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DOES THE JONES POLYNOMIAL DETECT THE UNKNOT?

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ABSTRACT. We address the question: Does there exist a non-trivial knot with a trivial Jones polynomial? To find such a knot, it is almost certainly sufficient to find a non-trivial braid on four strands in the kernel of the Burau representation. I will describe a computer algorithm to search for such a braid.

1. INTRODUCTION

The Jones polynomial $V_K(q)$ of a knot K is one of the most famous and important knot invariants. It is not a perfect tool for distinguishing different knots and links. There are many distinct knots with the same Jones polynomial (see [11]), and Eliahou, Kauffman and Thistlethwaite have recently found an infinitely many distinct links with Jones polynomial equal to that of the corresponding unlink [6]. However the answer to the following question remains unknown.

Question 1.1. *Does there exist a non-trivial knot K with $V_K(q) \equiv 1$?*

This is given as Problem 1 in [9]. There have been many attempts to find such a knot. A brute force approach was used in [5] to check all knots with up to seventeen crossings. Another approach attempted in [1] and [10] is to start with a complicated diagram of the unknot and apply mutations which do not alter the Jones polynomial but may alter the knot type.

The approach described in this paper comes from the theory of braids. Any knot K can be obtained as the closure of some braid β. The Jones polynomial of K is a trace function of the representation of β into the Temperley-Lieb algebra. We are therefore led to ask the following question.

Question 1.2. *Is the representation of the braid group into the Temperley-Lieb algebra faithful?*

This is Problem 3 in [9]. A non-trivial braid in the kernel of the Temperley-Lieb representation could be used to construct a knot with Jones polynomial equal to one. I am not aware of any proof that the knot so obtained must be non-trivial, but this seems unlikely to pose a problem if a specific braid were known. The following conjecture is therefore widely assumed to be true.

Conjecture 1.3. *If the Temperley-Lieb representation of the braid group is unfaithful then there exists a non-trivial knot with Jones polynomial equal to one.*

The Temperley-Lieb representation of B_n appears as a summand in a larger representation into the Hecke algebra $H(q,n)$ of type A_{n-1}. We will call this latter representation the Jones representation, although some authors use this term for what we are calling the Temperley-Lieb representation. The Jones representation was used by Ocneanu in [7] to define a two-variable generalisation of the Jones polynomial called the HOMFLY polynomial. The following conjecture is also widely assumed to be true.

Conjecture 1.4. *If the Jones representation of the braid group is unfaithful then there exists a non-trivial knot with HOMFLY polynomial equal to one.*

We will focus on the braid group B_4. In this case the Jones and Temperley-Lieb representations both decompose into the Burau representation together with some very simple representations. Thus we have the following.

Proposition 1.5. *The following are equivalent:*

- *the Jones representation of B_4 is faithful,*
- *the Temperley-Lieb representation of B_4 is faithful, and*
- *the Burau representation of B_4 is faithful.*

We are therefore led to ask the following question.

Question 1.6. *Is the Burau representation of B_4 faithful?*

A negative answer would almost certainly lead to a non-trivial knot whose HOMFLY polynomial is equal to one. As far as I know, a positive answer would have no such dramatic consequences other than finally determining for which values of n the Burau representation of B_n is faithful. Krammer [12] has already shown that B_4 is linear.

The Burau representation of B_n is known to be faithful for $n \leq 3$ [4] and unfaithful for $n \geq 5$ [2]. The case $n = 4$ seems to lie very close to the border between faithfulness and unfaithfulness.

The main aim of this paper is to propose a computer search for a non-trivial braid in the kernel of the Burau representation of B_4. This might seem overly ambitious. After all, it amounts to a search for a very special case of a non-trivial knot whose HOMFLY polynomial is equal to one (assuming Conjecture 1.4). Many people have tried and failed to find a non-trivial knot whose weaker Jones polynomial is equal to one. However there is some reason for optimism. A knot constructed by the methods of this paper would have thousands of crossings. Thus we are searching in relatively unexplored territory which might contain unexpected treasures. This is probably enough to justify the expenditure of some computer time, but perhaps not too much human time or brain power.

2. The Burau Representation

We now define the braid groups B_n and the Burau representation.

