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Principal graph stability and the jellyfish algorithm

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Abstract We show that if the principal graph of a subfactor planar algebra of modulus $\delta > 2$ is stable for two depths, then it must end in A_{finite} tails. This result is analogous to Popa's theorem on principal graph stability. We use these theorems to show that an $n-1$ supertransitive subfactor planar algebra has jellyfish generators at depth n if and only if its principal graph is a spoke graph. This is the published version of `arxiv:1208.1564`.

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1 Introduction

Every subfactor planar algebra embeds in the graph planar algebra of its principal graph [14,21]. Thus one can construct a subfactor planar algebra by finding candidate generators in the appropriate graph planar algebra, and then showing the planar algebra they generate is a subfactor planar algebra with the correct principal graph. Since a graph planar algebra satisfies all the unitarity conditions of a subfactor planar algebra, one must only show the planar subalgebra $P_\bullet$ is *evaluable*, i.e., $\dim(P_{0,\pm}) = 1$, to get some subfactor planar algebra. Additional arguments are needed to verify the principal graph of $P_\bullet$ is the graph with which we started.

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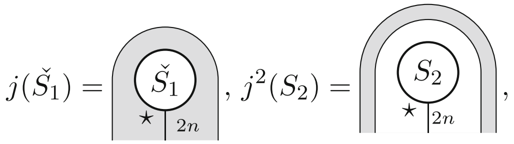
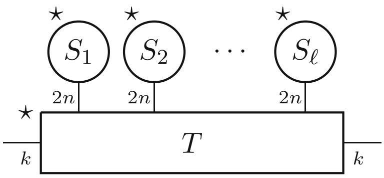
The *jellyfish algorithm* of [3] is an evaluation algorithm with two main ingredients:

(1) Elements in a set of generators $\mathcal{S}_\pm \subseteq P_{n,\pm}$ satisfy *jellyfish relations*, i.e., diagrams like

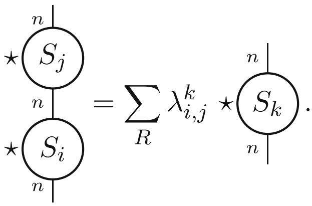


where $\check{S}_1 \in \mathcal{S}_-$, $S_2 \in \mathcal{S}_+$, can be written as linear combinations of *trains*, which are diagrams where any region meeting the distinguished interval of a generator meets the distinguished interval of the external disk, e.g.,



where $S_1, \ldots, S_\ell \in \mathcal{S}_\pm$ and T is a single Temperley-Lieb diagram (we suppress the external disk, and the external star goes in the upper left corner).

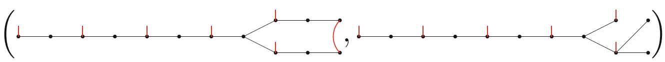
(2) The generators in $\mathcal{S}_\pm$, together with the Jones–Wenzl projection $f^{(n)}$, form an algebra under the usual multiplication



Given these two ingredients, one can evaluate any closed diagram in two steps.

(1) Pull all generators S to the outside of the diagram using the jellyfish relations, possibly getting diagrams with more S's, and
(2) Iteratively reduce the number of generators using the algebra property and an inner-most disk argument.

The jellyfish algorithm was first used in [3] to construct the extended Haagerup subfactor planar algebra with principal graphs

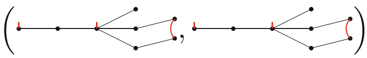


(the red markings at the even depths give the dual data), which completed the classification of non-A_∞ subfactors in the index range $(4, 3+\sqrt{3})$. They found 2-*strand jellyfish relations*

$$j(\check{S}) \in \operatorname{span}(\text{trains}_{5,+}(\{S\})) \quad \text{and} \quad j^2(S) \in \operatorname{span}(\text{trains}_{6,+}(\{S\}))$$

to evaluate all diagrams that are unshaded on the outside (see Definition 4.1 for the relevant notation).

The algorithm was used again in Han's thesis [6] to give a planar algebra construction of the Izumi-Xu 2221 subfactor planar algebra with principal graphs

 ,

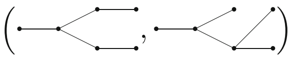
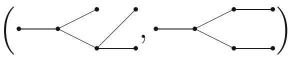
but with simpler 1-*strand jellyfish relations*:

$$j(\check{S}_1),\ j(\check{S}_2) \in \operatorname{span}(\text{trains}_{4,+}(\{S_1, S_2\})) \quad \text{and}$$
$$j(S_1),\ j(S_2) \in \operatorname{span}(\text{trains}_{4,-}(\{\check{S}_1, \check{S}_2\})).$$

(Note that these relations immediately imply relations for $j^2(S_i)$, $i = 1, 2$).

In recent work [16], Morrison and Penneys use the jellyfish algorithm to automate the construction of certain subfactor planar algebras whose principal and dual principal graphs are *spoke graphs*, which are trees with at most one vertex of degree greater than 2 (possibly with some multiple edges near the central vertex. See Definition 4.6). They constructed a new 4442 spoke subfactor along with a number of known spoke subfactors, including the Izumi-Xu 2221 (automating Han's thesis), the Goodman-de la Harpe-Jones 3311, and the Izumi 3333. Again, simpler 1-strand jellyfish relations were found.

Bigelow, Morrison, Peters, and Snyder [3] noticed that 1-strand jellyfish generators did not exist for the (extended) Haagerup subfactor planar algebra. Morrison and Penneys also noticed their non-existence for all known examples of subfactor planar algebras with annular multiplicities $*10$, i.e., for which the principal graphs (Γ_+, Γ_-) are a translated extension of

 or 

(*translating* a principal graph means attaching an A_k graph to the left, and *extending* means adding additional edges and vertices to the right). For more details on annular multiplicities $*10$, see [5,12,18].

In this paper, we give necessary and sufficient conditions that jellyfish relations exist for an $n-1$ supertransitive subfactor planar algebra with generators at depth n. (Of course, actually calculating these relations requires additional work, e.g., computations in the appropriate graph planar algebra after obtaining the generators.)

Theorem 1.1 *An $n-1$ supertransitive subfactor planar algebra has jellyfish generators at depth n if and only if its principal graph is a spoke graph. There are 1-strand jellyfish generators if and only if both the principal graph and dual principal graph are spoke graphs. See Theorems 4.9 and 4.10 for more details.*

To prove this result, we use techniques from Sect. 4 of Popa's paper [25]. Popa calls a (dual) principal graph Γ *stable at depth n* if Γ does not merge or split between depths n and $n+1$, and all edges between depths n and $n+1$ are simple. He proves a remarkable result, which we call *Popa's Principal Graph Stability Theorem*. For context, let $P_\bullet$

be a subfactor planar algebra of modulus δ with principal graphs (Γ_+, Γ_-), and let $\Gamma_\pm(k)$ denote the truncation of $\Gamma_\pm$ to depth k.

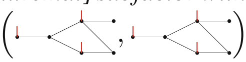
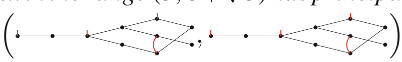
Theorem 1.2 (Popa's Principal Graph Stability Theorem 4.5 of [25]) *If* (Γ_+, Γ_-) *is stable at depth n, the truncation* $\Gamma_\pm(n+1) \neq A_{n+2}$*, and* $\delta > 2$*, then* (Γ_+, Γ_-) *is stable at depth k for all* $k \geq n$*, and* Γ_+, Γ_- *are finite.*

In examining this theorem, we found that trains first appeared in [25] in the language of λ-lattices! Using Popa's techniques along with trains and ideas stemming from the jellyfish algorithm, we prove an analogous result only looking at the principal graph, which is a strengthening of (a) of Lemma 4.7 of [25].

Theorem 1.3 *If* Γ_+ *is stable at depths n and* $n+1$*, the truncation* $\Gamma_+(n+1) \neq A_{n+2}$*, and* $\delta > 2$*, then* (Γ_+, Γ_-) *is stable at depth k for all* $k \geq n+1$*, and* Γ_+, Γ_- *are finite.*

Planar algebras are essential to our approach. We use the 1-click rotation (also known as the Fourier Transform), which is natural from a planar algebra viewpoint, in the important Lemma 3.2.

