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The Lawrence-Krammer Representation

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ABSTRACT. The Lawrence-Krammer representation of the braid groups recently came to prominence when it was shown to be faithful by myself and Krammer. It is an action of the braid group on a certain homology module $H_2(\tilde{C})$ over the ring of Laurent polynomials in q and t. In this paper we describe some surfaces in $\tilde{C}$ representing elements of homology. We use these to give a new proof that $H_2(\tilde{C})$ is a free module. We also show that the $(n-2,2)$ representation of the Temperley-Lieb algebra is the image of a map to relative homology at $t = -q^{-1}$, clarifying work of Lawrence.

1. Introduction

The Lawrence-Krammer representation is the action of the braid group B_n on a certain homology module $H_2(\tilde{C})$ over the ring $\Lambda = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$. It was introduced by Lawrence [**Law90**], except that she worked over $\mathbf{C}$ instead of Λ. It recently came to prominence when it was shown to be faithful for $n = 4$ [**Kra00**], and then for all n [**Big01**], [**Kra02**]. Since then, a number of papers have appeared which closely examine certain aspects of the representation. See [**PP01**], [**Son02**], and [**Bud02**]. We now continue in this tradition, with the specific goal of understanding the connection with the Temperley-Lieb algebra.

This paper was partly motivated by an attempt to clarify two points from [**Big01**]. First, the pairing between a "noodle" and a "fork" involved an algebraic intersection number between two non-compact surfaces in $\tilde{C}$. Such a thing is not necessarily well-defined, so I gave an indirect proof of the existence of a certain closed surface corresponding to a fork. I now have an explicit description of this surface, which will be given in Section 3.4. In fact it is possible to define the pairing without reference to this surface by using results from [**Kaw96**, Appendix E]. (Thanks to Won Taek Song, whose paper [**Son02**] drew my attention to this.)

Second, to compute matrices for the representation, I tensored $H_2(\tilde{C})$ with a field containing Λ. The resulting vector space contains $H_2(\tilde{C})$, but strictly speaking, the action of B_n on this vector space should not be called the Lawrence-Krammer representation. In Section 5 we give a new proof that $H_2(\tilde{C})$ is a free Λ-module. This is originally due to Paoluzzi and Paris [**PP01**], but our proof uses an explicit

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description of surfaces representing elements of a free basis of $H_2(\tilde{C})$. The correct matrices for the Lawrence-Krammer representation could be computed from this, but we do not do so since they are quite complicated.

By addressing these two issues from [**Big01**] we also shed new light on the work of Lawrence [**Law90**]. There, the Lawrence-Krammer representation was used to give a topological interpretation to a representation of the Hecke algebra. The idea is to specialise t to the value $-q^{-1}$, at which point the Lawrence-Krammer representation becomes reducible and the desired representation appears as a quotient. Somewhat complicated methods are used in [**Law90**] to define the required quotient. In Section 6 we show that it is simply the image of a map to a certain relative homology module.

Throughout this paper n is a positive integer, D is the unit disk centred at the origin in the complex plane, $-1 < p_1 < \cdots < p_n < 1$ are real numbers, and $D_n = D \setminus \{p_1, \ldots, p_n\}$ is the n-times punctured disk. The braid group B_n is the mapping class group of D_n. We also assume familiarity with the presentation of B_n using Artin generators σ_i, and the interpretation of B_n as the fundamental group of a configuration space in the plane.

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2. The Lawrence-Krammer representation

We briefly review the definition of the Lawrence-Krammer representation as given in [**Big01**]. Let C be the space of unordered pairs of distinct points in D_n. Let $c_0 = \{d_1, d_2\}$ be a basepoint in C, where d_1 and d_2 are distinct points on the boundary of the disk.

We define a homomorphism

$$\Phi\colon\ \pi_1(C, c_0) \to \langle q \rangle \oplus \langle t \rangle$$

as follows. Suppose $\alpha\colon I \to C$ is a closed loop in C representing an element of $\pi_1(C, c_0)$. By ignoring the puncture points we can consider α as a loop in the space of unordered pairs of points in the disk, and hence as a braid in B_2. Let j be the exponent of this braid in the Artin generator σ_1. Similarly, the map

$$s \mapsto \{p_1, \ldots, p_n\} \cup \alpha(s)$$

determines a braid in B_{n+2}. Let j' be the the exponent sum of this braid in the Artin generators of B_{n+2}. Note that j and j' have the same parity. Let $i = \frac{1}{2}(j' - j)$. We define

$$\Phi(\alpha) = q^i t^j.$$

Let $\tilde{C}$ be the connected covering space of C such that $\pi_1(\tilde{C}) = \ker(\Phi)$. Fix a choice of $\tilde{c}_0$ in the fibre over c_0. The homology group $H_2(\tilde{C})$ admits a Λ-module structure, where q and t act by covering transformations.

Suppose f is a homeomorphism from D_n to itself, representing an element of B_n. Let f_* be the induced map from $\pi_1(C, c_0)$ to itself. I claim that

$$\Phi \circ f_* = \Phi.$$

To see this, think of $\pi_1(C, c_0)$ as the subgroup of B_{n+2} consisting of braids whose first n strands are straight. Then f_* acts on $\pi_1(C, c_0)$ by conjugation with a braid on the first n strands. This preserves the image under Φ, since Φ maps to an abelian group.

It follows that f has a unique lift

$$\tilde{f}\colon (\tilde{C}, \tilde{c}_0) \to (\tilde{C}, \tilde{c}_0),$$

and that the induced map

$$\tilde{f}_*\colon H_2(\tilde{C}) \to H_2(\tilde{C})$$

is a Λ-module automorphism. We define the Lawrence-Krammer representation to be this action of B_n on $H_2(\tilde{C})$.

We now introduce some relative homology modules. For $\epsilon > 0$, let ν_ϵ be the set of points $\{x, y\} \in C$ such that either x and y are within distance ϵ of each other, or at least one of them is within distance ϵ of a puncture point. Let $\tilde{\nu}_\epsilon$ be the preimage of ν_ϵ in $\tilde{C}$. The relative homology modules $H_2(\tilde{C}, \tilde{\nu}_\epsilon)$ are nested by inclusion. Let

$$H_2(\tilde{C}, \tilde{\nu}) = \lim_{\epsilon \to 0} H_2(\tilde{C}, \tilde{\nu}_\epsilon),$$

and

$$H_2(\tilde{C}, \partial\tilde{C} \cup \tilde{\nu}) = \lim_{\epsilon \to 0} H_2(\tilde{C}, \partial\tilde{C} \cup \tilde{\nu}_\epsilon).$$

The braid group B_n acts on these, and on $H_2(\tilde{C}, \partial\tilde{C})$, by Λ-module automorphisms.

3. Some surfaces

In this section we describe some immersed surfaces in $\tilde{C}$. These will be used to represent elements of homology or relative homology. Each immersed surface will be specified by giving a map from a surface to C and showing that it can be lifted to $\tilde{C}$. We will not specify a choice of lift, and we will not pay much attention to issues of orientation. Thus the resulting element of homology or relative homology will only be defined up to multiplication by a unit in Λ. This will be sufficient for our purposes.

3.1. Squares and triangles. We describe some properly embedded surfaces representing elements of the second homology of $\tilde{C}$ relative to $\partial\tilde{C}$, $\tilde{\nu}$ and $\partial\tilde{C} \cup \tilde{\nu}$. These will be represented by one or two embedded edges in the disk.

