Skein theory for the *ADE* planar algebras

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ABSTRACT

We give generators and relations for the planar algebras corresponding to the E_8 subfactor. We also give an explicit basis, and an algorithm to express an arbitrary diagram as a linear combination of basis diagrams. We summarize the analogous results for the other *ADE* subfactors, which can be obtained by the same methods.

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1. Introduction

The notion of a planar algebra is due to Jones [1]. The roughly equivalent notion of a spider is due to Kuperberg [2]. Planar algebras arise in many contexts where there is a reasonably nice category with tensor products and duals. Examples are the category of representations of a quantum group, and the category of bimodules coming from a subfactor.

The subfactor planar algebras of index less than 4 can be classified into the two infinite families A_N and D_{2N}, and the two sporadic examples E_6 and E_8. See [3–6] for the story of this "ADE" classification.

The *Kuperberg program* can be summarized as follows.

> Give a presentation for every interesting planar algebra, and prove as much as possible about the planar algebra using only its presentation.

The planar algebras corresponding to subfactors of type A_N are fairly well understood. In [7], Morrison, Peters and Snyder basically complete the Kuperberg program in the D_{2N} case.

The aim of this paper is to extend the results of [7] to types E_6 and E_8. However, our approach is different. Whereas [7] use only combinatorial arguments starting from their presentation, we will also use known properties of the desired subfactor planar algebra. Thus this paper is not completely in the spirit of the Kuperberg program. As a compensation, we will address the following program, suggested by Jones in Appendix B of [8].

> Give a basis for every interesting planar algebra, and an algorithm to express any given diagram as a linear combination of basis elements.

Most of this paper concerns the subfactor of type E_8. In Section 3, we introduce the planar algebras $\mathcal{P}$ and $\mathcal{P}'$. Here, $\mathcal{P}$ is defined by a presentation with one generator and five relations, and $\mathcal{P}'$ is the subfactor planar algebra whose principal graph is the E_8 graph. In Section 4, we prove that $\mathcal{P}$ is isomorphic to $\mathcal{P}'$. In Section 5, we define a set of diagrams that will form a basis for our planar algebra. The proof that the basis spans is constructive, although we have not tried to give an efficient algorithm. Finally, in Section 6, we explain how our methods can be applied to types E_6, A_N and D_{2N}.

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2. Planar algebras

We give a brief and impressionistic review of the definition of a planar algebra. For the details, see the preprint [1] at Vaughan Jones' website.

Something that is called an "algebra" is usually a vector space together with one or more additional operations. A planar algebra $\mathcal{P}$ consists of infinitely many vector spaces (or one graded vector space if you prefer), together with infinitely many operations. For every non-negative integer k, we have a vector space $\mathcal{P}_k$. For every planar arc diagram T, we have a multilinear n-ary operation

$$\mathcal{P}(T)\colon \mathcal{P}_{k_1} \otimes \cdots \otimes \mathcal{P}_{k_n} \to \mathcal{P}_{k_0},$$

where n is the number of internal disks in T, $k_1, \ldots, k_n$ are the numbers of ends of strands on the internal disks of T, and k_0 is the number of ends of strands on the external disk.

In practice, $\mathcal{P}_k$ will be spanned by diagrams of some sort in a disk. A diagram may include embedded edges or loops called *strands*. Every diagram in $\mathcal{P}_k$ has k endpoints of strands on its boundary. A planar arc diagram T is a diagram consisting of strands in a disk with holes. The action of T is given by gluing diagrams into the holes of T, matching up endpoints of strands on the boundary of the diagrams with the endpoints of strands in T. To determine "which way up" to glue the diagrams, we use basepoints on the boundaries of diagrams and on the boundaries of input disks of T. These basepoints are indicated by a star, and are never allowed to coincide with the endpoints of strands.

Note that, for ease of exposition, we only work with "unshaded" planar algebras.