One of the biggest hurdles in the classification of subfactors to index 5 [7,18,20,27] were weeds with initial quadruple points. (A *weed* represents an infinite family of potential principal graphs obtained from a fixed subgraph by translating and extending. See [20] for more details.) Arguments to rule out $\mathcal{Q}$, $\mathcal{Q}'$ in [7] were case specific; they knew no general techniques for quadruple points to go beyond index 5. The theorems in this paper and [25] not only simplify eliminating $\mathcal{Q}$, $\mathcal{Q}'$ in [7] (and $\mathcal{B}$ in [18]), but also eliminate all remaining weeds with initial quadruple points up to index $3+\sqrt{5}$, providing more evidence for [17, Conjecture 2.2] of Morrison–Peters:

Conjecture *Any [extremal] subfactor with index in the range* $(5, 3+\sqrt{5})$ *has principal graphs* (A_∞, A_∞),  *or* .

Using our results, Morrison and Penneys have shown that to prove the conjecture of Morrison–Peters, one needs to eliminate roughly 10 weeds with initial triple points. These new weeds are similar to weeds eliminated in [18,20], but they are more complex.

Numerous other applications of our results are given in Sect. 4. We anticipate that our results will prove strong new obstructions to possible principal graphs.

1.1 Outline

Section 2 contains the background for this paper. Subsection 2.1 briefly recalls how to get a rigid, unitary, spherical 2-category $\mathcal{G}(P_\bullet)$ from a subfactor planar algebra $P_\bullet$ and how to define the principal graphs (Γ_+, Γ_-) from $\mathcal{G}(P_\bullet)$. Subsection 2.2 gives Popa's definition of stability for planar algebras and principal graphs and shows they are compatible.

In Sect. 3, we go through the proof of Popa's Theorem 1.2 using planar algebras and trains to prove Theorem 1.3. In Subsect. 3.1, we define trains, and we prove the important Lemma 3.2. In Subsect. 3.2, we show that stability is equivalent to trains spanning. Subsection 3.3 connects trains and jellyfish, and Subsect. 3.4 contains the proof of Theorem 1.3.

In Sect. 4, we give a number of applications of our results. Subsection 4.1 explains the connection between the jellyfish algorithm and spoke principal graphs, proving Theorem 1.1. Afterward, we give a few quick corollaries and a remark which uses the classification of subfactors to index 5 [7,18,20,27] to classify all simply laced, acyclic principal graphs of subfactors with at most 2 triple points. Subsection 4.2 gives a simple proof of Jones' quadratic tangles obstruction for annular multiplicities $*10$ subfactor planar algebras.

2 Background

We refer the reader to [3,12,13] for the definition of a (subfactor) planar algebra.

Remark 2.1 When we draw planar diagrams, we often suppress the external boundary disk. In this case, the external boundary is assumed to be a large rectangle whose distinguished interval contains the upper left corner. We draw one string with a number next to it instead of drawing that number of parallel strings. We shade the diagrams as much as possible, but if the parity is unknown, we often cannot know how to shade them. Finally, projections are usually drawn as rectangles with the same number of strands emanating from the top and bottom, while other elements may be drawn as circles.

2.1 2-categories and fusion graphs

We recall how to get a rigid, unitary, spherical 2-category $\mathcal{G}(P_\bullet)$ from a subfactor planar algebra $P_\bullet$ and how to define the principal graphs (Γ_+, Γ_-) from $\mathcal{G}(P_\bullet)$ (see also Sect. 4.1 of [19]).

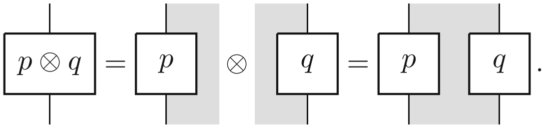
Definition 2.2 The *paragroup* $\mathcal{G}(P_\bullet)$ of $P_\bullet$, a rigid, unitary, spherical 2-category, is defined as follows.

The *objects* of $\mathcal{G}(P_\bullet)$ are the symbols $\bigcirc$ and $\circledcirc$.

The 1-*morphisms* of $\mathcal{G}(P_\bullet)$ are the projections of $P_\bullet$.

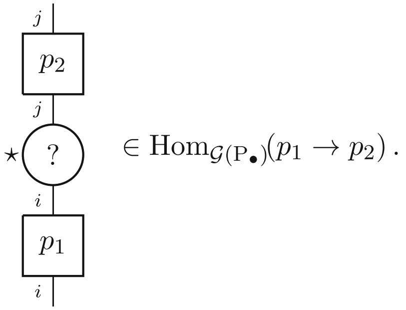
$$\mathrm{Hom}_{\mathcal{G}(\mathrm{P}_\bullet)}(\bigcirc \to \bigcirc) = \{p \in P_{i,+} | p \textit{ is a projection and } i \textit{ is even}\},$$
$$\mathrm{Hom}_{\mathcal{G}(\mathrm{P}_\bullet)}(\bigcirc \to \circledcirc) = \{p \in P_{i,+} | p \textit{ is a projection and } i \textit{ is odd}\},$$
$$\mathrm{Hom}_{\mathcal{G}(\mathrm{P}_\bullet)}(\circledcirc \to \bigcirc) = \{p \in P_{i,-} | p \textit{ is a projection and } i \textit{ is odd}\}, \textit{ and}$$
$$\mathrm{Hom}_{\mathcal{G}(\mathrm{P}_\bullet)}(\circledcirc \to \circledcirc) = \{p \in P_{i,-} | p \textit{ is a projection and } i \textit{ is even}\}.$$

The identity 1-morphisms are the empty diagrams. Composition of 1-morphisms, denoted $\otimes$, is given by horizontal concatenation; e.g., if $p \in \mathrm{Hom}_{\mathcal{G}(\mathrm{P}_\bullet)}(\bigcirc \to \circledcirc)$ and $q \in \mathrm{Hom}_{\mathcal{G}(\mathrm{P}_\bullet)}(\circledcirc \to \bigcirc)$, then

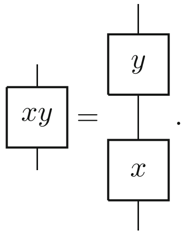


A 1-morphism p is called *simple* if $\dim(\mathrm{Hom}_{\mathcal{G}(\mathrm{P}_\bullet)}(p \to p)) = 1$.

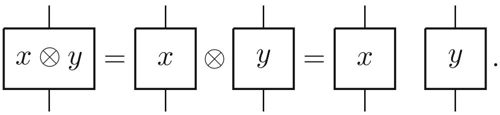
The 2-*morphisms* of $\mathcal{G}$ are as follows. If $p_1 \in P_{i,\pm}$ and $p_2 \in P_{j,\pm}$ then $\mathrm{Hom}_{\mathcal{G}(\mathrm{P}_\bullet)}(p_1 \to p_2)$ is $p_2 P_{j \to i} p_1$, where $P_{j \to i}$ is P_{i+j} with j strings on the bottom and i strings on the top. Note that $\mathrm{Hom}_{\mathcal{G}(\mathrm{P}_\bullet)}(p_1 \to p_2) = (0)$ if i and j do not have the same parity.