Suppose $\alpha_1, \alpha_2\colon I \to D$ are disjoint embeddings of the interior of I into D_n, and map the endpoints of I to puncture points (not necessarily injectively). Let f be the map from the interior of $I \times I$ to C given by $f(x, y) = \{\alpha_1(x), \alpha_2(y)\}$. A lift of f to $\tilde{C}$ will represent an element of $H_2(\tilde{C}, \tilde{\nu})$. Similarly, we can define an element of $H_2(\tilde{C}, \partial\tilde{C})$ corresponding to a pair of disjoint edges in D_n with endpoints on ∂D. Finally, we can define an element of $H_2(\tilde{C}, \partial\tilde{C} \cup \tilde{\nu})$ corresponding to a pair of edges which have a mixture of endpoints on ∂D and on puncture points. For all of these examples, call the resulting element of relative homology the *square* corresponding to the edges α_1 and α_2.

Now suppose $\alpha\colon I \to D$ is an embedding of the interior of I into D_n, and maps the endpoints of I to the puncture points. Define a map

$$f\colon \{(x, y) \in I \times I : 0 < x < y < 1\} \to C$$

by $f(x, y) = \{f(x), f(y)\}$. A lift of f to $\tilde{C}$ will represent an element of $H_2(\tilde{C}, \tilde{\nu})$. Similarly, we can obtain an element of $H_2(\tilde{C}, \partial\tilde{C} \cup \tilde{\nu})$ if we allow one or both endpoints of the edge to be on ∂D. For all of these examples, call the resulting element of relative homology the *triangle* corresponding to α.

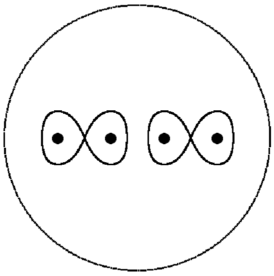
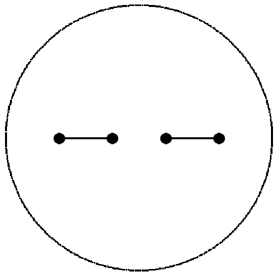
 $\mapsto (1-q)^2$  .

FIGURE 1. A torus and a square.

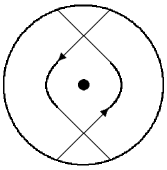


FIGURE 2. The disk B and edges $\alpha_1(I)$ and $\alpha_2(I)$.

3.2. A genus one surface. Suppose $\alpha_1, \alpha_2 \colon S^1 \to D_n$ are disjoint figure-eights, each going around two puncture points, as in Figure 1. Define a map f from the torus $S^1 \times S^1$ to C by $f(x, y) = \{\alpha_1(x), \alpha_2(y)\}$. Both the meridian and longitude of the torus are mapped into the kernel of Φ, so f lifts to a map

$$\tilde{f} \colon S^1 \times S^1 \to \tilde{C}.$$

We obtain an element $[\tilde{f}]$ of $H_2(\tilde{C})$.

It will be useful to know the image of $[\tilde{f}]$ in $H_2(\tilde{C}, \tilde{\nu})$. I claim this is $(1-q)^2$ times the square as indicated in Figure 1. To see this, homotope each figure-eight α_i so as to map the points $\pm 1 \in S^1$ into respective ϵ-neighbourhoods of the two puncture points enclosed by α_i. Then f maps the four lines $\{\pm 1\} \times S^1$ and $S^1 \times \{\pm 1\}$ into ν_ϵ. These lines cut the torus into four squares. The restriction of $\tilde{f}$ to each of these squares represents an element of $H_2(\tilde{C}, \tilde{\nu})$. The claim now follows from a careful comparison of the lifts and orientations of these squares, which is left to the reader.

3.3. A genus two surface. Suppose $\alpha_1, \alpha_2 \colon S^1 \to D_n$ are figure-eights such that α_1 passes around p_i and p_j, α_2 passes around p_j and p_k, and α_1 intersects α_2 twice, as in Figure 3. Let $B \subset D$ be a disk centred at p_j, containing the two points of intersection, and meeting each of α_1 and α_2 in a single edge. For $i = 1, 2$, let $I_i \subset S^1$ be the interval $\alpha_i^{-1}(B)$. For convenience, assume $I_1 = I_2$, and identify both with $I = [0, 1]$, oriented as shown in Figure 2.

Let T be the closure of

$$(S^1 \times S^1) \setminus (I_1 \times I_2).$$

Note that for $(x, y) \in T$ we have $\alpha_1(x) \neq \alpha_2(y)$. We can therefore define $f \colon T \to C$ by $f(x, y) = \{\alpha_1(x), \alpha_2(y)\}$.

Let f_0 be the restriction of f to $\partial T = \partial(I \times I)$. For $s \in I$, let f_s be the composition of f_0 with an anticlockwise rotation of B by an angle of $s\pi$ about the centre p_j. We can assume B has a rotational symmetry so that $f_1(x, y) = f_0(y, x)$

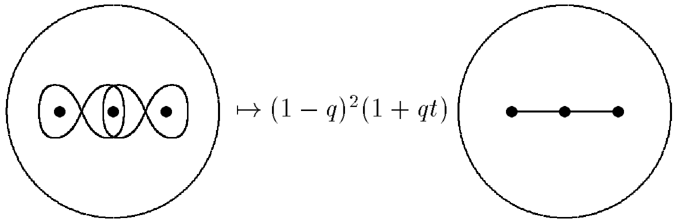


FIGURE 3. A genus two surface and a square.

for all $(x, y) \in \partial(I \times I)$. Thus f_0 and f_1 represent the same loops, but with opposite orientations.

We now build a closed genus two surface Σ_2 by gluing together $\partial T \times I$ and two copies of T as follows. First glue $\partial T \times I$ to T by $((x, y), 0) \sim (x, y)$. Then glue $\partial T \times I$ to a second copy T' of T by $((x, y), 1) \sim (y, x)$. Let Σ_2 be the surface so obtained. Let $g\colon \Sigma_2 \to C$ be given by $g|_T = g|_{T'} = f$ and $g|_{(\partial T \times \{s\})} = f_s$.

The fundamental group of Σ_2 is generated by the meridian and longitude of T and T'. Each of these is mapped by g into the kernel of Φ, so g lifts to a map $\tilde{g}\colon \Sigma_2 \to \tilde{C}$. We obtain an element $[\tilde{g}]$ of $H_2(\tilde{C})$.

We now compute the image of $[\tilde{g}]$ in $H_2(\tilde{C}, \tilde{\nu})$. For any $\epsilon > 0$ we can assume that the disk B has radius less than ϵ. Then g maps $\partial T \times I$ into ν_ϵ, so $\tilde{g}|_T$ represents an element of $H_2(\tilde{C}, \tilde{\nu}_\epsilon)$. This element is $(1-q)^2$ times a square, by a similar argument to the one given for the genus one surface. It remains to figure out how $\tilde{g}|_{T'}$ is related to $\tilde{g}|_T$.

Consider the path $\delta\colon I \to \Sigma_2$ given by

$$\delta(s) = ((0, 0), s) \in \partial T \times I.$$

This goes from $(0, 0) \in T$ to $(0, 0) \in T'$. Now $g \circ \delta$ is a loop in C in which the pair of points switch places by an anticlockwise rotation through an angle of π around p_j. Thus $\Phi(g \circ \delta) = qt$. It follows that $\tilde{g}|_{T'} = qt \circ \tilde{g}|_T$.