2.1. Composition

It is often convenient to draw an element of a planar algebra in a rectangle instead of a round disk. When we do this, the basepoint will always be at the top left corner, and the endpoints of strands will be on the top and bottom edges.

Let $\mathcal{P}_b^a$ denote the elements of $\mathcal{P}_{a+b}$, drawn in a rectangle, with a endpoints at the top and b at the bottom. If $A \in \mathcal{P}_b^a$ and $B \in \mathcal{P}_c^b$, then the *composition* of A and B is the element $AB \in \mathcal{P}_c^a$ obtained by stacking A on top of B. Note that the meaning of this composition depends on the value of b, which must be made clear from context. (Here, we are blurring the distinction between the planar algebra and the corresponding category, as defined in [7].)

2.2. Quantum integers

Suppose q is a non-zero complex number. The *quantum integer* $[n]$ is given by

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

These appear only briefly in this paper, and they can be treated as constants whose precise value is unimportant. However, they play an important role behind the scenes, for example in the definition of the Jones–Wenzl idempotents.

2.3. Temperley–Lieb planar algebra

A *Temperley–Lieb diagram* is a finite collection of disjoint properly embedded edges in a disk, together with a basepoint on the boundary. These form a planar algebra as follows. Suppose T is a planar arc diagram with n holes, and $D_1, \ldots, D_n$ are Temperley–Lieb diagrams with the appropriate numbers of endpoints. We can create a new Temperley–Lieb diagram D by inserting $D_1, \ldots, D_n$ into the holes in T, and deleting any resulting strands that form closed loops. Let m be the number of closed loops that were deleted. Then T maps the n-tuple $(D_1, \ldots, D_n)$ to $[2]^m D$.

The planar algebra of Temperley–Lieb diagrams is called the *Temperley–Lieb planar algebra*, and will be denoted $\mathcal{TL}$. It can be defined more briefly as the planar algebra with no generators and a single defining relation

 $= [2]$ $.$

Every planar algebra we consider will satisfy the above relation, and hence contain an image of $\mathcal{TL}$.

We now list some important examples of Temperley–Lieb diagrams. The *identity diagram* $\mathrm{id}_n \in \mathcal{TL}_n^n$ is the diagram consisting of n vertical strands in a rectangle.

Suppose $D \in \mathcal{TL}_n^m$ is a Temperley–Lieb diagram drawn in a rectangle. We say D *contains* a cup if it contains a strand that has both endpoints on the top edge of the rectangle. We say D *is* a cup if it is consists of n vertical strands and one strand that has endpoints on the top of the rectangle.

Similarly, a *cap* is a diagram in $\mathcal{TL}_{n+2}^n$ that has n vertical strands and one strand with both endpoints on the bottom of the rectangle.

The *Jones–Wenzl idempotent* p_n is the unique element of $\mathcal{TL}_n^n$ with the following properties.
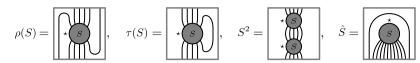
$$\rho(S) = \quad , \quad \tau(S) = \quad , \quad S^2 = \quad , \quad \hat{S} = $$

Fig. 1. The diagrams $\rho(S)$, $\tau(S)$, S^2 and $\hat{S}$, respectively.

- When p_n is expressed as a linear combination of Temperley–Lieb diagrams, the diagram id_n occurs with coefficient 1.
- If $X \in \mathcal{TL}_n^{n-2}$ is any cap then $X p_n = 0$.
- If $Y \in \mathcal{TL}_{n-2}^n$ is any cup then $p_n Y = 0$.

In all of our examples, q will be of the form $\mathrm{e}^{\mathrm{i}\pi/N}$. In this case the element p_n exists and is unique for all $n \leq N - 1$, but does not exist for $n \geq N$.

Let a *crossing* be the following element of $\mathcal{P}_4$.

$$= \mathrm{i}q^{\frac{1}{2}} \qquad - \mathrm{i}q^{-\frac{1}{2}} \qquad .$$

This allows us to consider knot and tangle diagrams as representing elements of the Temperley–Lieb planar algebra. We can express a diagram with k crossings as a linear combination of 2^k diagrams that have no crossings. This process is called *resolving the crossings*.