The two types of composition of 2-morphisms are given by vertical and horizontal concatenation of diagrams. If we have $x \in \mathrm{Hom}_{\mathcal{G}(\mathrm{P}_\bullet)}(p_1 \to p_2)$ and $y \in \mathrm{Hom}_{\mathcal{G}(\mathrm{P}_\bullet)}(p_2 \to p_3)$, then the vertical multiplication xy is given by

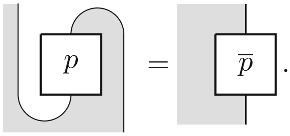


If $x \in \mathrm{Hom}_{\mathcal{G}(\mathrm{P}_\bullet)}(p_1 \to p_2)$ and $y \in \mathrm{Hom}_{\mathcal{G}(\mathrm{P}_\bullet)}(p_3 \to p_4)$ and p_1, p_2 are composable with p_3, p_4 respectively, then the horizontal multiplication $x \otimes y$ is given by

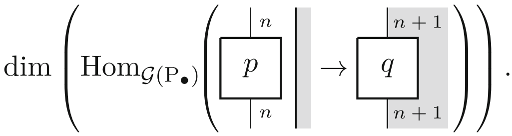


The *adjoint* operation in $\mathcal{G}(P_\bullet)$ is the identity on objects and 1-morphisms. The adjoint of a 2-morphism is the same as the adjoint operation in the planar algebra $P_\bullet$. If $x \in \mathrm{Hom}_{\mathcal{G}(\mathrm{P}_\bullet)}(p_1 \to p_2)$, where $p_1 \in P_{i,\pm}$ and $p_2 \in P_{j,\pm}$, then we can consider x as an element of $P_{i+j,\pm}$, take the adjoint, and consider the result x^* as an element of $\mathrm{Hom}_{\mathcal{G}(\mathrm{P}_\bullet)}(p_2 \to p_1)$.

The *duality* operation on $\mathcal{G}(P_\bullet)$ is the identity on all objects. On 1-morphisms and 2-morphisms, duality is rotation by π.



Definition 2.3 The *principal graph* Γ_+ of $P_\bullet$ is defined as follows. The even vertices of Γ_+ are the isomorphism classes of simple 1-morphisms in $\mathrm{Hom}(○ \to ○)$. The odd vertices of Γ_+ are the isomorphism classes of simple 1-morphisms in $\mathrm{Hom}(○ \to ◍)$. The number of edges between vertices corresponding to simple projections $p \in \mathrm{Hom}(○ \to ○)$ and $q \in \mathrm{Hom}(○ \to ◍)$, is



The *basepoint* $\star$ of Γ_+ is the vertex corresponding to the unshaded empty diagram. The *depth* of a vertex of Γ_+ is its distance from $\star$. This is equal to the minimum n such that the vertex is the equivalence class of a projection $p \in P_{n,+}$.

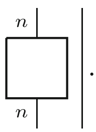
The *dual principal graph* Γ_- is defined in exactly the same way as Γ_+, but reversing the roles of ◍ and ○. The *basepoint* $\star$ of Γ_- is the vertex corresponding to the shaded empty diagram.

Our graphs are always drawn with the basepoint $\star$ at the left.

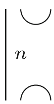
Remark 2.4 The "plus or minus" symbol $\pm$ is meant to be read respectively throughout an entire statement.

Remark 2.5 If $\Gamma_\pm$ is simply laced, and $p \in P_{n,\pm}$ is a minimal projection such that the vertex $[p]$ has depth n, then we identify $[p]$ with p.

Definition 2.6 Alternatively, from an operator algebras viewpoint, we can define the (dual) principal graph as the principal part of the Bratteli diagram of the tower of finite dimensional von Neumann algebras $P_\pm = (P_{n,\pm})$, where $P_{n,\pm}$ includes into $P_{n+1,\pm}$ unitally via the right inclusion



If $z_{n+1,\pm}$ is the central support of the Jones projection

$$e_{n,\pm} = \delta^{-1}$$  $$\in P_{n+1,\pm},$$

then for each $n \in \mathbb{N}$,

$$z_{n+1,\pm}P_{n-1,\pm} \subset z_{n+1,\pm}P_{n,\pm} \subset z_{n+1,\pm}P_{n+1,\pm}$$

is the Jones basic construction of finite dimensional von Neumann algebras [4,9]. Hence the Bratteli diagram of $P_\pm$ between depths n and $n+1$ consists of the reflection of the Bratteli diagram between depths $n-1$ and depth n, which is referred to as the "old part," and a "new part," which can be identified with the Bratteli diagram of the inclusion

$$(1 - z_{n+1,\pm})P_{n,\pm} \subset (1 - z_{n+1,\pm})P_{n+1,\pm}.$$

The principal graph is formed from only the "new parts." See [4] for more details.

2.2 Popa's stability criterion

In [25, Sect. 4], Popa gives a stability criterion for λ-lattices that has very strong consequences. We define the criterion, summarize the proof, and list some consequences.

Let $P_\bullet$ be a subfactor planar algebra, let $P_\pm = (P_{n,\pm})$ be the respective towers of algebras, and let (Γ_+, Γ_-) be the principal and dual principal graphs. Let $TL_\bullet \subset P_\bullet$ be the Temperley-Lieb planar subalgebra.

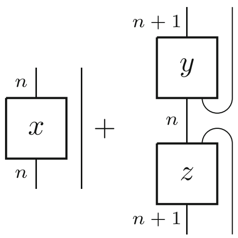
Definition 2.7 The (dual) principal graph $\Gamma_\pm$ of $P_\pm$ is said to be *stable at depth n* if every vertex at depth n connects to at most one vertex at depth $n+1$, no two vertices at depth n connect to the same vertex at depth $n+1$, and all edges between depths n and $n+1$ are simple. We say (Γ_+, Γ_-) is *stable at depth n* if both Γ_+ and Γ_- are stable at depth n.

Definition 2.8 (*Popa's stability criterion*) We say P_+ is *stable at depth n* if and only if

$$P_{n+1,+} = P_{n,+} + P_{n,+}e_{n,+}P_{n,+},$$

where we identify $P_{n,\pm}$ with its image in $P_{n+1,\pm}$ under the right inclusion (see Definition 2.6). We say $P_\bullet$ is *stable at depth n* if both P_+ and P_- are stable at depth n.

Remark 2.9 We remark that $P_{n,+} + P_{n,+}e_{n,+}P_{n,+}$ is the set of linear combination of diagrams of the form



where $x, y, z \in P_{n,\pm}$. We say $P_\bullet$ is *stable at depth n* if both P_+ and P_- are stable at depth n.

Lemma 2.10 *When we identify $P_{n,\pm}$ with its image in $P_{n+1,\pm}$ by adding one vertical string to the right,*

$$P_{n,\pm} + P_{n,\pm} e_{n,\pm} P_{n,\pm} = \langle P_{n,\pm}, TL_{n+1,\pm} \rangle,$$

where the angled brackets denote the algebra generated by $P_{n,\pm}$ and $TL_{n+1,\pm}$ under the usual multiplication.

Proof Let $e_{1,\pm}, \ldots, e_{n,\pm}$ be the standard algebra generators of $TL_{n+1,\pm}$. All of these lie in $P_{n,\pm}$ except for $e_{n,\pm}$, so

$$\langle P_{n,\pm}, TL_{n+1,\pm} \rangle = \langle P_{n,\pm}, e_{n,\pm} \rangle.$$

For any $x \in P_{n,\pm}$, we have $e_{n,\pm} x e_{n,\pm} = E_{P_{n-1,\pm}}(x) e_{n,\pm}$, where $E_{P_{n-1,\pm}}(x)$ is the conditional expectation (partial trace) of x. We can use this to reduce any word in $P_{n,\pm}$ and $e_{n,\pm}$ until it has at most one occurrence of $e_{n,\pm}$.

The following is [25, Proposition 4.3, Corollary 4.4]. We include a short proof for the reader's convenience.

Proposition 2.11 (Popa) *The following are equivalent:*

(1) *$P_\pm$ is stable at depth n.*
(2) *$\Gamma_\pm$ is stable at depth n.*

Proof As in Definition 2.6, let $z_{n+1,\pm}$ be the central support of $e_{n,\pm}$ in $P_{n+1,\pm}$, and identify $P_{n,\pm}$ with its image in $P_{n+1,\pm}$ under the right inclusion. Then

$$\begin{aligned} P_\pm \text{ is stable at depth } n &\iff P_{n+1,\pm} = P_{n,\pm} + P_{n,\pm} e_{n,\pm} P_{n,\pm} \\ &\iff (1 - z_{n+1,\pm}) P_{n+1,\pm} = (1 - z_{n+1,\pm}) P_{n,\pm} \\ &\iff \Gamma_\pm \text{ is stable at depth } n. \end{aligned}$$

□

Definition 2.12 Let $\Gamma_\pm(k)$ be the truncation of $\Gamma_\pm$ to depth k consisting of all vertices with depth at most k and all edges connecting them.