Let D be a disk. Let $p_1, \ldots, p_n$ be distinct points in the interior of D. We call these "puncture points". Let $D_n = D \setminus \{p_1, \ldots, p_n\}$. Let d_0 be a basepoint on ∂D_n. For concreteness, take D to be the unit disk in the complex plane centred at the origin, take $p_1, \ldots, p_n$ to be real numbers satisfying $-1 < p_1 < \cdots < p_n < 1$, and take d_0 to be $-i$.

The braid group B_n is defined to be the group of homeomorphisms from D_n to itself which act as the identity on ∂D_n, taken up to isotopy relative to ∂D_n. It is generated by $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i exchanges p_i and p_{i+1} by a counterclockwise half twist.

The fundamental group $\pi_1(D_n, d_0)$ is a free group with basis $x_1, \ldots, x_n$, where x_i is a loop based at d_0 which passes counterclockwise around p_i and no other puncture points. Let $\phi\colon\ \pi_1(D_n, d_0) \to \langle q \rangle$ be the homomorphism given by $\phi(x_i) = q$. Let $\tilde{D}_n$ be the covering space corresponding to the subgroup $\ker(\phi)$ of $\pi_1(D_n)$. Fix a point $\tilde{d}_0$ in the fibre over d_0.

A more concrete description of $\tilde{D}_n$ can be given as follows. Make a bi-infinite stack of $\mathbf{Z}$ copies of D_n. On each copy, make a series of vertical cuts in the upper half plane connecting each of the puncture points p_i to the boundary. Glue the left-hand side of each cut to the right-hand side of the corresponding cut on the copy of D_n one level lower.

The group of covering transformations of $\tilde{D}_n$ is $\langle q \rangle$. The $\mathbf{Z}$-module $H_1(\tilde{D}_n)$ can be considered as a $\mathbf{Z}[q^{\pm 1}]$-module, where multiplication by q is the induced action of the covering transformation q. Thought of in this way, $H_1(\tilde{D}_n)$ turns out to be a free $\mathbf{Z}[q^{\pm 1}]$-module of rank $n-1$.

The Burau representation is the induced action of B_n by $\mathbf{Z}[q^{\pm 1}]$-module homomorphisms on $H_1(\tilde{D}_n)$. We make this more precise as follows. Let $\beta\colon\ D_n \to D_n$ be a homeomorphism representing a braid $[\beta]$ in B_n. The induced action of β on $\pi_1(D_n)$ satisfies $\phi\beta = \phi$. It follows by some basic algebraic topology that there exists a unique lift $\tilde{\beta}$ which makes the following diagram commute.

$$\begin{array}{ccc} (\tilde{D}_n, \tilde{d}_0) & \xrightarrow{\tilde{\beta}} & (\tilde{D}_n, \tilde{d}_0) \\ \downarrow & & \downarrow \\ (D_n, d_0) & \xrightarrow{\beta} & (D_n, d_0) \end{array}$$

Furthermore, $\tilde{\beta}$ commutes with the action of q on $\tilde{D}_n$ by a covering transformation. Thus $\tilde{\beta}$ induces a $\mathbf{Z}[q^{\pm 1}]$-module homomorphism

$$\tilde{\beta}_*\colon\ H_1(\tilde{D}_n) \to H_1(\tilde{D}_n).$$

The Burau representation is the map

$$\mathrm{Burau}([\beta]) = \tilde{\beta}_*.$$

For example, using an appropriate choice of basis for $H_1(\tilde{D}_4)$, the Burau representation of B_4 maps the generators σ_1, σ_2 and σ_3 to the matrices

$$\begin{pmatrix} -q & q & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & -q & q \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -q \end{pmatrix},$$

respectively.

3. The Jones and HOMFLY polynomials

In this section we make the connection between the Burau representation of B_4 and the Jones and HOMFLY polynomials of a knot. We start by proving Proposition 1.5, which states that in the case of B_4 the Jones, Temperley-Lieb, and Burau representations are either all faithful or all unfaithful.

Proof of Proposition 1.5. We will not define the Temperley-Lieb or Jones representations, but we will use some of their basic properties, all of which can be found in [8].

The Jones representation of B_n can be decomposed into irreducible summands, one corresponding to each Young diagram with n boxes. The Young diagrams with 4 boxes are (4), $(3,1)$, $(2,2)$, $(2,1,1)$ and $(1,1,1,1)$. The Temperley-Lieb representation is the sum of those representations corresponding to Young diagrams with one or two rows, namely (4), $(3,1)$ and $(2,2)$. Let V_λ denote the representation corresponding to the Young diagram λ.