The crossing satisfies Reidemeister moves two and three. In place of Reidemeister one, we have the following.

$$= \mathrm{i}q^{\frac{3}{2}} \qquad .$$

3. The E_8 planar algebra

The purpose of this section is to define the planar algebras $\mathcal{P}$ and $\mathcal{P}'$. In the next section, we will show that they are isomorphic.

3.1. The presentation

We define $\mathcal{P}$ in terms of generators and relations. There is one generator $S \in \mathcal{P}_{10}$.
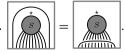
Before we list the relations, we define some notation. Let $q = \mathrm{e}^{\mathrm{i}\pi/30}$. Let $\omega = \mathrm{e}^{6\mathrm{i}\pi/5}$. Let $\rho(S)$, $\tau(S)$, S^2 and $\hat{S}$ be as shown in Fig. 1. We call $\rho(S)$ the *rotation* of S and $\tau(S)$ the *partial trace* of S.

The defining relations of $\mathcal{P}$ are as follows.

- $\bigcirc = [2] \bigcirc$,
- $\rho(S) = \omega S$,
- $\tau(S) = 0$,
- $S^2 = S + [2]^2 [3] p_5$,
- $\hat{S} p_{12} = 0$.

We call the first four relations the *bubble bursting*, *chirality*, *partial trace*, and *quadratic* relation, respectively. The fifth relation is equivalent to the following *braiding relation*.

Lemma 3.1. $\quad = \quad .$

Proof. Let X denote the diagram on the right-hand side of the above equation. We must show that $\hat{S} = X$.

We show that $\hat{S}Y = XY$ for any cup Y. First suppose Y is the cup

$$Y = \qquad .$$

Then

$$\hat{S}Y = \rho(S) = \omega S.$$

To compute XY, first apply Reidemeister one to the rightmost crossing, giving a factor of $iq^{3/2}$. Now resolve the remaining nine crossings. Each of these contributes a factor of $iq^{1/2}$. The easiest way to see this is to resolve the crossings one by one, working from right to left. In the end, we obtain

$$XY = (iq^{3/2})(iq^{1/2})^9 S = -q^6 S.$$

By comparing the above expressions, we find that $\hat{S}Y = XY$.

Similarly, if Y is the cup

$$Y = \boxed{\text{∪|||||||}},$$

then $\hat{S}Y = XY$. If Y is any other cup, then $\hat{S}Y$ and XY are both zero.

Recall that p_{12} is a linear combination of Temperley–Lieb diagrams. Every term in this linear combination contains a cup, except for the identity, which occurs with coefficient one. Thus

$$\hat{S} = X \iff \hat{S}p_{12} = Xp_{12}.$$

But Xp_{12} is clearly zero. Thus

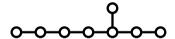$$\hat{S} = X \iff \hat{S}p_{12} = 0,$$

as required. □

The braiding relation says that it is possible to pass a strand over the generator S. However, it is not possible to pass a strand under S, so $\mathcal{P}$ is not a *braided planar algebra*. It is not even possible to pass a strand under S up to a change of sign, so $\mathcal{P}$ is not *partially braided* in the sense of [7, Theorem 3.2]. However, the above braiding relation is enough for many purposes.

3.2. The subfactor planar algebra

We now define the subfactor planar algebra $\mathcal{P}'$ and give some of its properties. This will require some basic knowledge of subfactor planar algebras and principal graphs. Most of this can be found in [7], or it can be taken on faith.

The E_8 graph is as follows.

Let $\mathcal{P}'$ be a subfactor planar algebra with principal graph E_8. It is proved in [9] that there are exactly two such planar algebras. However, we are working with unshaded planar algebras, so there are actually four possibilities for $\mathcal{P}'$, by the discussion in Appendix A of [8].

