughout this section, M will denote a countable model of ZFC.

The following lemma reduces the problem of constructing a model of $\neg\,\mathrm{MNP}(\kappa, \lambda)$ to a problem of constructing a model with a sufficiently large covering number. Everything else we need in order to construct a counterexample can easily be obtained by forcing, whereas covering numbers are not so easily altered.

THEOREM 6.1. *Suppose, in M, we have that $\mu < \lambda \leq \nu \leq \kappa$ are cardinals such that* $\mathrm{cov}(\nu, \mu) > \kappa$. *Then there exists a notion of forcing $P \in M$ which preserves cofinalities and cardinalities $\leq\kappa^+$, such that if $\mathscr{G}$ is P-generic over M then $M[\mathscr{G}] \models \neg\,\mathrm{MNP}(\kappa, \lambda)$.*

PROOF. We will use some standard forcing results, all of which can be found in [6].

We use a two-stage iterated forcing construction. Working in M, and using the notation of [6], let $P_0 := \mathrm{Fn}(\kappa^+, 2, \kappa^+)$, let $P_1 := \mathrm{Fn}(\kappa^+, 2)$ and let $P := P_0 \times P_1$. Let $\mathscr{G}$ be P-generic over M. Note that both P_0 and P_1, and hence also P, preserve cofinalities and cardinalities $\leq\kappa^+$. By standard forcing arguments, $M[\mathscr{G}] \models 2^{\aleph_0} = 2^\kappa = \kappa^+$. By Theorem 1.2, it remains only to show that $M[\mathscr{G}] \models \mathrm{cov}(\nu, \mu) > \kappa$. We use the following claim, which follows from Theorems VIII.1.4, VII.5.5 and VII.6.14 in [6].

CLAIM. *Suppose $A \in M[\mathscr{G}]$ is such that $A \subset M$ and $M[\mathscr{G}] \models |A| \leq \kappa$. Then there exists $B \in M$ such that $A \subset B$ and $M[\mathscr{G}] \models |A| = |B|$.*

Now suppose, seeking a contradiction, that $M[\mathscr{G}] \models \mathrm{cov}(\nu, \mu) \leq \kappa$. Working in $M[\mathscr{G}]$, let $\mathscr{C}$ be a μ-covering of ν such that $|\mathscr{C}| = \kappa$. By the above claim, we can replace every set $A \in \mathscr{C}$ with a set $B \in N$ such that $A \subset B$ and $M[\mathscr{G}] \models |A| = |B|$. Thus we can assume that $\mathscr{C} \subset M$. Applying the claim once again allows us to assume that $\mathscr{C} \in M$. Since P preserves cardinals $\leq\kappa$ it follows that, in M, we have that $\mathscr{C}$ is a μ-covering of ν and $|\mathscr{C}| = \kappa$. This contradicts our assumption that $M \models \mathrm{cov}(\nu, \mu) > \kappa$. Thus $M[\mathscr{G}] \models \mathrm{cov}(\nu, \mu) > \kappa$, and we are done. $\dashv$

Lemma 1.2 now yields the following.

COROLLARY 6.1. *If $M \models \mathrm{cf}(\kappa) < \lambda \leq \kappa$ then there exists a notion of forcing $P \in M$ which preserves cofinalities and cardinalities $\leq\kappa^+$, such that if $\mathscr{G}$ is P-generic over M then $M[\mathscr{G}] \models \neg\,\mathrm{MNP}(\kappa, \lambda)$.*

PROOF. Suppose $M \models \mathrm{cf}(\kappa) < \lambda \leq \kappa$. By Lemma 1.2, we also have $M \models \mathrm{cov}(\kappa, \mathrm{cf}(\kappa)) > \kappa$. Now apply Theorem 6.1 with $\mu = \mathrm{cf}(\kappa)$ and $\nu = \kappa$. ⊣

The next theorem shows that, in order to construct counterexamples to $\mathrm{MNP}(\kappa, \lambda)$ with $\lambda \leq \mathrm{cf}(\kappa)$, we will require covering numbers that are larger than Lemma 1.2 can provide.

In line with [7], we say that a group $G \leq \mathrm{Sym}(\Omega)$ is *ample* if there exists $\Delta \subset \Omega$ such that $|\Delta| < |\Omega|$ and Δ^c is full for G.

THEOREM 6.2. *The following are equiconsistent.*
(a) *There exist cardinals λ and κ such that $\lambda \leq \mathrm{cf}(\kappa)$ and $\neg\,\mathrm{MNP}(\kappa, \lambda)$;*
(b) *There exists an ample counterexample to* MNP.
(c) *There exist cardinals $\mu < \nu$ such that* $\mathrm{cov}(\nu, \mu) > \nu^+$.

PROOF. Suppose (a) holds. By Theorem 1.2 and Lemma 1.3, there exist $\nu \in [\lambda, \kappa]$ and $\mu < \lambda$ such that $\mathrm{cov}(\nu, \mu) > \kappa$. By Lemma 1.1 we have $\mathrm{cov}(\kappa, \mu) = \mathrm{cf}(\kappa)$, so $\nu \neq \kappa$. Thus $\nu < \kappa$, so $\mathrm{cov}(\nu, \mu) > \nu^+$, and hence (c) holds.

Conversely, suppose M is a model of (c). Working in M, let $\lambda := \mu^+$ and $\kappa := \nu^+$. By Theorem 6.1, we can extend M to a model N of $\neg\,\mathrm{MNP}(\kappa, \lambda)$ in which κ remains regular. Then N is a model of (a).