The representation $V_{(3,1)}$ is the Burau representation. If this is faithful then so are the Temperley-Lieb and Jones representations. Conversely, suppose β is a non-trivial braid in the kernel of the Burau representation of B_4. Let γ be the commutator $[(\sigma_1\sigma_2)^3, \beta]$. We will show that γ lies in the kernel of each summand of the Jones representation, and hence of the entire Jones representation.

The representations $V_{(4)}$ and $V_{(1,1,1,1)}$ are both one-dimensional. Being a commutator, γ must lie in the kernel of both of these.

The representation $V_{(2,2)}$ can be defined by composing the Burau representation of B_3 with the map from B_4 to B_3 given by $\sigma_1 \mapsto \sigma_1$, $\sigma_2 \mapsto \sigma_2$, and $\sigma_3 \mapsto \sigma_1$. Now $(\sigma_1\sigma_2)^3$ lies in the centre of B_3. Since γ is the commutator of a braid with $(\sigma_1\sigma_2)^3$, it must lie in the kernel of $V_{(2,2)}$.

Finally, consider the representation $V_{(2,1,1)}$. The Young diagram $(2,1,1)$ is a reflection of $(3,1)$. Reflection of the Young diagram has the effect of substituting

$$\sigma_i \mapsto -q\sigma_i^{-1}$$

in the corresponding representation. It is shown in [13] that the kernel of the Burau representation is invariant under the substitution

$$\sigma_i \mapsto \sigma_i^{-1}.$$

Thus the image of γ in $V_{(2,1,1)}$ must be the identity matrix multiplied by some power of $-q$. Since γ is a commutator, its determinant in any representation must equal one. Thus γ is in the kernel of $V_{(2,1,1)}$.

It remains to show that γ is a non-trivial braid. This is not difficult, but would take us too far afield. We therefore omit this part of the proof. □

Suppose β lies in the kernel of the Temperley-Lieb representation. Then the closures of the braids

$$\beta\sigma_1\sigma_2\ldots\sigma_{n-1}$$

and

$$\sigma_1\sigma_2\ldots\sigma_{n-1}.$$

have the same Jones polynomials. The closure of the braid $\sigma_1\sigma_2\ldots\sigma_{n-1}$ is the unknot. Thus the closure of $\beta\sigma_1\sigma_2\ldots\sigma_{n-1}$ has Jones polynomial equal to one. If we could be sure that this was a non-trivial knot then Conjecture 1.3 would be proved. If it is the unknot then all is not lost, since we could use any power β^k in place of β. Thus the following conjecture implies Conjecture 1.3.

Conjecture 3.1. *Let β be a non-trivial braid in B_n. There exists some integer k such that the closure of $\beta^k\sigma_1\sigma_2\ldots\sigma_{n-1}$ is a non-trivial knot.*

As well as powers of β, we also have products of conjugates of β at our disposal. And in place of $\sigma_1\sigma_2\ldots\sigma_{n-1}$ we could use any braid whose closure is the unknot. Thus we can weaken the above conjecture to the following.

Conjecture 3.2. *Let H be a non-trivial normal subgroup of B_n. Then there exists $\beta_1 \in H$ and $\beta_2 \in B_n$ such that the closure of β_2 is the unknot but the closure of $\beta_1\beta_2$ is a non-trivial knot.*

The above discussion applies equally well to the Jones representation and the HOMFLY polynomial. Thus Conjecture 3.2 also implies Conjecture 1.4.

A counterexample to Conjecture 3.2 would be truly astonishing, implying an unprecedented correlation between the algebraic structure of H and the geometric structure of the knots constructed. However it might be quite difficult to prove this "obvious" conjecture. This problem would probably be easily overcome in the case of a specific non-trivial braid in the kernel of the Burau representation of B_4.

4. The case $n = 3$

The aim of this section is to prove the following.

Theorem 4.1. *The Burau representation of B_3 is faithful.*

This is a well-known result and has been proved in many different ways (see, for example, [4]). The proof given here is a warm-up for the ideas that will be used later.

A *fork* is an embedded tree F in D with four vertices d_0, p_i, p_j and z such that

- F meets the puncture points only at p_i and p_j,
- F meets the ∂D_n only at d_0, and
- all three edges of F have z as a vertex.

The edge of F which contains d_0 is called the *handle* of F. The union of the other two edges forms a single edge which we call the *tine edge* of F and denote by $T(F)$. Orient $T(F)$ so that the handle of F lies to the right of $T(F)$.