Lemma 3.2. $\mathcal{P}'$ *is generated by an element* $R \in \mathcal{P}'_{10}$*. We can choose* $\mathcal{P}'$ *and* R *so that* R *satisfies the same defining relations as* S *in* $\mathcal{P}$*.*

Proof. Let R be a generator for the space of morphisms from p_{10} to the empty diagram in $\mathcal{P}'$.

The chirality relation is proved in [1, Theorem 4.2.13]. The "chirality" in that theorem is our value of ω^2. The two shaded planar algebras with principal graph E_8 give two complex conjugate values for ω^2. For each of these, the two corresponding unshaded planar algebras correspond to a choice of sign for ω. We have chosen our value of ω so that the braiding relation holds with our definition of a crossing.

The quadratic relation is Eq. (4.3.5) in [10]. This equation currently contains a misprint: it should read $R^2 = (1-r)R + rp_n$. However, the proof is correct. In our context, $r = [2]^2/[3]$, and we have rescaled R by a factor of $-[3]$.

The remaining three relations are easy. The constant $[2]$ in the bubble bursting relation is the positive eigenvalue of the E_8 graph. The partial trace relation is immediate from our choice of R. The relation $\hat{R}p_{12} = 0$ follows from the fact that there is no non-zero morphism from p_{12} to the empty diagram. To see this, use the information encoded in the principal graph to decompose p_{12} into minimal idempotents, and observe that the empty diagram does not occur as a factor. □

Lemma 3.3. *In* $\mathcal{P}'$*,* id_7 *is equal to a linear combination of diagrams of the form* AB*, where* $A \in (\mathcal{P}')^7_m$ *and* $B \in (\mathcal{P}')^m_7$ *for some* $m < 7$*.*

Proof. This comes down to the fact that the E_8 graph has diameter less than seven. Every minimal idempotent is a summand of id_m for some $m < 7$. □

Lemma 3.4. $p_{29} = 0$ *in* $\mathcal{P}'$*.*

Proof. This holds in any subfactor planar algebra where $[30] = 0$. □

Definition. Suppose Y is a Temperley–Lieb diagram. Suppose x is a point on the boundary of Y that is not the endpoint of any strand. Draw an embedded arc in Y from the basepoint to x. Now count the number of strands in Y that have one endpoint on either side of this arc. We say Y is *JW-reduced* if, for every such point x, the number of such strands is strictly less than 29.

Lemma 3.5. *The space of JW-reduced Temperley–Lieb diagrams is a basis for the space of Temperley–Lieb diagrams in $\mathcal{P}'$.*

Proof. Suppose Y is a Temperley–Lieb diagram. For convenience, draw Y as an element of $\mathcal{TL}_n^0$. Let $x_0, \ldots, x_n$ be points in the $n+1$ spaces between endpoints on the bottom edge of Y. For $i = 0, \ldots, n$, let a_i be the number of strands that have endpoints on either side of x_i. Call $(a_0, \ldots, a_n)$ the *sequence corresponding to Y*. This sequence starts and ends at 0, and satisfies $a_{i+1} = a_i \pm 1$.

Now Y is JW-reduced if and only if $a_i < 29$ for all i. Suppose Y is not JW-reduced. Then $a_i = 29$ for some i. Let L be a vertical line at x_i. We can assume L intersects the strands of Y in exactly 29 points. Let Y' be the result of inserting a sideways copy of p_{29} into L. This is zero by Lemma 3.4. Thus Y is a linear combination of diagrams that are obtained by inserting non-identity Temperley–Lieb diagrams into L. Each such diagram has a corresponding sequence that is smaller than $(a_0, \ldots, a_n)$ in lexicographic order. This process must terminate with a linear combination of JW-reduced Temperley–Lieb diagrams.