If $\Gamma_\pm$ is stable at depth k for all $k \geq n$ then $\Gamma_\pm$ can be obtained by attaching graphs of type A to $\Gamma_\pm(n)$. The following theorem implies that, with some simple exceptions, these attached graphs of type A have finite length. We call them A_{finite} *tails*.

Theorem 2.13 **([26])** *If a connected component of $\Gamma_\pm \backslash \Gamma_\pm(n) = A_\infty$ for some $n \geq 0$, then $\Gamma_\pm \in \{A_\infty, A_{\infty,\infty}, D_\infty\}$.*

Note that this theorem also follows from Theorem 6.5 in [23], which applies to infinite depth subfactors by [21].

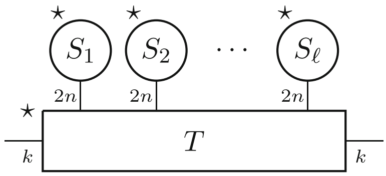
3 Principal graph stability via trains

In this section, we prove Popa's Principal Graph Stability Theorem 1.2 and our Theorem 1.3 using planar algebras and trains.

3.1 Trains

Let $P_\bullet$ be a subfactor planar algebra, and let $TL_\bullet \subset P_\bullet$ be its Temperley-Lieb planar subalgebra.

Definition 3.1 Given a set $\mathcal{S}_\pm \subset P_{n,\pm}$, a *train from* $\mathcal{S}_\pm$ is a planar tangle $\mathcal{T}$ labeled by elements from $\mathcal{S}_\pm$ such that for each input disk of $\mathcal{T}$, its distinguished interval meets the region that meets the distinguished interval of the output disk. A train in $P_{k,\pm}$ can be drawn in the form

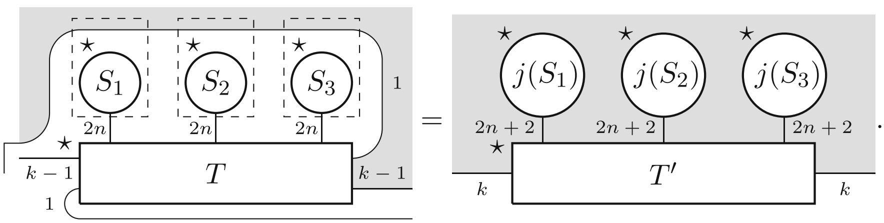


where $S_1, \ldots, S_\ell \in \mathcal{S}_\pm$, T is a single Temperley-Lieb diagram, and the distinguished interval of the external disk is at the top.

An *ℓ-car train from* $\mathcal{S}_\pm$ is a train from $\mathcal{S}_\pm$ with ℓ labeled input disks. Note that any single diagram from $TL_{k,\pm}$ is a 0-car train from $\mathcal{S}_\pm$. We let $\text{trains}_{k,\pm}(\mathcal{S}_\pm)$ denote the set of trains from $\mathcal{S}_\pm$ in $P_{k,\pm}$. We say *trains from* $\mathcal{S}_\pm$ *span* $P_\pm$ if $P_{k,\pm} = \text{span}(\text{trains}_{k,\pm}(\mathcal{S}_\pm))$ for all $k \geq n$.

Lemma 3.2 *Suppose $k > n$. If trains from $P_{n,+}$ span $P_{k,+}$, then trains from $P_{n+1,-}$ span $P_{k,-}$.*

Proof Consider the Fourier transform (one click rotation) of a train from $P_{n,+}$, which can be drawn with an arc passing over the ℓ labelled disks $S_1, \ldots, S_\ell \in P_{n,+}$. We then combine each S_i with a segment of this arc to obtain $j(S_i) \in P_{n+1,-}$, and thus obtain a train from $P_{n+1,-}$. For example, in the case of a 3-car train, we have the following:



Since trains from $P_{n,+}$ span $P_{k,+}$, and the one click rotation is a vector space isomorphism, it follows that trains from $P_{n+1,-}$ span $P_{k,-}$. □

3.2 Trains and stability

Trains first appeared in [25]. The following lemma allows us to translate between the above planar algebra definition of trains and Popa's λ-lattice formalism.

Lemma 3.3 *For all* $k > n$, $\mathrm{span}(\mathrm{trains}_{k,+}(P_{n,+})) = \langle P_{n,+}, TL_{k,+}\rangle$.

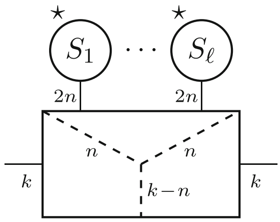
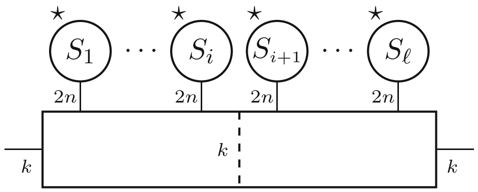
Here, $P_{n,+}$ is considered as a subalgebra of $P_{k,+}$ by the inclusion operation of adding $k-n$ vertical strands on the right, and the angled brackets denote the associative algebra generated by $P_{n,+}$ and $TL_{k,+}$ under the usual multiplication.

Proof The inclusion

$$\langle P_{n,+}, TL_{k,+}\rangle \subseteq \mathrm{span}(\mathrm{trains}_{k,+}(P_{n,+}))$$

is obvious. For the other inclusion, suppose $X \in \mathrm{trains}_{k,+}(P_{n,+})$ is an ℓ-car train.

Claim Either

- $X =$  , or
- $X =$  .

Here, the rectangle in each diagram indicates a Temperley-Lieb diagram T, and the dashed lines inside the rectangle partition T into Temperley-Lieb subdiagrams, i.e., they intersect the indicated number of strands of T transversely. Note that in the second case, $\ell \geq 2$.

If X is as in the first diagram of the claim then $X = T_1 S T_2$, where $T_1, T_2 \in TL_{k,+}$ and $S \in P_{n,+}$. If X is as in the second diagram of the claim then the result follows by induction on ℓ.

It remains only to prove the claim. First we define a metric on the regions of the Temperley-Lieb diagram T. Suppose x and y are two points in T that do not lie on the edges of T. A path in T from x to y is a *geodesic* if it crosses the edges of T transversely and a minimum number of times. The *distance* $d(x, y)$ is the number of crossings in a geodesic from x to y. This determines a metric on the regions of T. This is the same as the graph metric on the tree dual to T. We will use basic properties of metrics on trees, which we defer to two technical Lemmas 3.4 and 3.5.

Let p be a point on the bottom edge of T. Let $x_0, \dots, x_\ell$ be points along the top edge of T that separate $S_1, \dots, S_\ell$. Here, x_0 is the top left corner, x_ℓ is the top right corner, and every other x_i separates S_i and S_{i+1}.

We have that

- $d(x_i, x_{i+1}) \leq 2n$ for all $i \in \{0, \dots, \ell - 1\}$, and
- $d(x_0, p), d(x_\ell, p) \leq k$.

By Lemma 3.4, either

- $d(x_0, x_\ell) \leq 2n$, or
- $d(x_i, p) \leq k$ for some $i \in \{1, \dots, \ell - 1\}$.

First suppose $d(x_0, x_\ell) \leq 2n$. By Lemma 3.5, there is a point y in T such that

- $d(x_0, y), d(x_\ell, y) \leq n$, and
- $d(y, p) \leq k - n$.

Furthermore, we can assume these inequalities are equalities modulo two. We now define the Y-shaped graph as in the first case of the claim. The central vertex is y. For the spokes, start with geodesics and introduce switchbacks as needed to increase the number of intersection points.