Also note that T' and T inherit the same orientation from Σ_2. This is because the ends of the annulus $\partial T \times I$ were attached with opposite orientations. We conclude that the image of $[\tilde{g}]$ in $H_2(\tilde{C}, \tilde{\nu})$ is $(1-q)^2(1+qt)$ times a square, as shown in Figure 3.

3.4. A genus three surface. Suppose $\alpha_1, \alpha_2\colon S^1 \to D_n$ are figure-eights, both passing around p_i and p_j, and intersecting transversely at four points, as in Figure 4. We construct a map from a genus three surface Σ_3 into $\tilde{C}$ by a slight modification of the procedure used above for Σ_2. This time the surface T will be a torus with two disks removed, one for each of p_i and p_j. Two annuli are then needed to glue T to another copy T' of T. We obtain a genus three surface Σ_3 and a map $g\colon \Sigma_3 \to C$.

We now show that g lifts to a map $\tilde{g}\colon \Sigma_3 \to \tilde{C}$. The image of the longitude, meridian, and both boundary components of T all lie in the kernel of Φ. Thus $g|_T$ lifts to $\tilde{g}|_T$. This lift can be extended to the annuli $\partial T \times I$. Finally, define $\tilde{g}|_{T'}$ to be the covering transformation qt applied to $\tilde{g}|_T$.

We now compute the image of $[\tilde{g}]$ in $H_2(\tilde{C}, \tilde{\nu})$. This is $(1-q)^2(1+qt)$ times a square, by a similar argument to the one given for the genus two surface. This square corresponds to a parallel pair of edges from p_i to p_j. These edges can be homotoped

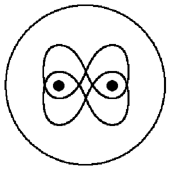
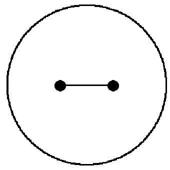
 $\mapsto (1-q)^2(1+qt)(1-t)$  .

FIGURE 4. A genus three surface and a triangle.

so as to lie within ϵ of each other. Then a diagonal of the square in $\tilde{C}$ will lie in $\tilde{\nu}_\epsilon$. This diagonal cuts the square into two triangles, which have opposite orientations and differ by the covering transformation t. Thus $[\tilde{g}]$ represents $(1-q)^2(1+qt)(1-t)$ times a triangle, as shown in Figure 4.

4. An intersection pairing

It is not immediately obvious that the surfaces described in Section 3 represent non-trivial elements of homology or relative homology. In Sections 5 and 6 we will need to prove even stronger results concerning the linear independence of various sets of such elements. Our main tool will be the following intersection pairing.

For $x \in H_2(\tilde{C})$ and $y \in H_2(\tilde{C}, \partial\tilde{C} \cup \tilde{\nu})$ let $(x \cdot y) \in \mathbf{Z}$ denote the standard intersection number. We define an intersection pairing

$$\langle \cdot, \cdot \rangle \colon\ H_2(\tilde{C}) \times H_2(\tilde{C}, \partial\tilde{C} \cup \tilde{\nu}) \to \Lambda$$

by

$$\langle x, y \rangle = \sum_{i,j \in \mathbf{Z}} (x \cdot q^i t^j y) q^i t^j.$$

A similar definition gives a pairing

$$\langle \cdot, \cdot \rangle' \colon\ H_2(\tilde{C}, \tilde{\nu}) \times H_2(\tilde{C}, \partial\tilde{C}) \to \Lambda.$$

To check that these are well defined requires some elementary homology theory. See [**Kaw96**, Appendix E], where the following properties are also proved.

For $x \in H_2(\tilde{C})$, $y \in H_2(\tilde{C}, \partial\tilde{C} \cup \tilde{\nu})$, $\sigma \in B_n$, and $\lambda \in \Lambda$, we have

$$\langle \sigma x, \sigma y \rangle = \langle x, y \rangle,$$

and

$$\langle \lambda x, y \rangle = \lambda \langle x, y \rangle = \langle x, \bar{\lambda} y \rangle,$$

where $\bar{\lambda}$ is the image of λ under the automorphism of Λ taking q to q^{-1} and t to t^{-1}. Similar identities hold for $\langle \cdot, \cdot \rangle'$.

Note that the above definition of $\langle \cdot, \cdot \rangle$ differs from that of [**Big01**] in that the order of the entries is reversed. The above is consistent with [**Kaw96**], and with the usual definition of sesquilinear.

It will frequently be necessary to compute the intersection pairing in specific examples, up to multiplication by a unit in Λ. The rest of this section is devoted to discussion of how to do this.

Let α_1 and α_2 be edges in D with endpoints on puncture points, representing a square $a \in H_2(\tilde{C}, \tilde{\nu})$. Let β_1 and β_2 be edges in D with endpoints on ∂D, representing a square $b \in H_2(\tilde{C}, \partial\tilde{C})$. We discuss how to compute $\langle a, b \rangle'$ up to multiplication by a unit in Λ.

Let A and B be the surfaces in C corresponding to a and b respectively. The intersection $A \cap B$ is the set of pairs of the form $\{x, y\}$ where either

- $x \in \alpha_1 \cap \beta_1$ and $y \in \alpha_2 \cap \beta_2$, or
- $x \in \alpha_1 \cap \beta_2$ and $y \in \alpha_2 \cap \beta_1$.

Assume all edges, intersect transversely, so A intersects B transversely. Let $\tilde{A}$ and $\tilde{B}$ be lifts of A and B to $\tilde{C}$.

If A and B do not intersect then neither will any of their lifts, so $\langle a, b\rangle' = 0$. If A and B intersect at one point $\{x, y\}$ then choose lifts and orientations so that $\tilde{A}$ intersects $\tilde{B}$ with positive sign at a point in the fibre of $\{x, y\}$. Then $\langle a, b\rangle' = 1$.

Now suppose A and B intersect in two points, $\{x, y\}$ and $\{x', y'\}$. Choose lifts and orientations so that $\tilde{A}$ intersects $\tilde{B}$ with positive sign at a point in the fibre of $\{x, y\}$. Let i and j be the integers such that $\tilde{A}$ intersects $q^i t^j \tilde{B}$ at a point in the fibre of $\{x', y'\}$, and let $\epsilon = \pm 1$ be the sign of that intersection. Then $\langle a, b\rangle' = 1 + \epsilon q^i t^j$. It remains to compute i, j and ϵ.

Let γ_A and γ_B be paths from $\{x, y\}$ to $\{x', y'\}$ that lie in A and B respectively. Let $\tilde{\gamma}_A$ and $\tilde{\gamma}_B$ be the lifts that start in $\tilde{A} \cap \tilde{B}$. Then $\tilde{\gamma}_A(1)$ and $\tilde{\gamma}_B(1)$ are the points in the fibre of $\{x', y'\}$ lying in $\tilde{A}$ and $\tilde{B}$ respectively. Thus

$$q^i t^j \tilde{\gamma}_B(1) = \tilde{\gamma}_A(1),$$

so

$$q^i t^j = \Phi(\gamma_A^{-1} \gamma_B).$$

In practice this will be easy to compute. It remains to compute ϵ.