We prove linear independence by a dimension count. The space of Temperley–Lieb diagrams in $\mathcal{P}'$ is the subfactor planar algebra with principal graph A_{29}. Number the vertices of the A_{29} graph in order $0, 1, \ldots, 28$. The dimension of the space of Temperley–Lieb diagrams in $\mathcal{P}'_n$ is the number of paths in the A_{29} graph of length n that start and end at vertex number 0. Such a path is the sequence corresponding to exactly one JW-reduced Temperley–Lieb diagram. This completes the proof. □

Definition. For $m, n \geq 0$, let $\mathcal{B}_n^m$ be the set of JW-reduced diagrams in $\mathcal{TL}_n^m$ that do not contain a cup.

Lemma 3.6. $\dim(\mathcal{P}'_n) = \sharp\mathcal{B}_n^0 + \sharp\mathcal{B}_n^{10} + \sharp\mathcal{B}_n^{18} + \sharp\mathcal{B}_n^{28}$.

Proof. The idea is to decompose id_n into a direct sum of minimal idempotents in the category corresponding to $\mathcal{P}'$. The dimension of $\mathcal{P}'_n$ is the number of copies of the empty diagram in this decomposition.

First, work in the image of the Temperley–Lieb planar algebra in $\mathcal{P}'$. Here, the minimal idempotents are the Jones–Wenzl idempotents. The space of morphisms from id_n to p_m has as a basis the set of $p_m Y$ such that $Y \in \mathcal{B}_n^m$. Thus the number of copies of p_m in the decomposition of id_n is the number of elements of $\mathcal{B}_n^m$.

Now we work in $\mathcal{P}'$, and further decompose p_m into minimal idempotents. This is easy to do using the information encoded in the principal graph. The number of copies of the empty diagram in the decomposition of p_m is one if $m \in \{0, 10, 18, 28\}$ and zero otherwise. This completes the proof. □

4. The isomorphism

The aim of this section is to prove the following.

Theorem 4.1. $\mathcal{P}'$ *is isomorphic to $\mathcal{P}$.*

By Lemma 3.2, there is a surjective planar algebra morphism Φ from $\mathcal{P}$ to $\mathcal{P}'$, taking S to R. It remains to show that Φ is injective.

Lemma 4.2. *Every element of $\mathcal{P}_0$ is a scalar multiple of the empty diagram.*

Proof. Suppose D is a diagram in $\mathcal{P}_0$. Let m be the number of copies of S in D. We will use induction on m.
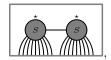
By the braiding relation, we can assume the copies of S lie on the vertices of a regular m-gon, and that every strand lies inside this m-gon. By resolving all crossings, we can assume D contains no crossings. By the bubble bursting relation, we can assume D contains no closed loops. By the chirality and partial trace relations, we can assume D contains no strand with both endpoints on the same copy of S.

We have now reduced to the case in which the copies of S lie on the vertices of a regular m-gon, and every strand lies inside this m-gon and connects two distinct copies of S. We can think of D as a triangulated m-gon, where the edges have multiplicities. (We may need to add some edges with multiplicity zero in order to literally triangulate the m-gon.) Any triangulated polygon has a vertex with valency two, not counting multiplicities. However, every vertex has valency 10 if we count multiplicities. Thus there is an edge with multiplicity at least 5. This gives us a copy of S^2 inside D, up to rotation of the copies of S. The result now follows from the quadratic relation and induction on m. □

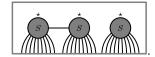**Lemma 4.3.** *Every element of $\mathcal{P}_{10}$ is a linear combination of Temperley–Lieb diagrams and S.*

Fig. 2. Possible values for *X*.

Proof. Suppose D is a diagram in $\mathcal{P}_{10}^0$. Let m be the number of copies of S in D. We will use induction on m.

By the braiding relation, we can assume the copies of S lie in a row at the top of D, and all strands of D lie entirely below the height of the tops of the copies of S. As in the proof of the previous lemma, we can assume that every strand connects two distinct copies of S, or has at least one endpoint on the bottom edge of D.

Suppose there is a strand that connects a non-adjacent pair of copies of S. Between these copies of S there must exist a copy of S that is connected only to its two adjacent copies. It must be connected to at least one of these by at least 5 strands. The result now follows from the quadratic relation and induction on m.