A *noodle* is an embedded oriented edge N in D_n such that

- N goes from d_0 to another point on ∂D_n,
- N meets ∂D_n only at its endpoints, and
- a component of $D_n \setminus N$ contains precisely one puncture point.

This last requirement was not included in the definition given in [3]. Without it, Theorem 5.1 is not true, as far as I know.

Let F be a fork and let N be a noodle. We define a pairing $\langle N, F\rangle$ in $\mathbb{Z}[q^{\pm 1}]$ as follows. If necessary, apply a preliminary isotopy of F so that $T(F)$ intersects N

transversely. Let $z_1, \dots, z_k$ denote the points of intersection between $T(F)$ and N (in no particular order). For each $i = 1, \dots, k$, let γ_i be the arc in D_n which goes from d_0 to z_i along F, then back to d_0 along N. Let a_i be the integer such that $\phi(\gamma_i) = q^{a_i}$. In other words, a_i is the sum of the winding numbers of γ_i around each of the puncture points p_j. Let ϵ_i be the sign of the intersection between N and F at z_i. Let

$$\langle N, F\rangle = \sum_{i=1}^{k} \epsilon_i q^{a_i}. \tag{1}$$

We should really check that this is independent of our choice of preliminary isotopy of F. This is easy enough to prove directly. It is also a special case of the following lemma.

Lemma 4.2 (The Basic Lemma). *Let $\beta \colon D_n \to D_n$ represent an element of the kernel of the Burau representation. Then $\langle N, F\rangle = \langle N, \beta(F)\rangle$ for any noodle N and fork F.*

Proof. We can assume that the tine edges of F and $\beta(F)$ both intersect N transversely.

Let $\tilde{F}$ be the lift of F to $\tilde{D}_n$ which contains $\tilde{d}_0$. Let $\tilde{T}(F)$ be the corresponding lift of $T(F)$. Then $\tilde{T}(F)$ intersects $q^a\tilde{N}$ transversely for any $a \in \mathbf{Z}$. Let $(q^a\tilde{N}, \tilde{T}(F))$ denote the algebraic intersection number of these two arcs. Then the following definition of $\langle N, F\rangle$ is equivalent to Eq. (1).

$$\langle N, F\rangle = \sum_{a \in \mathbf{Z}} (q^a\tilde{N}, \tilde{T}(F)) q^a. \tag{2}$$

Suppose $T(F)$ goes from p_i to p_j. Let $\nu(p_i)$ and $\nu(p_j)$ be disjoint small regular neighbourhoods of p_i and p_j respectively. Let γ be a subarc of $T(F)$ which starts in $\nu(p_i)$ and ends in $\nu(p_j)$. Let δ_i be a loop in $\nu(p_i)$ based at $\gamma(0)$ which passes counterclockwise around p_i. Similarly, let δ_j be a loop in $\nu(p_j)$ based at $\gamma(1)$ which passes counterclockwise around p_j. Let $T_2(F)$ be the "figure eight"

$$T_2(F) = \gamma\delta_j\gamma^{-1}\delta_i^{-1}.$$

Let $\tilde{T}_2(F)$ be the lift of $T_2(F)$ which is equal to $(1-q)\tilde{T}(F)$ outside a small neighbourhood of the puncture points. Then the following definition of $\langle N, F\rangle$ is equivalent to Eq. (2).

$$\langle N, F\rangle = \frac{1}{1-q} \sum_{a \in \mathbf{Z}} (q^a\tilde{N}, \tilde{T}_2(F)) q^a. \tag{3}$$

Note that $\tilde{T}_2(F)$ is a closed loop in $\tilde{D}_n$. Since β is in the kernel of the Burau representation, the loops $\tilde{T}_2(F)$ and $\tilde{T}_2(\beta(F))$ represent the same element of $H_1(\tilde{D}_n)$. They therefore have the same algebraic intersection number with any lift $q^a\tilde{N}$ of N. Thus Eq. (3) will give the same result for $\langle N, \beta(F)\rangle$ as for $\langle N, F\rangle$. □

We now use the assumption that $n = 3$.