Relabel if necessary so that $x, x' \in \alpha_1$ and $y, y' \in \alpha_2$. Orient the edges α_1, α_2, β_1 and β_2. The sign of the intersection of A with B at $\{x, y\}$ is determined by

- the sign of the intersection of α_1 with $\beta_1 \cup \beta_2$ at x,
- the sign of the intersection of α_2 with $\beta_1 \cup \beta_2$ at y, and
- whether $x \in \beta_2$ or $x \in \beta_1$.

By assumption, the conventions are such that this intersection is positive. Each of the following will reverse the sign of the intersection of A with B at $\{x', y'\}$.

- α_1 intersects $\beta_1 \cup \beta_2$ with different signs at x and x',
- α_2 intersects $\beta_1 \cup \beta_2$ with different signs at y and y',
- either $x \in \beta_1$ and $y \in \beta_2$, or $x \in \beta_2$ and $y \in \beta_1$.

If none or two of these hold then $\epsilon = 1$. If one or three of these hold then $\epsilon = -1$.

We will sometimes need to compute intersection pairings between surfaces that are not squares. First consider the case A is a triangle corresponding to an edge α, B is a square as before, and A and B intersect transversely at two points. Then the above discussion applies almost unchanged. In place of the assumption that $x, x' \in \alpha_1$ and $y, y' \in \alpha_2$, make the assumption that x occurs before y and x' occurs before y' with respect to the orientation of α. We will also compute pairings between surfaces which are neither squares nor triangles, but the same ideas will apply.

Finally, suppose A and B intersect at more than two points. Then each point of intersection contributes a monomial to the pairing. We can assume one of these monomials is 1, and compute the others by the methods discussed above.

5. A basis

In this section, we analyse the Lawrence-Krammer representation using the tools developed in Sections 3 and 4. We give a new proof of the following theorem.

THEOREM 5.1. *$H_2(\tilde{C})$ is a free Λ-module of rank $\binom{n}{2}$.*

This is originally due to Paoluzzi and Paris [**PP01**]. Our proof gives a more explicit description of surfaces representing elements of a free basis for $H_2(\tilde{C})$. The matrices usually given for the Lawrence-Krammer representation use a basis for $\mathbf{Q}(q,t) \otimes H_2(\tilde{C})$, not for $H_2(\tilde{C})$. This subtle distinction will be discussed at the end of this section.

We use the following basic result about $H_2(\tilde{C})$.

LEMMA 5.2. *The vector space $\mathbf{Q}(q,t) \otimes H_2(\tilde{C})$ has dimension $\binom{n}{2}$. The natural map from $H_2(\tilde{C})$ to $\mathbf{Q}(q,t) \otimes H_2(\tilde{C})$ is injective.*

The proof uses a finite 2-complex that is homotopy equivalent to C. Various methods for constructing the required complex can be found in [**Law90**], [**Big01**], [**PP01**], and [**Bud02**]. Since the complex has dimension two, $H_2(\tilde{C})$ is the kernel of a matrix ∂ with entries in Λ. Now $\mathbf{Q}(q,t) \otimes H_2(\tilde{C})$ is the kernel of the matrix ∂, considered as a matrix over $\mathbf{Q}(q,t)$. Thus $H_2(\tilde{C})$ is the submodule of $\mathbf{Q}(q,t) \otimes H_2(\tilde{C})$ consisting of those vectors whose entries lie in Λ. The dimension of $\mathbf{Q}(q,t) \otimes H_2(\tilde{C})$ can be computed from an explicit description of the matrix ∂, which can be found in any of the papers mentioned above.

5.1. Proof of the theorem. We now prove Theorem 5.1. For $1 \leq i < j \leq n$, we define $v'_{i,j} \in H_2(\tilde{C}, \tilde{\nu})$ as follows. If $j - i > 2$, let $v'_{i,j}$ be the square corresponding to the edges $[p_i, p_{i+1}]$ and $[p_{j-1}, p_j]$. If $j - i = 2$ and $i > 1$, let $v'_{i,j}$ be the square corresponding to the edge $[p_i, p_{i+1}]$ and an edge from p_{i-1} to p_j whose interior lies in the lower half plane. Let $v'_{1,3}$ be the square corresponding to the edges $[p_1, p_2]$ and $[p_2, p_3]$. If $j - i = 1$, let $v'_{i,j}$ be the triangle corresponding to the edge $[p_i, p_j]$.

For $1 \leq i < j \leq n$, let $v_{i,j}$ be an element of $H_2(\tilde{C})$ whose image in $H_2(\tilde{C}, \tilde{\nu})$ is

- $(1-q)^2(1+qt)(1-t)v'_{i,j}$ if $j = i+1$,
- $(1-q)^2(1+qt)v'_{i,j}$ if $i = 1$ and $j = 3$,
- $(1-q)^2 v'_{i,j}$ otherwise.

Such an element exists by Section 3. We will show that the $v_{i,j}$ form a free basis for $H_2(\tilde{C})$.

For every $1 \leq i < j \leq n$, let $x_{i,j} \in H_2(\tilde{C}, \partial\tilde{C})$ be the square corresponding to a pair of vertical edges, one passing just to the right of p_i, the other passing just to the left of p_j, and both having endpoints on ∂D.

LEMMA 5.3. *The following identities hold up to multiplication by a unit in Λ.*

- $\langle v'_{i,j}, x_{i,j} \rangle' = 1$ *for* $1 \leq i < j \leq n$,
- $\langle v'_{i,i+2}, x_{i-1,i+1} \rangle' = 1$ *for* $i = 2, \ldots, n-2$,
- $\langle v'_{i,i+2}, x_{i,i+1} \rangle' = 1 - t$ *for* $i = 2, \ldots, n-2$.

All other pairings $\langle v'_{i',j'}, x_{i,j} \rangle'$ are zero.

PROOF. Use the methods described in Section 4. □

For convenience, choose lifts and orientations so that

$$\langle v'_{i,j}, x_{i,j} \rangle' = 1$$

for $1 \leq i < j \leq n$.

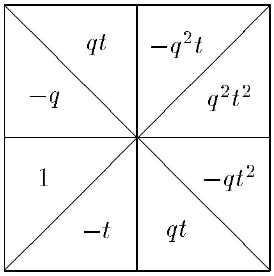


FIGURE 5. The function representing $x_{n-1,n}$ breaks into triangles.

From Lemma 5.3 it follows that the $v'_{i,j}$ are linearly independent. Hence the $v_{i,j}$ are linearly independent. By Lemma 5.2, they span $\mathbf{Q}(q,t) \otimes H_2(\tilde{C})$ as a $\mathbf{Q}(q,t)$-module. It remains to show that they span $H_2(\tilde{C})$ as a Λ-module. In other words, we must prove the following.

LEMMA 5.4. *Let $c_{i,j} \in \mathbf{Q}(q,t)$ for $1 \leq i < j \leq n$ be such that*

$$v = \sum_{1 \leq i < j \leq n} c_{i,j} v_{i,j}$$

lies in $H_2(\tilde{C})$. Then $c_{i,j} \in \Lambda$ for all $1 \leq i < j \leq n$.

We use the following facts about the $x_{i,j}$.

LEMMA 5.5. *The image of $x_{n-1,n}$ in $H_2(\tilde{C}, \partial\tilde{C} \cup \tilde{\nu})$ is $(1-q)(1+qt)(1-t)$ times a triangle.*

PROOF. Let $\alpha\colon I \to D$ be a straight edge from ∂D to p_n. Let α_2 be an edge in an ϵ-neighbourhood of α with endpoints on ∂D, passing anticlockwise around α. Let α_1 be an edge in an ϵ-neighbourhood of α with endpoints on ∂D, passing anticlockwise around α_2. Define $f\colon I \times I \to C$ by $f(x,y) = \{\alpha_1(x), \alpha_2(y)\}$. Then f lifts to a map $\tilde{f}$ which represents $x_{n-1,n}$.