Now suppose every strand either connects adjacent copies of S or has at least one endpoint on the bottom edge of D. If $m = 1$ then $D = S$, and we are done. Suppose $m > 1$. Either the leftmost or rightmost copy of S is connected to the bottom of D by at most 5 strands. This copy of S is connected to its only adjacent copy of S by at least 5 strands. The result now follows from the quadratic relation and induction on m. □

Definition. Suppose $X \in \mathcal{P}_n^0$ for some $n < 29$. Then X is a *morphism* from p_n to the empty diagram if $Xp_n = X$, or equivalently, if $XY = 0$ for every cup $Y \in \mathcal{P}_{n-2}^n$.

Lemma 4.4. *If* $n < 29$ *and* $n \notin \{0, 10, 18, 28\}$ *then every morphism from* p_n *to the empty diagram is zero.*

Proof. First note that $\mathcal{P}_n$ is zero for odd values of n. Thus we can assume $n = 2k$. We have the following identities, shown in the case $k = 1$.

The first equality is an isotopy. The second follows from the braiding relation. To prove the third, resolve each crossing and eliminate all terms that contain a cup attached to X. There are $2k$ instances of a strand crossing itself. Each of these contributes a factor of $iq^{3/2}$, by Reidemeister one. There are $2k(2k - 1)$ instances of two distinct strands crossing. Each of these contributes a factor of $iq^{1/2}$.

Thus we have $X = q^{2k(k+1)}X$. If X is non-zero then $q^{2k(k+1)} = 1$, so $2k(k + 1)$ is a multiple of 60. The result now follows from simple case checking. □

Lemma 4.5. Φ *is injective on* $\mathcal{P}_n$ *for* $n \leq 16$.

Proof. The case $n = 0$ follows from Lemma 4.2, and the case $n = 10$ follows from Lemma 4.3. Suppose $X \in \mathcal{P}_n^0$ is in the kernel of Φ, where $n \leq 16$ and $n \notin \{0, 10\}$. If Y is a cup then XY is in the kernel of Φ, so $XY = 0$ by induction on n. Thus X is a morphism from p_n to the empty diagram. The result now follows from Lemma 4.4. □

We are now ready to prove Theorem 4.1. Suppose $X \in \mathcal{P}_n$ is in the kernel of Φ. We must show $X = 0$. We can assume $n > 16$.

Write X as an element of $\mathcal{P}_7^{n-7}$. The relation in Lemma 3.3 also holds in $\mathcal{P}$, since Φ is an isomorphism on $\mathcal{P}_{14}$. Thus X is a linear combination of diagrams of the form XAB, where $A \in \mathcal{P}_m^7$ and $B \in \mathcal{P}_7^m$ for some $m > 7$. For any such A, XA is in the kernel of Φ, so $XA = 0$ by induction on n. Thus $X = 0$. This completes the proof that Φ is an isomorphism.

5. A basis

We now define a set of diagrams that will form a basis for $\mathcal{P}_n$. Recall the definition of $\mathcal{B}_n^m$ from Section 3.2.

Definition. Let $\mathcal{B}_n$ be the set of diagrams of the form XY, where X is one of the four diagrams shown in Fig. 2, and $Y \in \mathcal{B}_n^m$ for some m. Here, m is the appropriate element of $\{0, 10, 18, 28\}$ in order for XY to be well defined.

The aim of this section is to prove the following.

Theorem 5.1. $\mathcal{B}_n$ *is a basis for* $\mathcal{P}_n$.

First we establish some more consequences of the defining relations of $\mathcal{P}$.

Lemma 5.2. *The diagram*

is a linear combination of diagrams that have at most one copy of S.