The proof that (b) and (c) are equiconsistent requires a slightly more careful reworking of the proof of Theorem 1.2, and will not be covered here. ⊣

Results due to Dodd and Jensen show that if $\mathrm{cov}(\nu, \mu) > \nu^+$ for any cardinals $\mu < \nu$ then there exists an inner model of ZFC which has a measurable cardinal. (This fact follows from [3, Corollary 6.10] and [4, Theorem 5.17].) Thus none of the items in Theorem 6.2 can be shown to be consistent without assuming the consistency of some large cardinal axiom. The following lemma eliminates the need for us to deal directly with large cardinals by allowing us instead to borrow results from the field of cardinal arithmetic.

LEMMA 6.1. *If ν is a singular strong limit cardinal and $\mathrm{cf}(\nu) \leq \mu < \nu$ then* $\mathrm{cov}(\nu, \mu) = 2^\nu$.

PROOF. Under the assumptions, we have that $2^\nu = \nu^\mu$. (See, for example, [5, Lemma 6.5].)

Let $\mathscr{C}$ be a μ-covering of ν such that $|\mathscr{C}| = \mathrm{cov}(\nu, \mu)$. Every μ-subset of ν is a subset of an element of $\mathscr{C}$. But ν is a strong limit cardinal, so each element of $\mathscr{C}$ has fewer than ν subsets. Thus there are at most $|\mathscr{C}|$ distinct μ-subsets of ν. Thus $\nu^\mu \leq \mathrm{cov}(\nu, \mu)$. But clearly $\mathrm{cov}(\nu, \mu) \leq \nu^\mu$, so we are done. ⊣

The question of what values covering numbers can take is therefore closely related to the question of what values 2^ν can take when ν is a singular strong limit cardinal. This is the so called "singular cardinals problem". Recent years have seen explosion of consistency results in this area. I will list only a couple of examples here.

The following is a special case of [8, Theorem 2]. The general result is harder to state, and only slightly more useful for our purposes.

THEOREM 6.3 (Magidor). *Suppose, in M, we have that θ is a supercompact cardinal, $\beta < \theta$ is a limit ordinal and $n < \omega$. Then there exists an extension of M in which $\aleph_\beta$ is a strong limit cardinal and $2^{\aleph_\beta} = \aleph_{\beta+n}$. Furthermore, if $|\beta| < \aleph_{|\beta|}$ then we can assume that cofinalities and cardinalities are preserved $\leq|\beta|$.*

For example, this yields the following result.

COROLLARY 6.2. *For all $n < \omega$ we have that $\neg\,\mathrm{MNP}(\aleph_{\omega+n}, \aleph_1)$ is consistent with the existence of a supercompact cardinal.*

PROOF. Let M be a model of ZFC containing a supercompact cardinal. By Theorem 6.3, there exists an extension N of M in which $\aleph_\omega$ is a strong limit cardinal and $2^{\aleph_\omega} = \aleph_{\omega+n+1}$. By Lemma 6.1, N satisfies $\mathrm{cov}(\aleph_\omega, \aleph_0) = \aleph_{\omega+n+1}$. By Theorem 6.1, we can extend N to a model of $\neg\,\mathrm{MNP}(\aleph_{\omega+n}, \aleph_1)$. $\dashv$

Our next example requires us to define a rather big cardinal (though not a "large" cardinal, as such).

DEFINITION 6.1. Let $\mathscr{C}_0$ be the class of all cardinals. For all $n < \omega$, let $\mathscr{C}_{n+1}$ be the class of cardinals $\nu \in \mathscr{C}_n$ such that $\{\lambda \in \mathscr{C}_n : \lambda < \nu\}$ has cardinality ν. Let $\mathscr{C}_\omega = \bigcap_{n<\omega} \mathscr{C}_n$.

Thus $\mathscr{C}_1$ is the class of cardinal fixed points, $\mathscr{C}_2$ is the class of fixed points of $\mathscr{C}_1$, and so on. Note that $\min \mathscr{C}_\omega$ has countable cofinality. The following is taken from [11, Theorem 2.6].

THEOREM 6.4 (Shelah). *Suppose, in M, we have that the GCH holds, ν is a supercompact cardinal whose supercompactness is preserved by ν-directed complete forcing, and $\kappa > \nu$ is such that $(\nu, \kappa]$ contains no inaccessible cardinals. Then there exists an extension of M in which no cardinals in $[\nu, \kappa^+]$ are collapsed, except possibly successors of singular cardinals, and ν is a singular strong limit cardinal satisfying $\nu = \min \mathscr{C}_\omega$ and $2^\nu = \kappa^+$.*

Although somewhat cumbersome to state rigorously, this theorem basically tells us that if we assume the consistency of supercompact cardinals then there is no upper bound on the size of $\mathrm{cov}(\min \mathscr{C}_\omega, \aleph_0)$. It follows that $\neg\,\mathrm{MNP}(\kappa, \lambda)$ is consistent with the existence of supercompact cardinals for any "suitably specified" cardinals λ and κ satisfying $\lambda < \min \mathscr{C}_\omega < \kappa$.

The following, given as [1, Problem 5.26], is an open problem.

PROBLEM 6.1. *Suppose the first cardinal fixed point $\delta = \aleph_\delta$ is a strong limit cardinal. Is there an upper bound on $2^{\aleph_\delta}$?*

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