We can assume that for all $s \in I$ the four points $\alpha_1(s)$, $\alpha_2(s)$, $\alpha_1(1-s)$, and $\alpha_2(1-s)$ lie within distance ϵ of each other. Then f maps the lines $\{(x,x)\}$, $\{(x,1-x)\}$, $\{(\frac{1}{2},y)\}$, and $\{(x,\frac{1}{2})\}$ into ν_ϵ. These lines cut $I \times I$ into eight pieces. The restriction of $\tilde{f}$ to each piece represents an element of $H_2(\tilde{C}, \partial\tilde{C} \cup \tilde{\nu_\epsilon})$. Each of these elements is a multiple of the triangle corresponding to α, as shown in Figure 5. Combining these, we see that $x_{n-1,n}$ is $(1-q)(1+qt)(1-t)$ times the triangle corresponding to α. □

LEMMA 5.6. *For all $1 \leq i < j \leq n$, The image of $x_{i,j}$ in $H_2(\tilde{C}, \partial\tilde{C} \cup \tilde{\nu})$ is $(1-q)^2$ times a linear combination of squares.*

PROOF. Use a similar argument to Lemma 5.5, as suggested by Figure 6. □

PROOF OF LEMMA 5.4. First consider the case $n = 3$. Then

$$\langle v, x_{2,3} \rangle = (1-q)^2(1+qt)(1-t)c_{2,3}.$$

By Lemma 5.6, $x_{2,3}$ is a multiple of $(1-q)^2$, and hence of $(1-\bar{q})^2$. Since $\langle \cdot, \cdot \rangle$ is sesquilinear, it follows that $(1+qt)(1-t)c_{2,3} \in \Lambda$. Similarly, Lemma 5.5 implies that $(1-q)c_{2,3} \in \Lambda$. Since Λ is a unique factorisation domain, it follows that $c_{2,3} \in \Lambda$. By a symmetrical argument, $c_{1,2} \in \Lambda$.

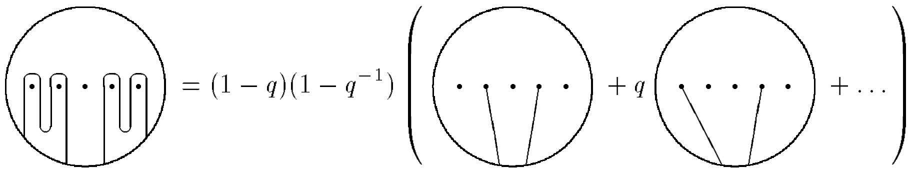


FIGURE 6. $x_{2,4}$ is $(1-q)^2$ times a linear combination of squares.

Now

$$v - (c_{1,2}v_{1,2} + c_{2,3}v_{2,3})$$

still lies in $H_2(\tilde{C})$, so we can reduce to the case $v = c_{1,3}v_{1,3}$. The following holds up to multiplication by a unit in Λ.

$$\langle v'_{1,3}, \sigma_2 x_{2,3}\rangle' = (1-t).$$

Thus

$$\langle v, \sigma_2 x_{2,3}\rangle = (1-t)(1-q)^2(1+qt)c_{1,3}.$$

By Lemma 5.5, $\sigma_2 x_{2,3}$ is a multiple of $(1-t)(1-q)(1+qt)$. Thus $(1-q)c_{1,3} \in \Lambda$. But

$$\langle v, x_{1,3}\rangle = (1-q)^2(1+qt)c_{1,3}.$$

By Lemma 5.6, $(1+qt)c_{1,3} \in \Lambda$. Thus $c_{1,3} \in \Lambda$.

Now suppose $n > 3$. For $i = 1, \ldots, n-2$ we have

$$\langle v, x_{i,n}\rangle = (1-q)^2 c_{i,n}.$$

By Lemma 5.6, $c_{i,n} \in \Lambda$. Also

$$\langle v, x_{n-1,n}\rangle = (1-q)^2(1+qt)(1-t)c_{n-1,n}.$$

By Lemma 5.6, $(1+qt)(1-t)c_{n-1,n} \in \Lambda$. By Lemma 5.5, $(1-q)c_{n-1,n} \in \Lambda$. Thus $c_{n-1,n} \in \Lambda$. We can now subtract the terms $c_{i,n}v_{i,n}$, and so reduce to the case $c_{i,n} = 0$ for all $i = 1, \ldots, n-1$. Then v represents an element of homology of the preimage in $\tilde{C}$ of the space of unordered pairs of distinct points in an $(n-1)$-times punctured disk. The result now follows by induction on n. $\square$

This completes the proof of Theorem 5.1.

5.2. The Krammer representation. There is some confusion as to the exact definition of the "Lawrence-Krammer representation". In an attempt to clarify the situation, we now compare and contrast a slightly different representation which we will call the "Krammer representation".

Let the *Krammer representation* be the following action of B_n on a free Λ-module V of rank $\binom{n}{2}$ with basis $\{F_{i,j} : 1 \le i < j \le n\}$.

$$\sigma_i(F_{j,k}) = \begin{cases} F_{j,k} & i \notin \{j-1, j, k-1, k\}, \\ qF_{i,k} + (q^2-q)F_{i,j} + (1-q)F_{j,k} & i = j-1, \\ F_{j+1,k} & i = j \neq k-1, \\ qF_{j,i} + (1-q)F_{j,k} + (1-q)qtF_{i,k} & i = k-1 \neq j, \\ F_{j,k+1} & i = k, \\ -tq^2F_{j,k} & i = j = k-1. \end{cases}$$

This representation is given in [**Kra00**], but with different conventions as explained below. It is also in [**Big01**], but with a sign error. The name "Krammer representation" was chosen because Krammer seems to have initially found this independently of Lawrence and without any use of homology.

The vector spaces $\mathbf{Q}(q,t)\otimes V$ and $\mathbf{Q}(q,t)\otimes H_2(\tilde{C})$ are isomorphic representations of B_n, by [**Big01**, Theorem 4.1]. The isomorphism is given by

$$\phi\colon\ F_{i,j}\mapsto(\sigma_{i-1}\dots\sigma_2\sigma_1)(\sigma_{j-1}\dots\sigma_3\sigma_2)v_{1,2}.$$

Thus ϕ maps $F_{i,j}$ to the genus three surface corresponding to an edge from p_i to p_j in the lower half plane.

The basis elements in [**Kra00**] correspond to similar edges in the *upper* half plane. Thus Krammer's $v_{i,j}$ should be identified with my

$$(\sigma_{j-1}\dots\sigma_2\sigma_1)(\sigma_{k-1}\dots\sigma_3\sigma_2)F_{1,2}.$$

Also, Krammer's t is my $-t$.

For $n\geq 3$, the map $\phi\colon V\to H_2(\tilde{C})$ defined as above is not an isomorphism. To see this, note that the composition of ϕ with the natural map from $H_2(\tilde{C})$ to $H_2(\tilde{C},\tilde{\nu})$ sends V into $(1-t)H_2(\tilde{C},\tilde{\nu})$. However the image of $v_{1,3}$ in $H_2(\tilde{C},\tilde{\nu})$ does not lie in $(1-t)H_2(\tilde{C},\tilde{\nu})$. Thus $v_{1,3}$ is not in the image of ϕ.