Now suppose $d(x_i, p) \leq k$. In this case we can find a vertical edge as in the second case of the claim. Start with a geodesic from x_i to p and introduce switchbacks as needed to increase the number of intersection points. □

Lemma 3.4 *Suppose $x_0, \dots, x_\ell$ and p are vertices in a tree such that*

- $d(x_i, x_{i+1}) \leq 2n$ *for all* $i \in \{0, \dots, \ell - 1\}$, *and*
- $d(x_0, p), d(x_\ell, p) \leq k$.

Then either

- $d(x_0, x_\ell) \leq 2n$, *or*
- $d(x_i, p) \leq k$ *for some* $i \in \{1, \dots, \ell - 1\}$.

Proof If $d(x_{i-1}, x_{i+1}) \leq 2n$ for some i, then we can omit x_i from the sequence and the hypotheses will still hold. Thus, without loss of generality,

$$d(x_{i-1}, x_{i+1}) > 2n \quad \text{for all } i \in \{1, \dots, \ell - 1\}.$$

Under this assumption, we show $d(x_i, p) \leq k$ for *every* $i \in \{1, \dots, \ell - 1\}$.

Fix $i \in \{1, \dots, \ell - 1\}$. The geodesics connecting the three points x_{i-1}, x_i, x_{i+1} form a Y-shaped subtree, and the spoke ending at x_i must be the shortest of the three spokes. It follows that

$$d(x_i, p) < \max(d(x_{i-1}, p), d(x_{i+1}, p)).$$

Thus largest value of $d(x_j, p)$ occurs when either $j = 0$ or $j = \ell$. In particular, $d(x_i, p) \leq k$ for all $i \in \{1, \dots, \ell - 1\}$. □

Lemma 3.5 *Suppose x_0, x_ℓ and p are vertices on a tree such that*

- $d(x_0, x_\ell) \leq 2n$, *and*
- $d(x_0, p), d(x_\ell, p) \leq k$,

and these inequalities are equalities modulo two. Then there is a vertex y such that

- $d(y, x_0), d(y, x_\ell) \leq n$, *and*
- $d(y, p) \leq k - n$,

and these inequalities are equalities modulo two.

Proof The geodesics connecting the three points x_0, x_ℓ and p form a Y-shaped subtree. Let z be the central vertex of this subtree.

Suppose $d(x_0, z) \geq n$. Let y be the point on the geodesic from x_0 to z such that $d(y, x_0) = n$. Then $d(y, x_\ell) = d(x_0, x_\ell) - n$ and $d(y, p) = d(x_0, p) - n$. Thus y is the required vertex.

The cases $d(x_\ell, z) \geq n$ and $d(p, z) \geq k - n$ are similar.

Finally, suppose $d(x_0, z) < n$, $d(x_\ell, z) < n$, and $d(p, z) < k - n$. Either all three or none of these inequalities are equalities modulo two. Thus we can take y to be either z or any vertex adjacent to z. □

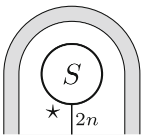
We now summarize our results on trains and stability in the following theorem, which follows by a simple induction argument together with Lemma 2.10, Proposition 2.11, and Lemma 3.3.

Theorem 3.6 *The following are equivalent:*

(1) $P_\pm$ *is stable at depth* $n, n+1, \ldots, k-1$,
(2) $\Gamma_\pm$ *is stable at depth* $n, n+1, \ldots, k-1$.
(3) $P_{k,\pm} = \langle P_{n,\pm}, TL_{k,\pm}\rangle$, *and*
(4) *Trains from* $P_{n,\pm}$ *span* $P_{k,\pm}$.

3.3 Trains and jellyfish

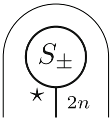
Lemma 3.7 *Suppose $\mathcal{S}_+ \subset P_{n,+}$ generates $P_\bullet$ as a planar algebra. Then trains from $\mathcal{S}_+$ span P_+ if and only if*

$j^2(S) =$  $\in \mathrm{span}(\mathrm{trains}_{n+2,+}(\mathcal{S}_+))$

for all $S \in \mathcal{S}_+$.

Proof The "only if" direction is trivial. The "if" direction is the first part of the jellyfish algorithm from Section 4 of [3]. Suppose $\mathcal{S}_+$ satisfies the above *jellyfish relations*. Given an element of $P_{k,+}$ that is a tangle labeled by elements of $\mathcal{S}_+$, we use the jellyfish relation to pull a copy of $S \in \mathcal{S}_+$ closer to the region that touches the distinguished interval of the outside boundary. This will typically give a linear combination of labeled tangles that contain more elements $S \in \mathcal{S}_+$. Nevertheless, the algorithm terminates with an element of $\mathrm{span}(\mathrm{trains}_{k,+}(\mathcal{S}_+))$.

Lemma 3.8 *Suppose $\mathcal{S}_+ \subset P_{n,+}$, $\mathcal{S}_- \in P_{n,-}$, and $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$ generates $P_\bullet$ as a planar algebra. Then trains from $\mathcal{S}$ span $P_\bullet$ if and only if*

$$j(S_\pm) =$$


$$\in \operatorname{span}(\text{trains}_{n+1,\mp}(\mathcal{S}_\mp))$$

for all $S_\pm \in \mathcal{S}_\pm$.

Proof This is similar to the proof of Lemma 3.7. □

Proposition 3.9 *Suppose $P_\bullet$ is generated as a planar algebra by $P_{n,+}$. Then*

(1) If (Γ_+, Γ_-) is stable at depth n, then (Γ_+, Γ_-) is stable at depth k for all $k \geq n$.
(2) If Γ_+ is stable at depths n and $n+1$, then (Γ_+, Γ_-) is stable at depth k for all $k \geq n+1$.

Proof

(1) The first statement is [25, Proposition 4.2]. In our terminology, the proof is as follows. Suppose (Γ_+, Γ_-) is stable at depth n. If $S \in P_{n,\pm}$ then, by Theorem 3.6,

$$j(S) \in P_{n+1,\pm} = \langle P_{n,\pm}, TL_{n+1,\pm}\rangle = \operatorname{span}(\text{trains}_{n+1,\pm}(P_{n,\pm})).$$

By Lemma 3.8, trains from $P_{n,\pm}$ span $P_\pm$, so again by Theorem 3.6, (Γ_+, Γ_-) is stable at depth k for all $k \geq n$.

(2) Suppose Γ_+ is stable at depths n and $n+1$. If $S \in P_{n,+}$ then, by Theorem 3.6,

$$j^2(S) \in P_{n+2,+} = \langle P_{n,+}, TL_{n+2,+}\rangle = \operatorname{span}(\text{trains}_{n+2,+}(P_{n,+})).$$

By Lemma 3.7, trains from $P_{n,+}$ span P_+, so again by Theorem 3.6, Γ_+ is stable at depth k for all $k \geq n$.
It remains to show that Γ_- is stable at depth k for all $k \geq n+1$. Since trains from $P_{n,+}$ span P_+, trains from $P_{n+1,-}$ span P_- by Lemma 3.2. Once more, by Theorem 3.6, Γ_- is stable at depth k for all $k \geq n+1$. □

3.4 The proof of Theorem 1.3

The proof of Popa's Principal Graph Stability Theorem has three main ingredients. First, the stability of (Γ_+, Γ_-) is used in Proposition 3.10 to construct a planar subalgebra $Q_\bullet \subset P_\bullet$ whose principal graphs (Λ_+, Λ_-) are stable at all higher depths. Second, by Theorem 2.13, the main result of [26], $\Lambda_\pm$ has no A_∞ tails, so $Q_\bullet$ is finite depth. Finally, the graph norm argument in Theorem 3.11 (and Corollary 3.12) shows $\Gamma_+ = \Lambda_+$, so $Q_\bullet = P_\bullet$. Theorem 3.11 is distilled from the last statement in the proof of Popa's Principal Graph Stability Theorem. We provide a proof for the convenience of the reader.