Lemma 4.3 (The Key Lemma). *In the case $n = 3$, $\langle N, F\rangle = 0$ if and only if $T(F)$ is isotopic to an arc which is disjoint from N.*

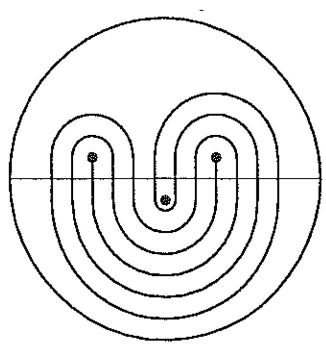

FIGURE 1. A tine edge and a noodle in D_3.

Proof. Apply an isotopy to F so that $T(F)$ intersects N at a minimum number of points, which we denote $z_1, \ldots, z_k$ (in no particular order). Recall the definition given in Eq. (1).

$$\langle N, F\rangle = \sum_{i=1}^{k} \epsilon_i q^{a_i}.$$

If $k = 0$ then clearly $\langle N, F\rangle = 0$. We now assume that $k > 0$ and prove that $\langle N, F\rangle \neq 0$.

By applying a homeomorphism to our picture, we can take N to be a horizontal straight line through D_3 with two puncture points above it and one puncture point below it. (The noodle has been pulled straight and the fork is twisted!) Let D_n^+ and D_n^- be the upper and components of $D_n \setminus N$ respectively. Relabel the puncture points so that D_n^+ contains p_1 and p_2 and D_n^- contains p_3.

Consider the intersection of $T(F)$ with D_n^-. This consists of a disjoint collection of arcs which have both endpoints on N, and possibly one arc with an endpoint on p_3. An arc in $T(F) \cap D_n^-$ which has both endpoints on N must enclose p_3, since otherwise it could be slid off N to reduce the number of points of intersection between $T(F)$ and N. Thus $T(F) \cap D_n^-$ must consist of a collection of parallel arcs enclosing p_3, and possibly one arc with an endpoint on p_3.

Similarly, each arc in $T(F) \cap D_n^+$ either encloses one of the puncture points p_1 or p_2, or has an endpoint on one of p_1 or p_2. There can be no arc in $T(F) \cap D_n^+$ which encloses both p_1 and p_2, since the outermost such arc together with the outermost arc in $T(F) \cap D_n^-$ would form a closed loop.

An example of a noodle and a tine edge in D_3 is shown in Figure 1. We have omitted the handle of the fork, which plays no role in our argument.

Let z_i and z_j be two points of intersection between $T(F)$ and N which are joined by an arc in $T(F) \cap D_n^+$ or $T(F) \cap D_n^-$. This arc, together with a subarc of N, encloses one puncture point. Thus

$$a_j = a_i \pm 1.$$

Also, $T(F)$ intersects N with opposite signs at z_i and z_j, so

$$\epsilon_j = -\epsilon_i.$$

Thus

$$\epsilon_j(-1)^{a_j} = \epsilon_i(-1)^{a_i}.$$

Proceeding along $T(F)$, we conclude that the values of $\epsilon_i(-1)^{a_i}$ are the same for all $i = 1, \ldots, k$. Thus $\langle N, F \rangle$ evaluated at $q = -1$ is equal to $\pm k$. Thus $\langle N, F \rangle$ is not equal to zero. □

We are now ready to prove that the Burau representation of B_3 is faithful.

Proof of Theorem 4.1. Let $\beta\colon D_3 \to D_3$ be a homeomorphism which represents an element of the kernel of the Burau representation. We will show that β is isotopic relative to ∂D_n to the identity map, and so represents the trivial braid.

Let N be a noodle. As before, take N to be a horizontal line through D_n such that the puncture points p_1 and p_2 lie above N and p_3 lies below N. Let F be a fork such that $T(F)$ is a straight line from p_1 to p_2 which does not intersect N. Then $\langle N, F \rangle = 0$. By the Basic Lemma, $\langle N, \beta(F) \rangle = 0$. By the Key Lemma, $\beta(T(F))$ is isotopic to an arc which is disjoint from N. By applying an isotopy to β relative to ∂D_n, we can assume that $\beta(T(F)) = T(F)$.

By a similar argument using different noodles, we can assume that each of the edges of the triangle with vertices p_1, p_2 and p_3 is fixed by β. Thus β must be some power of Δ^2, the Dehn twist about a curve parallel to ∂D_n. It is easy to show that the Burau representation of Δ^2 is the scalar matrix $q^3 I$. Thus the only power of Δ^2 which lies in the kernel of the Burau representation is the trivial braid. □

5. The case $n = 4$

We now address the question of whether the Burau representation of B_4 is faithful. If the Key Lemma holds for the case $n = 4$ then the same argument used for B_3 can be used to show that the Burau representation of B_4 is faithful. The converse is also true: if the Key Lemma is false for a given n then the Burau representation of B_n is unfaithful. In other words, the following theorem holds.