In fact Paoluzzi and Paris [**PP01**] showed that the modules V and $H_2(\tilde{C})$ are not isomorphic representations of B_n for $n\geq 3$. The distinction becomes important when we specialise q and t to values which are not algebraically independent, as we will in the next section.

It is possible to compute the matrices for the Lawrence-Krammer representation with respect to the basis $\{v_{i,j}\}$. We will not do this since they are quite complicated. They must be conjugate to those of the Krammer representation when considered as matrices over $\mathbf{Q}(q,t)$, but not when considered as matrices over Λ.

The following conjecture would give a nice topological interpretation of the Krammer representation.

Conjecture 5.7. *The Krammer representation is isomorphic to the action of B_n on $H_2(\tilde{C},\tilde{\nu})$.*

The isomorphism should be the map $\psi\colon\ V\to H_2(\tilde{C},\tilde{\nu})$ given by

$$\psi(F_{i,j})=(\sigma_{i-1}\dots\sigma_2\sigma_1)(\sigma_{j-1}\dots\sigma_3\sigma_2)v'_{1,2}.$$

This respects the action of B_n by the proof of [**Big01**, Theorem 4.1]. It remains to show that the terms $\psi(F_{i,j})$ form a basis for $H_2(\tilde{C},\tilde{\nu})$. The proof should follow the same method as that of Theorem 5.1. The only difficulty is showing that $\mathbf{Q}(q,t)\otimes H_2(\tilde{C},\tilde{\nu})$ has dimension $\binom{n}{2}$.

6. A representation of the Temperley-Lieb algebra

The aim of this section is to prove the following.

Theorem 6.1. *Let R be a domain containing invertible elements q and t, and let*

$$\iota\colon\ R\otimes H_2(\tilde{C})\to R\otimes H_2(\tilde{C},\tilde{\nu})$$

be induced by the natural map from homology to relative homology. Suppose $1+qt=0$, q has a square root, $q^2\neq 1$, and $q^3\neq 1$. Then the image of ι is the representation of $\mathrm{TL}_n(R)$ corresponding to the partition $(n-2,2)$.

Here, $\mathrm{TL}_n(R)$ is the Temperley-Lieb algebra. We will define this, and the desired representation in Section 6.4.

In [**Law90**, Theorem 5.1], Lawrence constructed the representation of the Hecke algebra corresponding to the partition $(n-k,k)$ for $k=1,\ldots,\lfloor n/2\rfloor$. These are the representations that factor through the Temperley-Lieb algebra. This was generalised still further in [**Law96**] to give the Hecke algebra representation corresponding to any partition of n. Theorem 6.1 only covers the case $(n-2,2)$, but also has a number of advantages over the work of Lawrence. Firstly, it gives a more elementary description of the required quotient of $R\otimes H_2(\tilde{C})$. Secondly, it works over a fairly general ring, whereas Lawrence worked over $\mathbf{C}$ and used the matrices we gave in Section 5.2 for the "Krammer representation". Finally, our proof gives an explicit isomorphism between the two representations. It is to be hoped that these advantages generalise to arbitrary partitions of n.

I suspect that the requirements on R in Theorem 6.1 can be weakened somewhat. The main example to keep in mind is $R=\mathbf{Z}[q^{\pm\frac{1}{2}}]$ with $t=-q^{-1}$.

6.1. A basis. For $1\le i<j\le n$, let $v_{i,j}$, $v'_{i,j}$ and $x_{i,j}$ be as defined in Section 5. Then

$$\iota(v_{i,j})=\begin{cases}0 & \text{if}\, j=i+1\,\text{or}\, j=3,\\ (1-q)^2v'_{i,j} & \text{otherwise.}\end{cases}$$

The pairing $\langle\cdot,\cdot\rangle'$ can be extended to a map

$$\langle\cdot,\cdot\rangle'\colon\ (R\otimes H_2(\tilde{C},\tilde{\nu}))\times H_2(\tilde{C},\partial\tilde{C})\to R.$$

Lemma 5.3 still holds, so the $v'_{i,j}$ are linearly independent in $R\otimes H_2(\tilde{C},\tilde{\nu})$. Thus the image of ι is the free module with basis

$$\{(1-q)^2v'_{i,j} : j>\max(i+1,3)\}.$$

Let H be the free module with basis

$$\{v'_{i,j} : j>\max(i+1,3)\}.$$

To ease the notation, we will work with H instead of the image of ι, since the two are isomorphic.

Let K be the field of fractions of R. Then $K\otimes H$ is a vector space of dimension $n(n-3)/2$, and contains H as an embedded submodule. It is sometimes easier to prove that a relation holds in H by proving that it holds in $K\otimes H$.

6.2. The Hecke algebra. We now prove that H is a representation of the Hecke algebra.

DEFINITION 6.2. The Hecke algebra $\mathcal{H}_n(R)$, or simply $\mathcal{H}_n$, is the R-algebra given by generators $1,\sigma_1,\ldots,\sigma_{n-1}$ and relations

- $\sigma_i\sigma_j=\sigma_j\sigma_i$ if $|i-j|>1$,
- $\sigma_i\sigma_j\sigma_i=\sigma_j\sigma_i\sigma_j$ if $|i-j|=1$,
- $(\sigma_i-1)(\sigma_i+q)=0$.

Thus $\mathcal{H}_n$ is the group algebra RB_n of the braid group modulo the relations

$$(\sigma_i-1)(\sigma_i+q)=0.$$

LEMMA 6.3. *$(\sigma_i-1)(\sigma_i+q)$ acts as the zero map on H.*

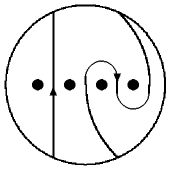
 $= x_{1,n-1} - q^{-1}x_{1,n}$

FIGURE 7. Showing that $\sigma_{n-1}^{-1}x_{i,n} = x_{i,n-1} - q^{-1}x_{i,n}$.

PROOF. Since the σ_i are all conjugate, we need only check the case $i = n-1$. To do this, we find a basis for $K \otimes H$ consisting of eigenvectors of σ_{n-1}, each having eigenvalue 1 or $-q$.

The elements $v'_{i,j}$ for $\max(i+1,3) < j \leq n-2$ are linearly independent eigenvectors of σ_{n-1} with eigenvalue 1. There are $(n-2)(n-5)/2$ of these.

Let α be a circular closed curve based at p_{n-2} and enclosing p_{n-1} and p_n. For $i = 1, \ldots, n-4$, let u_i be the square corresponding to the edges $[p_i, p_{i+1}]$ and α. Let u_{n-3} be the square corresponding to the edge $[p_{n-3}, p_{n-2}]$ and a circular closed curve based at p_{n-4} enclosing $p_{n-3}, \ldots, p_n$. These are eigenvalues of σ_{n-1} with eigenvalue 1. For $i, i' = 1, \ldots, n-3$, the following identities hold up to multiplication by a unit in Λ.

- $\langle u_i, x_{i,n}\rangle' = 1-q$,
- $\langle u_i, x_{i',n}\rangle' = 0$ for $i' \neq i$,
- $\langle v'_{i,j}, x_{i',n}\rangle' = 0$ for $1 \leq i < j \leq n-2$.