Proof. Let $\mathrm{Join}_2(S, S)$ denote the above diagram. Consider $\mathrm{Join}_2(S, S)p_{16}$. On the one hand, this is zero by Lemma 4.4. On the other hand, we can write p_{16} as a linear combination of Temperley–Lieb diagrams. The identity diagram occurs with coefficient one. Every other term contains a cup. This cup either connects the two copies of S or gives zero. Thus $\mathrm{Join}_2(S, S)$ is equal to a linear combination of diagrams that contain two copies of S connected by three parallel strands.
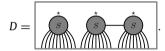
An analogous argument applies to $\mathrm{Join}_3(S, S)$ and $\mathrm{Join}_4(S, S)$. Thus we can work our way up, step by step, to a linear combination of diagrams in which the two copies of S are connected by five strands. Now apply the quadratic relation to obtain a linear combination of diagrams that have at most one copy of S. □

Lemma 5.3. *If $m \in \{0, 10, 18, 28\}$ then every morphism from p_m to the empty diagram is a scalar multiple of Xp_m, where X is one of the four diagrams shown in Fig. 2.*
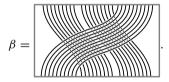
Proof. Suppose D is a diagram in $\mathcal{P}^0_m$. We must show that Dp_m is a scalar multiple of Xp_m, where X is the diagram from Fig. 2 that lies in $\mathcal{P}^0_m$.

Similarly to the proof of Lemma 4.3, we can assume the copies of S lie in a row at the top of D, with their basepoints at the top. If there is a strand with both endpoints on the bottom edge of D then $Dp_m = 0$. Thus we can assume every strand connects two distinct copies of S, or connects a copy of S to the bottom of D. By Lemma 5.2, we can assume that any pair of copies of S is connected by at most one strand.

If $m \in \{0, 10, 18\}$ then the only possible values of D are as shown in Fig. 2. Suppose $m = 28$, and let X be the fourth diagram in Fig. 2. The only other possibility for D is the horizontal reflection of X,

By the braiding relation, $D = X\beta$, where β is the braid

By resolving all 180 crossings in β,

$$\beta p_{28} = (iq^{1/2})^{180}p_{28} = -p_{28}.$$

Thus $Dp_{28} = X\beta p_{28} = -Xp_{28}$. □

Lemma 5.4. *Any diagram $X \in \mathcal{P}^0_n$ is a linear combination of diagrams that either contain a cap or are one of the four diagrams in Fig. 2.*

Proof. First consider the case $n \geq 29$. Let $p_{29} \otimes \mathrm{id}_{n-29}$ denote the Jones–Wenzl idempotent with extra vertical strands if necessary to bring the total up to n. Consider $X(p_{29} \otimes \mathrm{id}_{n-29})$. On the one hand, this is zero since it contains p_{29}. On the other hand, we can write $p_{29} \otimes \mathrm{id}_{n-29}$ as a linear combination of Temperley–Lieb diagrams. The identity diagram occurs with coefficient one, and every other term contains a cap. Thus X is equal to a linear combination of diagrams that contain a cap.

Now consider the case $n < 29$. If $n \in \{0, 10, 18, 28\}$ then Xp_n is a scalar multiple of Dp_n, where D is one of the four diagrams in Fig. 2. If $n \notin \{0, 10, 18, 28\}$ then Xp_n is zero by Lemma 4.4. The result now follows from a similar argument to the case $n \geq 29$. □

Lemma 5.5. $\mathcal{B}_n$ *spans* $\mathcal{P}_n$.

Proof. Suppose D is a diagram in $\mathcal{P}_n$. Draw D in a rectangle, with all endpoints on the bottom edge.

As in the proof of Lemma 4.3, we can assume the copies of S lie in a row at the top of D, and every strand either connects adjacent copies of S or connects a copy of S to the bottom edge of D. Let m be the number of strands that connect a copy of S to the bottom edge of D. We proceed by induction on m.

Write D in the form XY, where X is a diagram in $\mathcal{P}_m^0$ and Y is a Temperley–Lieb diagram in $\mathcal{P}_n^m$. Now apply Lemma 5.4 to X and Lemma 3.5 to Y. Thus XY is a linear combination of diagrams of the form $X'Y'$, where Y' is JW-reduced, and X' either contains a cap or is one of the four diagrams in Fig. 2.