Theorem 5.1. *The following are equivalent:*

- *the Burau representation of B_n is faithful,*
- *if N and F are any noodle and fork in D_n such that $\langle N, F \rangle = 0$ then $T(F)$ is isotopic to an arc which is disjoint from N.*

A proof can be found in [2, Theorem 1.4], or [13, Theorem 1.5], although neither of these use the terminology of forks and noodles. The arc α in [2] corresponds to the tine edge of F, and the arc β from the basepoint to a puncture corresponds to the noodle N. To be precise, N must lie in a regular neighbourhood of β which contains only one puncture point. (This is why, in the definition of a noodle, we required that a component of $D_n \setminus N$ contain precisely one puncture point.)

A simple generalisation of our argument for B_3 shows that the second condition of Theorem 5.1 implies the first. The proof of the other direction is constructive. Suppose $\langle N, F \rangle = 0$ but $T(F)$ is not isotopic to an arc which is disjoint from N. Let γ_1 be a simple closed curve which is parallel to the boundary of the component of $D_n \setminus N$ containing all but one puncture point. Let γ_2 be the boundary of a regular neighbourhood of $T(F)$. It is shown in [2] that the commutator of the Dehn twist

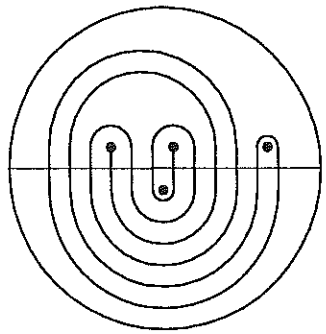

FIGURE 2. A tine edge and a noodle in D_4.

about γ_1 with the half Dehn twist about γ_2 is a non-trivial braid in the kernel of the Burau representation of B_n.

We now define a *standard form* for a noodle N and tine edge $T(F)$, similar to the one used in the proof of the Key Lemma. Let N be a horizontal straight line through D_4 with p_1, p_2 and p_3 above it, and p_4 below it. Let D_4^+ and D_4^- be the upper and lower halves of $D_4 \setminus N$, respectively. We say that N and $T(F)$ are in standard form if $T(F)$ intersects N transversely, each arc in $D_4^- \cap T(F)$ either encloses p_4 or has an endpoint on p_4, and each arc in $D_4^+ \cap T(F)$ either

- encloses one of p_1, p_2 or p_3,
- encloses p_1 and p_2, the two leftmost puncture points in D_4^+, or
- has an endpoint on a puncture point.

Figure 2 shows an example of a noodle and a tine edge in standard form in D_4.

Any noodle N and tine edge $T(F)$ can be put into standard form by first isotoping $T(F)$ so as to intersect N at a minimum possible number of points, and then applying some homeomorphism to the entire picture. The homeomorphism might need to be orientation-reversing. This would have the effect of changing the sign of $\langle N, F\rangle$ and substituting q^{-1} for q, so would not affect whether $\langle N, F\rangle$ is zero.

By some of the basic theory of curves on surfaces, if $T(F)$ and N are in standard form and intersect then $T(F)$ is not isotopic to an arc which is disjoint from N. Thus the Burau representation of B_4 is faithful if and only if, whenever N and $T(F)$ are in standard form and intersect, we have that $\langle N, F\rangle \neq 0$. The handle of F can be ignored because it has no effect on $\langle N, F\rangle$ up to sign and multiplication by a power of q.

The simple parity argument used to prove the Key Lemma in D_3 will not work for D_4 because of the existence of arcs enclosing two puncture points. In fact, in D_4 there can be some cancellation in the calculation of $\langle N, F\rangle$, whereas our argument showed that this cannot happen in D_3. We might attempt a more sophisticated argument which shows that there cannot be *complete* cancellation. Unfortunately, none of the obvious approaches seem to work. For example, it is possible to have complete cancellation of all of the highest and lowest powers of q that occur in the calculation of $\langle N, F\rangle$.

Conversely, we could attempt a computer search to find N and $T(F)$ in standard form which intersect, but give $\langle N, F\rangle = 0$. This approach has worked for B_5 [2]. We now outline the algorithm for B_4.