Thus the u_i are linearly independent and not in the span of $\{v'_{i,j} : j \leq n-2\}$. Now let u be the square corresponding to α and a circular closed curve based at p_{n-3} enclosing α. The following identities hold up to multiplication by a unit in Λ.

- $\langle u, x_{n-2,n-1}\rangle' = (1-q^2)(1+q^2t)(1-t)$,
- $\langle u_i, x_{n-2,n-1}\rangle' = 0$ for $i = 1, \ldots, n-3$,
- $\langle v'_{i,j}, x_{n-2,n-1}\rangle' = 0$ for $j \leq n-2$.

These can be computed using the methods of Section 4. The first is quite difficult, and we will discuss a less direct way to compute it later.

From the above identities, it follows that

$$\mathcal{B}_1 = \{v_{i,j} : \max(i+1,3) < j \leq n-2\} \cup \{u_1, \ldots, u_{n-3}\} \cup \{u\}$$

is a linearly independent set of eigenvectors of σ_{n-1} with eigenvalue 1. Each $v \in \mathcal{B}_1$ lies in H, as can be seen by constructing a surface in $\tilde{C}$ whose image in $H_2(\tilde{C}, \tilde{\nu})$ is $(1-q)^2 v$.

For $i = 1, \ldots, n-3$, note that $v'_{i,n}$ must be an eigenvector of σ_{n-1}, because the function σ_{n-1} can be chosen to fix setwise the square in C whose lift represents $v'_{i,n}$. To find the eigenvalue, note that

$$\sigma_{n-1}^{-1}x_{i,n} = x_{i,n-1} - q^{-1}x_{i,n},$$

as shown in Figure 7, so

$$\langle \sigma_{n-1}v'_{i,n}, x_{i,n}\rangle = \langle v'_{i,n}, \sigma_{n-1}^{-1}x_{i,n}\rangle = -q\langle v'_{i,n}, x_{i,n}\rangle.$$

Thus

$$\mathcal{B}_{-q} = \{v'_{i,n} : i = 1, \ldots, n-3\}.$$

is a set of eigenvectors of σ_{n-1} with eigenvalue $-q$. They are linearly independent, and since $-q \neq 1$, they are not in the span of $\mathcal{B}_1$. By a dimension count, $\mathcal{B}_1 \cup \mathcal{B}_{-q}$ is a basis for $K \otimes H$. Each element of this basis is annihilated by $(\sigma_{n-1}-1)(\sigma_{n-1}+q)$, so we are done. $\square$

In the above proof we used the fact that

$$\langle u, x_{n-2,n-1}\rangle' = (1-q^2)(1+q^2t)(1-t),$$

up to multiplication by a unit in Λ. We now describe an indirect argument to obtain this identity. Let λ denote $(1-q^2)(1+q^2t)(1-t)$.

Let P be an ϵ-neighbourhood of the horizontal edge $[p_{n-1}, p_n]$, where $\epsilon > 0$ is small. Let Z be the set of points $\{x, y\} \in C$ such that either x and y are within distance ϵ of each other, or at least one of them lies in P. Let $\tilde{Z}$ be the preimage of Z in $\tilde{C}$.

Let $x' \in H_2(\tilde{C}, \partial\tilde{C} \cup \tilde{Z})$ be a triangle represented by the horizontal edge $[p_n, 1]$. I claim that the image of $x_{n-2,n-1}$ in $H_2(\tilde{C}, \partial\tilde{C} \cup \tilde{Z})$ is $\lambda x'$. The proof is almost identical to that of Lemma 5.5. This time P is to be treated as one large puncture point. Since P actually contains two puncture points, it is necessary to substitute q^2 for q in the proof of Lemma 5.5.

The surface representing u does not meet $\tilde{Z}$, so it is possible to define an intersection pairing $\langle u, x'\rangle$. The surfaces representing u and x' intersect transversely at one point when projected to C. Thus $\langle u, x'\rangle$ is a unit in Λ. The pairing is sesquilinear, so $\langle u, x\rangle = \bar{\lambda}$. But λ is equal to $\bar{\lambda}$, up to multiplication by a unit in Λ, so we are done.

6.3. The Temperley-Lieb algebra. We now prove that H is a representation of the Temperley-Lieb algebra TL_n.

Definition 6.4. Let R be a domain containing an invertible element q with a square root $q^{\frac{1}{2}}$. The Temperley-Lieb algebra $\mathrm{TL}_n(R)$, or simply TL_n, is the R-algebra given by generators $1, e_1, \ldots, e_{n-1}$ and relations

- $e_i e_j = e_j e_i$ if $|i-j| > 1$,
- $e_i e_j e_i = e_i$ if $|i-j| = 1$,
- $e_i^2 = (-q^{\frac{1}{2}} - q^{-\frac{1}{2}})e_i$.

Consider the map from $\mathcal{H}_n$ to TL_n given by $\sigma_i \mapsto 1 + q^{\frac{1}{2}}e_i$. I claim this is well-defined and has kernel generated by

$$z_{i,j} = \sigma_i\sigma_j\sigma_i - \sigma_i\sigma_j - \sigma_j\sigma_i + \sigma_i + \sigma_j - 1$$

for $|i-j| = 1$. This is [**Jon87**, Equation 11.6], except using slightly different conventions. It can be verified by adding the relations $z_{i,j} = 0$ to the presentation of $\mathcal{H}_n$, substituting $1 + q^{\frac{1}{2}}e_i$ for σ_i, and simplifying to obtain the presentation for TL_n.

Lemma 6.5. *$z_{i,j}$ acts as the zero map on H.*

Proof. The $z_{i,j}$ are all conjugate to each other, so it suffices to prove that $z_{n-2,n-1}$ acts as the zero map on H. We have

$$\begin{aligned} z_{n-2,n-1} &= (\sigma_{n-2}\sigma_{n-1} - \sigma_{n-1} + 1)(\sigma_{n-2} - 1) \\ &= (\sigma_{n-1}\sigma_{n-2} - \sigma_{n-2} + 1)(\sigma_{n-1} - 1). \end{aligned}$$

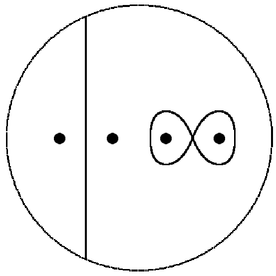


FIGURE 8. The arcs ξ_1 and γ used to define y_1.

Thus it suffices to find a basis for $K \otimes H$, each of whose elements is an eigenvector of either σ_{n-1} or σ_{n-2} with eigenvalue 1.

Let $\mathcal{B}_1$ be the linearly independent set of eigenvectors of σ_{n-1} with eigenvalue 1 as defined in the proof of Lemma 6.3.

Let α' be a circular closed curve based at p_n and enclosing p_{n-1} and p_{n-2}. For $i = 1, \ldots, n-4$, let w_i be the square corresponding to the edges $[p_i, p_{i+1}]$ and α'. Let w_{n-3} be the square corresponding to α' and a circular closed curve based at p_{n-3} and enclosing α'. The w_i all lie in H, and are eigenvectors for σ_{n-2} with eigenvalue 1.