Since we are proving an analogous result, we proceed in the same manner, but we will use the 1-click rotation argument from Lemma 3.2 in a crucial way.

Recall that $\Gamma_\pm(k)$ is the truncation of $\Gamma_\pm$ to depth k. The first part of the following proposition is similar to [25, Proposition 4.1].

Proposition 3.10 *Suppose $P_\bullet$ is a subfactor planar algebra, and fix $n \geq 0$. Let $Q_\bullet$ be the planar subalgebra generated by $P_{n,+}$. Let (Λ_+, Λ_-) be the principal and dual principal graph of $Q_\bullet$, and note that $\Lambda_\pm(n) = \Gamma_\pm(n)$.*

(1) If (Γ_+, Γ_-) is stable at depth n, then $\Lambda_\pm(n+1) = \Gamma_\pm(n+1)$, and (Λ_+, Λ_-) is stable at depth k for all $k \geq n$.
(2) If Γ_+ is stable at depths n and $n+1$, then $\Lambda_+(n+2) = \Gamma_+(n+2)$, Λ_+ is stable at depth j for all $j \geq n$, and Λ_- is stable at depth k for all $k \geq n+1$.

Proof

(1) Since (Γ_+, Γ_-) is stable at depth n, by Proposition 2.11,

$$P_{n+1,\pm} = \operatorname{span}(\text{trains}_{n+1,\pm}(P_{n,\pm})) = Q_{n+1,\pm},$$

and thus $\Lambda_\pm(n+1) = \Gamma_\pm(n+1)$. Since $Q_\bullet$ is generated by $Q_{n,+} = P_{n,+}$ and (Λ_+, Λ_-) is stable at depth n, trains from $Q_{n,\pm}$ span $Q_\pm$, and $\Lambda_\pm$ is stable at depth k for all $k \geq n$ by Proposition 3.9.

(2) Since Γ_+ is stable at depths n and $n+1$, by Theorem 3.6,

$$P_{n+2,+} = \langle P_{n,+}, TL_{n+2,+}\rangle = \operatorname{span}(\text{trains}_{n+2,+}(P_{n,+})) = Q_{n+2,+},$$

and thus $\Lambda_+(n+2) = \Gamma_+(n+2)$. Since $Q_\bullet$ is generated by $Q_{n,+} = P_{n,+}$ and Λ_+ is stable at depths n and $n+1$, trains from $Q_{n,+}$ span Q_+, Λ_+ is stable at depth j for all $j \geq n$, and Λ_- is stable at depth k for all $k \geq n+1$ by Proposition 3.9.

Theorem 3.11 *Suppose Λ and Γ are finite, connected bipartite graphs with base-points and have the same norm $\delta > 2$. Suppose we have Frobenius-Perron eigenvectors λ and γ for Λ and Γ respectively and there is some $n \geq 1$ such that*

- $\Lambda(n) = \Gamma(n) \neq A_{n+1}$,
- $\lambda|_{\Lambda(n)} = \gamma|_{\Gamma(n)}$, *and*
- *Λ is stable at depth k for all $k \geq n$.*

Then $\Lambda = \Gamma$.

Proof Fix a vertex a_1 of depth exactly n in Λ.

First, suppose a_1 has no adjacent vertices of depth $n+1$ in Λ. Now $\delta\lambda(a_1)$ is the sum of the values of λ over vertices adjacent to a_1. But a_1 and all vertices adjacent to it lie in $\Lambda(n) = \Gamma(n)$, and $\gamma = \lambda$ when restricted to $\Gamma(n)$. Thus a_1 also has no adjacent vertices of depth $n+1$ in Γ.

Now suppose a_1 has an adjacent vertex a_2 of depth $n+1$ in Λ. Since Λ is stable at depth n and higher, a_1 is attached to an A_{finite} tail $a_1, \ldots, a_k$ in Λ. The values of $\lambda(a_i)$ for all i are determined by the values of δ and $\lambda(a_k)$. The most important property for us is

$$\delta\lambda(a_{i+1}) < 2\lambda(a_i)$$

for $i = 1, \ldots, k-1$.

Now consider the set of vertices b in Γ that are adjacent to a_1 and have depth $n+1$. The sum of the values of γ over these vertices is equal to $\lambda(a_2)$. If there are at least two such vertices, or one with multiplicity at least two, then one of them must satisfy

$$\gamma(b) \leq \lambda(a_2)/2.$$

But then

$$\delta\gamma(b) \leq \delta\lambda(a_2)/2 < \lambda(a_1) = \gamma(a_1).$$

This contradicts the fact that $\delta\gamma(b)$ is the sum of the values of γ over the vertices adjacent to b. It follows that a_1 has exactly one adjacent vertex at depth $n+1$ in Γ, which we name a_2, and we have $\gamma(a_2) = \lambda(a_2)$.

Applying the same argument recursively gives a path $a_1, a_2, \dots, a_k$ in Γ where $a_2, \dots, a_{k-1}$ have valency two, a_k has valency one, and $\gamma(a_i) = \lambda(a_i)$ for all i.

Thus every vertex of depth n in Γ is attached to an A_{finite} tail with the same length as the corresponding vertex in Λ. We conclude that $\Gamma = \Lambda$. □

Corollary 3.12 *Suppose $Q_\bullet$ is a planar subalgebra of $P_\bullet$ with $\delta > 2$, and let Λ_+ be the principal graph of $Q_\bullet$. Assume that there is an $n \geq 1$ such that*

- *$\Lambda_+(n) = \Gamma_+(n) \neq A_{n+1}$, and*
- *Λ_+ is finite and stable at depth k for all $k \geq n$.*

Then $\Lambda_+ = \Gamma_+$, so $Q_\bullet = P_\bullet$.

Proof First, the depth of $P_\bullet$ is at most the depth of $Q_\bullet$. If $Q_\bullet$ is depth q, then $q+1$ parallel strings factor through q parallel strings, since any Pimsner-Popa basis for $Q_{q+1,+}$ over $Q_{q,+}$ is a Pimsner-Popa basis for $P_{q+1,+}$ over $P_{q,+}$. Hence Γ_+ is finite, and $\delta = \|\Lambda_+\| = \|\Gamma_+\|$ by [10].

Since $\Lambda_+(n) = \Gamma_+(n)$, $\dim(Q_{n,+}) = \dim(P_{n,+})$, as both are equal to the number of loops of length $2n$ on Γ_+ starting at $\star$. Thus $Q_{n,+} = P_{n,+}$. Since the traces agree on $Q_{n,+}$ and $P_{n,+}$, the resulting Frobenius-Perron eigenvectors on Λ_+ and Γ_+ agree up to depth n, and the hypotheses of Theorem 3.11 are satisfied. Thus $\Lambda_+ = \Gamma_+$.

Finally, by counting dimensions once more, we have $Q_{k,+} = P_{k,+}$ for all $k \geq 0$, and thus $Q_\bullet = P_\bullet$. □

We now have all the tools necessary to prove Popa's Principal Graph Stability Theorem 1.2 and our Theorem 1.3.

Proof of Theorem 1.2 By Proposition 3.10, there is a planar subalgebra $Q_\bullet \subseteq P_\bullet$ with principal graphs (Λ_+, Λ_-) such that $\Lambda_\pm(n+1) = \Gamma_\pm(n+1)$ and $\Lambda_\pm$ is stable at depth k for all $k \geq n$. By Theorem 2.13, $\Lambda_\pm$ is finite and obtained from the truncation $\Lambda_\pm(n+1)$ by adding A_{finite} tails. Finally, by Corollary 3.12, $\Gamma_\pm = \Lambda_\pm$, so $Q_\bullet = P_\bullet$. □

Proof of Theorem 1.3 By Proposition 3.10, there is a planar subalgebra $Q_\bullet \subseteq P_\bullet$ with principal graphs (Λ_+, Λ_-) such that $\Lambda_+(n+2) = \Gamma_+(n+2)$, Λ_+ is stable at depth j for all $j \geq n$, and Λ_- is stable at depth k for all $k \geq n+1$. By Theorem 2.13, $\Lambda_\pm$ is finite and obtained from the truncation $\Lambda_\pm(n+1)$ by adding A_{finite} tails. Finally, by Corollary 3.12, $\Gamma_\pm = \Lambda_\pm$, so $Q_\bullet = P_\bullet$. □

4 Applications

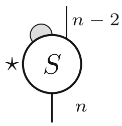
4.1 Jellyfish and spokes

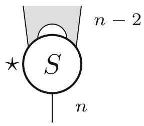
Recall that a subfactor planar algebra $P_\bullet$ is called k *supertransitive* if k is maximal such that $TL_{k,\pm} = P_{k,\pm}$. Let $P_\bullet$ be an $(n-1)$ supertransitive subfactor planar algebra with $n < \infty$. In particular, $P_\bullet \neq TL_\bullet$.