A tine edge $T(F)$ in standard form is determined up to isotopy by the following:

- four non-negative integers specifying the number of arcs in $T(F) \cap D_n^+$ of each of the four possible types, and
- which of the puncture points are endpoints of $T(F)$.

Given data defining $T(F)$, one can compute $\langle N, F\rangle$, up to sign and multiplication by a power of q, by progressing along $T(F)$ from one end to the other. We can thus embark upon an exhaustive open-ended search for a tine edge $T(F)$ in standard form which intersects N but gives $\langle N, F\rangle = 0$.

One of the most time-consuming aspects of this algorithm is the need to deal with arrays of integers representing the polynomial $\langle N, F\rangle$. There is a simple trick which can be used to eliminate this problem. Let M be a large integer. Consider a map

$$\mathbf{Z}[q^{\pm 1}] \to \mathbf{Z}/M\mathbf{Z}$$

sending q to some unit in $\mathbf{Z}/M\mathbf{Z}$. Instead of computing $\langle N, F\rangle$ we can compute its image in $\mathbf{Z}/M\mathbf{Z}$. This allows us to work with a single integer instead of an array. There will be some "false alarms" for which $\langle N, F\rangle$ is non-zero but its image in $\mathbf{Z}/M\mathbf{Z}$ is zero. However these are infrequent and easily checked separately.

This trick speeds up the search considerably. I have used it to check all forks for which $T(F)$ intersects N at up to 2000 points. By comparison, the example in D_5 consists of a noodle and a tine edge which intersect at 100 points.

There are some possibilities for further improvements in the algorithm. Perhaps the simplest way to speed up the search is to increase the number of searchers. I would like to take this opportunity to advertise my webpage

http://www.ms.unimelb.edu.au/~bigelow

where, at the time of writing, it is possible to donate computer time to this noble and possibly futile search.

6. Specialising q

We conclude this paper with an aside concerning the "false alarms" mentioned in the previous section. Recall that a false alarm occurs when $\langle N, F\rangle$ is non-zero but maps to zero in $\mathbf{Z}/M\mathbf{Z}$ when q is assigned some unit q_0. Usually this is not very interesting, since M was fairly arbitrary. But some false alarms occur when the integer q_0 is a root of $\langle N, F\rangle$. At first I thought that these more interesting false alarms should give rise to a non-trivial element of the kernel of the specialisation of the Burau representation to $q = q_0$. However it turns out that the correct theorem is as follows.

Theorem 6.1. *Let q_0 be a complex number which is neither zero nor a root of unity. The following are equivalent:*

- *the Burau representation of B_n is faithful when q is specialised to q_0,*
- *if N and F are any noodle and fork in D_n such that both q_0 and $1/q_0$ are roots of $\langle N, F\rangle$ then $T(F)$ is isotopic to an arc which is disjoint from N.*

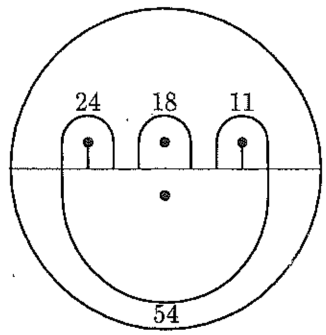

FIGURE 3. A tine edge and a noodle in D_4.

The proof is similar to that of Theorem 5.1. However it is necessary to work with $\langle F, N\rangle$ as well as $\langle N, F\rangle$. Here, $\langle F, N\rangle$ is defined similarly to $\langle N, F\rangle$, and is related by the equation

$$\langle F, N\rangle(q) = -\langle N, F\rangle(1/q).$$

If q is transcendental then $\langle F, N\rangle$ is zero if and only if $\langle N, F\rangle$ is, so it suffices to consider only $\langle N, F\rangle$. However if q is a specific value q_0 then the proof requires that both $\langle N, F\rangle$ and $\langle F, N\rangle$ are zero at q_0.

Note that the Burau representation of B_n is not faithful when q is specialised to a root of unity. To see this, let Δ^2, be the Dehn twist about a curve parallel to ∂D_n. The Burau representation of Δ^2 is the scalar matrix $q^n I$. At a kth root of unity, Δ^{2k} will lie in the kernel of the Burau representation.

Using Theorem 6.1, the computer search for an element of the kernel of the Burau representation of B_4 can be modified to search for an element of the kernel at some specific integer value of q. It took about half a minute for this program to find the following.