Let $\gamma\colon S^1 \to D_n$ be a figure-eight going around p_{n-1} and p_n in a regular neighbourhood of $[p_{n-1}, p_n]$. For $i = 1, \ldots, n-3$, let $\xi_i\colon I \to D_n$ be a vertical edge with endpoints on ∂D, passing just to the right of p_i. See Figure 8. Define

$$f_i\colon I \times S^1 \to C$$

by $f_i(x, y) = \{\xi_i(x), \gamma(y)\}$. This lifts to $\tilde{C}$. Let y_i be the corresponding element of $H_2(\tilde{C}, \partial\tilde{C})$. For $i, j = 1, \ldots, n-3$, the following holds up to multiplication by a unit in Λ.

$$\langle w_i, y_j\rangle' = \begin{cases} 1 - q^3 & i = j \neq n-3 \\ (1-q^3)(1-q^3t^2) & i = j = n-3 \\ 0 & i \neq j \end{cases}$$

Further, $\langle v, y_i\rangle' = 0$ for all $v \in \mathcal{B}_1$. Thus

$$\{w_1, \ldots, w_{n-3}\} \cup \mathcal{B}_1$$

is linearly independent. By a dimension count, it forms a basis of $K \otimes H$. Each vector in this basis is annihilated by $z_{n-2,n-1}$, so we are done. □

6.4. The $(n-2, 2)$ **representation.** We have shown that H can be considered as a representation of TL_n. We now define the representation S of TL_n corresponding to the partition $(n-2, 2)$, and show that it is isomorphic to H. A good introduction to the representation theory of the Temperley-Lieb algebra is [**Wes95**].

DEFINITION 6.6. Let M be the left-ideal of TL_n generated by e_1e_3, and let N be the left-ideal generated by $\{e_5, \ldots, e_{n-1}\}$. The representation of TL_n corresponding to the partition $(n-2, 2)$ is the TL_n-module $S = M/(M \cap N)$.

This is called $S(n, 2)$ in [**Wes95**]. It is a free R-module with basis

$$s_{i,j} = (e_i \ldots e_3e_2)(e_{j-1} \ldots e_5e_4)(e_1e_3)$$

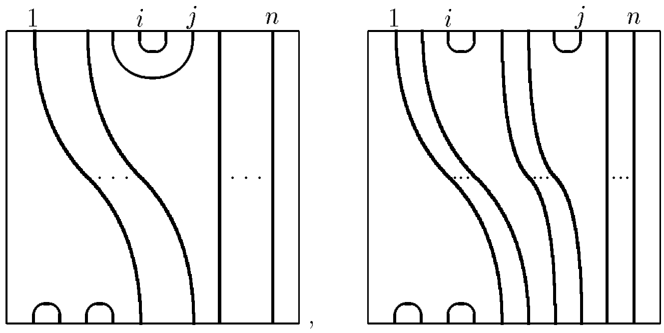


FIGURE 9. $s_{i,j}$ for $j = i+2$ and $j > i+2$.

for $1 \le i < j \le n$ such that $j > \max(i+1, 3)$. There is a diagrammatic interpretation of TL_n, in which $s_{i,j}$ corresponds to the diagram shown in Figure 9.

Let $\psi\colon \mathrm{TL}_n \to H$ be the unique map compatible with the action such that $\psi(1) = v'_{1,4}$. Then

$$\psi(e_3) = q^{-\frac{1}{2}}(\sigma_3 - 1)(v'_{1,4}) = (-q^{\frac{1}{2}} - q^{-\frac{1}{2}})v'_{1,4}.$$

Similarly $\psi(e_1) = (-q^{\frac{1}{2}} - q^{-\frac{1}{2}})v'_{1,4}$. Thus

$$\psi(e_1 e_3) = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^2 v'_{1,4}.$$

Let $\psi'\colon M \to H$ be given by

$$\psi' = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{-2}\psi|_M.$$

For $i = 5, \dots, n-1$ we have that

$$\psi(e_i) = q^{-\frac{1}{2}}(\sigma_i - 1)(v'_{1,4}) = 0.$$

Thus $\psi(N) = 0$, so $\psi'(M \cap N) = 0$. We therefore have a map $\phi\colon S \to H$ induced by ψ'.

Recall that S has a basis

$$s_{i,j} = (e_i \dots e_2)(e_{j-1} \dots e_4)(e_1 e_3)$$

for $1 \le i < j \le n$ such that $j > \max(i+1, 3)$. We will show that $\phi(s_{i,j}) = v'_{i,j}$ up to multiplication by a unit in Λ.

LEMMA 6.7. *For $j = 5, \dots, n$ we have $(\sigma_{j-1} - 1)v'_{1,j-1} = v'_{1,j}$ up to multiplication by a unit in Λ.*

PROOF. Let $\alpha_1\colon I \to D$ be the edge $[p_1, p_2]$. Let $\alpha_2\colon I \to D$ be the edge $[p_{j-2}, p_{j-1}]$. Define $f\colon I \times I \to C$ by $f(x,y) = \{\alpha_1(x), \alpha_2(y)\}$. Then f lifts to a map $\tilde{f}$ which represents $v'_{1,j-1}$.

We can assume that $\sigma_{j-1}\alpha_2(\frac{1}{2})$ is within distance ϵ of p_{j-1}. Then f maps the line $\{(x, \frac{1}{2})\}$ into ν_ϵ. The restriction of $\tilde{f}$ to $I \times [0, \frac{1}{2}]$ represents some unit multiple of $v_{1,j-1}$. The restriction of $\tilde{f}$ to $I \times [\frac{1}{2}, 1]$ represents some unit multiple of $v_{1,j}$. Thus

$$\sigma_{j-1} v'_{1,j-1} = \lambda v'_{1,j-1} + \mu v'_{1,j}$$

for some units $\lambda, \mu \in \Lambda$.

It remains to show that $\lambda = 1$. Recall that $\langle v'_{1,j-1}, x_{1,j-1}\rangle' = 1$. Thus $\langle \sigma_{j-1} v'_{1,j-1}, x_{1,j-1}\rangle' = \lambda$. But $\sigma_{j-1} x_{1,j-1} = x_{1,j-1}$, so

$$\begin{aligned}\langle \sigma_{j-1} v'_{1,j-1}, x_{1,j-1}\rangle' &= \langle \sigma_{j-1} v'_{1,j-1}, \sigma_{j-1} x_{1,j-1}\rangle' \\ &= \langle v'_{1,j-1}, x_{1,j-1}\rangle' \\ &= 1.\end{aligned}$$

Thus $\lambda = 1$, so $(\sigma_{j-1} - 1)v'_{1,j-1} = \mu v'_{1,j}$. □

LEMMA 6.8. *For $i = 2, \ldots, j-1$ we have $(\sigma_i - 1)v'_{i-1,j} = v'_{i,j}$ up to multiplication by a unit in Λ.*

PROOF. In the case $i < j - 1$, the proof is almost identical to that of the previous lemma. Suppose $i = j - 1$. Then

$$\sigma_i v'_{i-1,j} = \lambda v'_{i-1,j} + \mu v'_{i,j}$$

for some $\lambda, \mu \in \Lambda$. To see that $\lambda = 1$, use the same argument as in the proof of the previous lemma. To see that μ is a unit in Λ, note that it is equal to $\langle \sigma_i v'_{i-1,j}, x_{i,j}\rangle'$, and the surfaces representing $\sigma_i v'_{i-1,j}$ and $x_{i,j}$ intersect transversely at one point when projected to C. □

By these two lemmas we have $\phi(s_{i,j}) = v'_{i,j}$ up to multiplication by a unit in Λ. This implies that ϕ is an isomorphism, which completes the proof of Theorem 6.1.

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