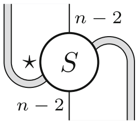
Definition 4.1 We call a set $\mathcal{S}_+ \subset P_{n,+}$ a set of 2-*strand jellyfish generators for* $P_\bullet$ if

(1) (Trains span) Trains from $\mathcal{S}_+$ span $P_\bullet$, and
(2) (Structure algebra) $\operatorname{span}(\mathcal{S}_+ \cup \{f^{(n)}\}) \subseteq P_{n,+}$ is an algebra under the usual multiplication.

Remark 4.2 Note that if $\mathcal{S}_+$ is a set of 2-strand jellyfish generators for $P_\bullet$, then we also have

- (TL-capping) 

, 

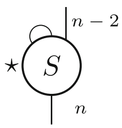
$\in TL_{n-1,\pm}$ for all $S \in \mathcal{S}_+$.
- (Rotational closure) For each $S \in \mathcal{S}_+$, 

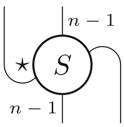
$\in \operatorname{span}(\mathcal{S}_+) \oplus TL_{n,+}$.

Definition 4.3 We call a set $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$ with $\mathcal{S}_\pm \subseteq P_{n,\pm}$ a set of 1-*strand jellyfish generators for* $P_\bullet$ if

(1) (Trains span) Trains from $\mathcal{S}_\pm$ span $P_\bullet$.
(2) (Structure algebra) $\operatorname{span}(\mathcal{S}_+ \cup \{f^{(n)}\}) \subset P_{n,+}$ and $\operatorname{span}(\mathcal{S}_- \cup \{\check{f}^{(n)}\}) \subset P_{n,-}$ are algebras under the usual multiplication.

Remark 4.4 As in Remark 4.2, if $\mathcal{S}$ is a set of 1-strand jellyfish generators, then we also have

- (TL-capping) 

$\in TL_{n-1,\pm}$ for all $S \in \mathcal{S}_\pm$.
- (Rotational closure) For each $S \in \mathcal{S}_\pm$, 

$\in \operatorname{span}(\mathcal{S}_\mp) \oplus TL_{n,\mp}$.

Remark 4.5 If $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$ is a set of 1-strand jellyfish generators for $P_\bullet$, then $\mathcal{S}_+$ is a set of 2-strand jellyfish generators for $P_\bullet$, and $\mathcal{S}_-$ is a set of 2-strand jellyfish generators for the dual of $P_\bullet$ (obtained by reversing the shading).

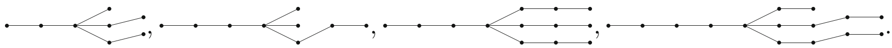
Definition 4.6 A *simply laced spoke graph* is a tree with two distinguished vertices $\star$ and c such that $\star$ has valence 1 and every vertex except possibly c has valence at most 2.

In general, a *spoke graph* is a graph obtained from a simply laced spoke graph by replacing some edges with multiple edges. Further, we require these multiple edges to be incident to c, but not include the edge from c in the direction of $\star$.

For a (dual) principal graph Γ to be a spoke graph, we require that $\star$ be the basepoint of Γ.

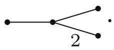
Remark 4.7 Since $P_\bullet$ is $n-1$ supertransitive, If $\Gamma_\pm$ is a spoke graph, then c is at depth $n-1$.

Example 4.8 Some examples of finite simply laced spoke graphs are the 2221, 3311, 3333, and 4442 principal graphs:



An example of an infinite simply laced spoke graph is the D_∞ principal graph

Examples of spoke graphs that are not simply laced are the principal graphs of fixed-point subfactors $R^G \subset R$ for G non-abelian, e.g., $G = S_3$:



Theorem 4.9 *Suppose $P_\bullet$ is an $(n-1)$ supertransitive subfactor planar algebra with $\delta > 2$ and principal graphs (Γ_+, Γ_-). The following are equivalent.*

(1) $P_{n,+} \cup P_{n,-}$ is a set of 1-strand jellyfish generators for $P_\bullet$.
(2) Γ_+ and Γ_- are finite spoke graphs.
(3) $\Gamma_+(n+1)$ and $\Gamma_-(n+1)$ are spoke graphs.

Proof

(1) $\Rightarrow$ (2): Since trains from $P_{n,\pm}$ span $P_\bullet$, $\Gamma_\pm$ is stable at depth k for all $k \geq n$ by Theorem 3.6. By Theorem 2.13, Γ_+, Γ_- are finite.

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Note that (Γ_+, Γ_-) is stable at depth n, so let $Q_\bullet$ be the subfactor planar algebra generated by $P_{n,\pm}$ as in Proposition 3.10, and note that trains from $P_{n,\pm}$ span $Q_\bullet$. Since $P_\bullet$ is $(n-1)$ supertransitive, $P_{n,+} \cup P_{n,-}$ is a set of 1-strand jellyfish generators for $Q_\bullet$. Finally, by Popa's Principal Graph Stability Theorem 1.2, $Q_\bullet = P_\bullet$. □

Theorem 4.10 *Suppose $P_\bullet$ is an $(n-1)$ supertransitive subfactor planar algebra with $\delta > 2$ and principal graph Γ_+. The following are equivalent.*

(1) $P_{n,+}$ is a set of 2-strand jellyfish generators for $P_\bullet$.
(2) Γ_+ is a finite spoke graph, and Γ_- is stable at depth k for all $k \geq n+1$.
(3) $\Gamma_+(n+2)$ is a spoke graph.

Proof

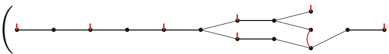
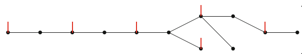
(1) $\Rightarrow$ (2): Since trains from $P_{n,+}$ span $P_\bullet$, Γ_+ is stable at depth k for all $k \geq n$ by Theorem 3.6. By Lemma 3.2, trains from $P_{n+1,-}$ span P_-, so again Γ_- is stable at depth k for all $k \geq n+1$. By Theorem 2.13, Γ_+ is finite (and thus so is Γ_-).

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Note that Γ_+ is stable at depths n and $n+1$, so let $Q_\bullet$ be the subfactor planar algebra generated by $P_{n,\pm}$ as in Proposition 3.10, and note that trains from $P_{n,+}$ span $Q_\bullet$. Since $P_\bullet$ is $(n-1)$ supertransitive, $P_{n,+}$ is a set of 2-strand jellyfish generators for $Q_\bullet$. Finally, by Theorem 1.3, $Q_\bullet = P_\bullet$. □

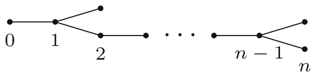
Remark 4.11 If $P_\bullet$ is a subfactor planar algebra with principal graphs (Γ_+, Γ_-), and if Γ_+ and Γ_- are simply laced spoke graphs, then $\Gamma_+ = \Gamma_-$. The traces of projections that are dual to each other must be equal, and thus the Frobenius-Perron dimensions of vertices of Γ_+, Γ_- at odd depths must agree.

Corollary 4.12 *There is no set of jellyfish generators in $P_{6,\pm}$ (1 or 2-strand) for the Asaeda-Haagerup subfactor planar algebra [1] with principal graphs*

$(\Gamma_+, \Gamma_-) = \Big($  ,  $\Big)$.