Corollary 6.2. *The Burau representation of B_4 is unfaithful at $q = 2$ and at $q = 1/2$.*

Proof. Let $T(F)$ be the tine edge in standard form as shown schematically in Figure 3. The endpoints of $T(F)$ at p_1 and p_3 are shown. Segments of $T(F)$ are labelled with numbers to indicate the number of parallel copies required.

A laborious computation or a short computer program can be used to check that

$$\langle N, F\rangle = -(q-1)(q-2)(2q-1)(q^2-q+1)(q^2+1),$$

up to multiplication by a power of q. Both 2 and $1/2$ are roots of this polynomial. □

The proof of Theorem 6.1 uses the same construction as that of Theorem 5.1. Thus we obtain a specific non-trivial braid β in the kernel of the Burau representation of B_4 at $q = 2$ and $q = 1/2$. To make things more readable, let $a = \sigma_1$, $b = \sigma_2$, and $c = \sigma_3$. Then

$$\beta = [(ba)^3, \psi^{-1} b \psi],$$

where

$$\psi = a^{-3}b^{-2}c^{-1}bc^4b^{-1}cbabc^2ba^{-1}b^{-1}c^{-2}.$$

Note, this uses the convention that braids compose from right to left.

The noodle and fork shown in Figure 3 are the simplest possible example in the sense that they have the fewest points of intersection. They also have the curious property that none of the subarcs of $T(F)$ above N enclose two puncture points, so there is no cancellation in the calculation of $\langle N, F\rangle$. I can think of no explanation for this.

Apart from 2, 1/2, and any root of unity, I know of no other values at which the Burau representation of B_4 is unfaithful, and certainly no values at which it is faithful. This is despite hundreds of hours of computer time spent on the case $q = 3$, and somewhat less on $q = 4$ and $q = 5$.

In the case of B_3, I know of no value of q other than roots of unity at which the Burau representation is unfaithful. We have shown that the Burau representation of B_3 is faithful at any transcendental value of q. The following lemma provides many other values at which it is faithful.

Lemma 6.3. *Let N and F be a noodle and a fork in D_3 such that $T(F)$ is not isotopic to an arc which is disjoint from N. Then the highest and lowest powers of q in the polynomial $\langle N, F\rangle$ both occur with coefficient ± 1.*

Corollary 6.4. *If the Burau representation of B_3 is unfaithful at $q = q_0$ then both q_0 and $1/q_0$ are roots of a monic polynomial. In particular, the Burau representation of B_3 is faithful at any rational number other than 0 or ± 1.*

Proof of Lemma 6.3. Put N and F in the standard form as in Figure 1. Thus N is a horizontal straight line with p_1 and p_2 above it and p_3 below it. Assume that d_0 is the left endpoint of N. We show that the lowest power of q in $\langle N, F\rangle$ occurs with coefficient ± 1. The highest power of q and the case where d_0 is the right endpoint of N are handled similarly.

Let $z_1, \ldots, z_k$ be the points of intersection between N and $T(F)$. Recall Eq. (1), which states that

$$\langle N, F\rangle = \sum_{i=1}^{k} \epsilon_i q^{a_i}.$$

Let z_i be such that a_i is minimal. We will show that there is only one such z_i. We proceed by induction on k. The case $k = 1$ is trivial, so assume $k > 1$.

If z_i were to the right of p_3 then there would be a subarc of $T(F)$ going from z_i around p_3 in the clockwise (negative) sense to intersect N at a point z_j. Then $a_j = a_i - 1$, which contradicts the minimality of a_i. Thus z_i must lie to the left of p_3.

Let P be a vertical line from the top of the disk to a point on N between the puncture points p_1 and p_2 such that P does not intersect $T(F)$. If z_i were to the left of p_3 but to the right of P then there would be a subarc of $T(F)$ going from z_i around p_2 in the clockwise sense, once again contradicting the minimality of a_i. Thus z_i lies to the left of P.

Let N' be the union of P with the portion of N which lies to the left of P. This is a noodle which intersects $T(F)$ at fewer than k points. The pairing $\langle N', F \rangle$ is the sum of those monomials $\epsilon_j q^{a_j}$ for which z_j lies to the left of P. Thus z_i is such that a_i is minimal in the calculation of $\langle N', F \rangle$. By the induction hypothesis, there is only one such z_i, so we are done. □

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