If X' contains a cap or Y' contains a cup then $X'Y'$ lies in the span of $\mathcal{B}_n$, by induction on m. If X' does not contain a cap and Y' does not contain a cup then $X'Y'$ is an element of $\mathcal{B}_n$. $\square$

By Lemma 3.6, the dimension of $\mathcal{P}_n$ is the number of elements of $\mathcal{B}_n$. This completes the proof of Theorem 5.1.

6. The other *ADE* planar algebras

We now consider the subfactor planar algebras of types A_N, D_{2N} and E_6. Each of these has a presentation and a basis similar to those we gave for E_8, and by similar arguments.

The subfactor planar algebra with principal graph A_N is the planar algebra with no generators and the defining relations

- $\bigodot = [2] \bigcirc$,
- $p_N = 0$,

where $q = e^{i\pi/(N+1)}$.

The subfactor planar algebra with principal graph D_{2N} is generated by a single element $S \in \mathcal{P}_{4N-4}$. A list of four defining relations is given in [7]. One alternative list of defining relations is as follows.

- $\bigodot = [2] \bigcirc$,
- $\rho(S) = \sqrt{-1}S$,
- $\tau(S) = 0$,
- $S^2 = p_{2N-2}$,
- $p_{4N-3} = 0$,

where $q = e^{i\pi/(4N-2)}$.

The subfactor planar algebra with principal graph E_6 is the planar algebra $\mathcal{P}$ with a single generator $S \in \mathcal{P}_6$ and the defining relations

- $\bigodot = [2] \bigcirc$,
- $\rho(S) = e^{4i\pi/3}S$,
- $\tau(S) = 0$,
- $S^2 = S + [2]^2[3]p_3$,
- $\hat{S}\, p_8 = 0$,

where $q = e^{i\pi/12}$.

These planar algebras all have braiding properties at least as strong as in the E_8 case. In the A_N case, Reidemeister moves two and three imply that you can drag a strand over or under any part of a diagram. In the D_{2N} case, [7] prove that you can drag a strand over any part of a diagram, and you can drag a strand under any part of the diagram, up to a possible change of sign. In the E_6 and E_8 cases, you can drag a strand over any part of a diagram, but you cannot drag a strand under a generator, even up to sign.

In each of these planar algebras, the JW-reduced Temperley–Lieb diagrams form a basis for the space of Temperley–Lieb diagrams. However, the number 29 in the definition of JW-reduced must be replaced by the number k such that $p_k = 0$. This is N in the A_N case, $4N - 3$ in the D_{2N} case, and 11 in the E_6 case.

Recall that our basis in the E_8 case consists of diagrams of the form XY, where X is one of a short list of possibilities, and Y is a JW-reduced Temperley–Lieb diagram with no cups. There are similar but simpler bases for the other planar algebras in the *ADE* classification. In the A_N case, X is simply the empty diagram. In the D_{2N} and E_6 cases, X is either the empty diagram or the generator S.

Note that [7] give a different basis in the D_{2N} case. Their basis elements are built out of minimal idempotents, each of which they define as an explicit linear combination of diagrams. Our basis is simpler from a diagrammatic point of view, while theirs is more natural from a purely algebraic point of view. Their construction also applies in the A_N, E_6, and E_8 cases. In the A_N case, the minimal idempotents are simply the Jones–Wenzl idempotents. We did not compute explicit minimal idempotents in the E_6 and E_8 cases, but our results reduce this to what should be a straightforward exercise in linear algebra.

Finally, we remark that [7] use purely combinatorial methods to prove that their presentation gives a non-trivial planar algebra. It would be interesting to do this in the E_6 and E_8 cases. Together this with explicit minimal idempotents, this would be a new proof of the existence of subfactor planar algebras with principal graph E_6 and E_8.

Acknowledgements

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