Proof This is immediate from Theorems 4.9 and 4.10 and Remark 4.5. □

Proposition 4.13 *Recall there are n non-isomorphic subfactor planar algebras with principal graphs $(D^{(1)}_{n+2}, D^{(1)}_{n+2})$ for $4 \le n < \infty$ [8,24].*

$D^{(1)}_{n+2} =$ 

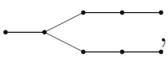
If $P_\bullet$ is such a subfactor planar algebra, then $P_\bullet$ is not generated by $P_{2,\pm}$.

Proof Let $Q_\bullet$ be the subfactor planar algebra generated by $P_{2,\pm}$ as in Proposition 3.10, and note that trains from $P_{2,\pm}$ span $Q_\bullet$. If $Q_\bullet$ has principal graphs (Λ_+, Λ_-), then $\Lambda_\pm$ is stable at depth k for all $k \ge 2$, so $\Lambda_\pm = D_\infty$. □

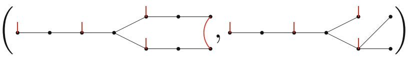
Remark 4.14 In [15], Morrison and Penneys give a planar algebra presentation by generators and relations for the $A^{(1)}_{2n-1}$ and $D^{(1)}_{n+2}$ planar algebras using jellyfish of different sizes. The $A^{(1)}_{2n-1}$ planar algebras are generated by one 2-box and two n-boxes, and the $D^{(1)}_{n+2}$ planar algebras are generated by one 2-box and one n-box. The differences in the relations for each of the n distinct subfactor planar algebras are the rotational eigenvalues of the n-boxes.

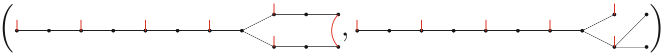
Definition 4.15 Recall from [20] that *translating* a principal graph means attaching an A_k graph to the left, and *extending* means adding additional edges and vertices to the right, where by convention, the basepoint $\star$ corresponding to the empty diagram is always at the left, and vertices are placed left to right corresponding to depth.

Corollary 4.16 *If Γ_+ is a translated extension of*

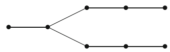
 ,

then (Γ_+, Γ_-) is one of

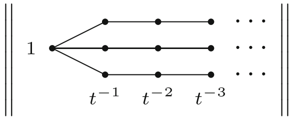
$\mathcal{H} =$  *or*

$\mathcal{EH} =$  .

Proof By the classification of subfactors with index 4 [24],



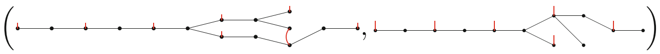
is not the principal graph of a subfactor, so Γ_+ must be a nontrivial translated extension, and thus $\delta > 2$. Thus by Theorem 4.10, Γ_+ is a finite spoke graph. By [10], the modulus of a finite depth subfactor planar algebra is equal to the norm of its principal graph, so

$\delta = \|\Gamma_+\| <$  $= \sqrt{4.5} < \sqrt{5}$

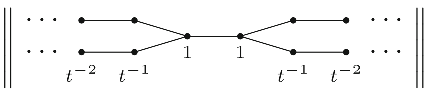
(by Lemma A.4 of [18], the infinite graph above has a strictly positive ℓ^2-eigenvector whose weights are given by the labels above corresponding to eigenvalue $t + t^{-1}$ where $t = \sqrt{2}$. The norm of the infinite graph is then $t + t^{-1}$ by Theorems 4.4 and 6.2 of [22]). By the classification of subfactors below index 5 [7,18,20,27], we know $(\Gamma_+, \Gamma_-) \in \{\mathcal{H}, \mathcal{EH}\}$. □

Remark 4.17 The classification of subfactors to index 5 can be used to completely classify all subfactor planar algebras $P_\bullet$ of modulus $\delta > 2$ whose principal graph Γ_+ is a tree with no vertices of degree greater than 3 and at most two triple points. Note that Γ_+ must be finite by Theorem 2.13, and $2 < \delta = \|\Gamma_+\|$ by [10].

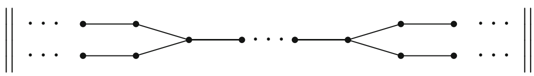
If Γ_+ has exactly one triple point, then the same argument as in Corollary 4.16 shows that $(\Gamma_+, \Gamma_-) \in \{\mathcal{H}, \mathcal{EH}\}$. If Γ_+ has exactly two triple points, and $(\Gamma_-, \Gamma_+) \notin \{\mathcal{H}, \mathcal{EH}\}$, then

$(\Gamma_\pm, \Gamma_\mp) = \mathcal{AH} =$  .

To see this, note that

 $= \sqrt{5}$

(once again, the infinite graph has a strictly positive ℓ^2-eigenvector corresponding to eigenvalue $t + t^{-1}$ where $t = \frac{1}{2}(1 + \sqrt{5})$). A simple induction argument shows that if we subdivide the simple edge between the two triple points in the infinite graph above, the norm will decrease, i.e.,

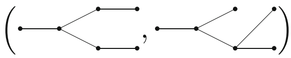
 $< \sqrt{5}$

(see [2, 3.1.2]). Hence if Γ_+ has exactly two triple points, then $\|\Gamma_+\| < \sqrt{5}$, and the claim follows. Finally, note there is exactly one subfactor planar algebra with each of the principal graphs $\mathcal{H}, \mathcal{EH}, \mathcal{AH}$ [1,3].

4.2 Another proof of the quadratic tangles formula

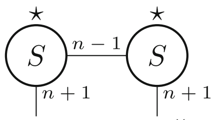
The annular multiplicities of a subfactor planar algebra with $\delta > 2$ were defined in [11,12] in terms of the decomposition of $P_\bullet$ into irreducible annular Temperley-Lieb modules. Using only this decomposition, Jones obtains a formula, [12, Theorem 5.1.11], for annular multiplicities $*10$ subfactor planar algebras which gives a strong restriction to possible principal graphs. In fact, this formula was used to rule out various weeds in the classification of subfactors to index 5 [18].

By Ocneanu's triple point obstruction [5,12,18], if $P_\bullet$ has annular multiplicities $*10$, then $(\Gamma_\pm, \Gamma_\mp)$ is a translated extension of



We now provide another proof of [12, Theorem 5.1.11] using the fact that one of the graphs above is not a spoke graph. By passing to the dual of $P_\bullet$ if necessary (i.e., by switching the shading), we may assume (Γ_+, Γ_-) are translated extensions of the above graphs.

The statement of [12, Theorem 5.1.11] uses the following notation.

- $[k] = (q^k - q^{-k})/(q - q^{-1})$, where $[2] = q + q^{-1} = \delta$,
- $n \in \mathbb{N}$ is such that $P_\bullet$ is $(n-1)$ supertransitive,
- $\check{r} \geq r \geq 1$ is the ratio of the projections at depth n of Γ_-, Γ_+ respectively (by calculating Frobenius-Perron dimensions, $\check{r} = [n+2]/[n]$),
- $S \in P_{n,+}$ is a low-weight rotational eigenvectors with eigenvalue ω_S,
- $\sigma_S = \omega_S^{1/2}$, which is determined by $\check{r} \geq r \geq 1$,
- $\check{S} = \sigma_S^{-1} \mathcal{F}(S) \in P_{n,-}$, where $\mathcal{F}(S)$ is the one click rotation of S,
- $\{\cup_i(S) | 0 \leq i \leq 2n+1\}$ is the basis of annular consequences of S, and $\{\widehat{\cup}_i(S) | 0 \leq i \leq 2n+1\}$ is the dual basis, and similarly for $\check{S}$,
- $S \circ S =$  is the quadratic tangle which lies in annular consequences, and similarly for $\check{S} \circ \check{S}$, and
- $W_{k,\omega_S} = q^k + q^{-k} - \omega_S - \omega_S^{-1}$.

Our proof of Jones' result only uses Jones' formulas for the dual basis $\widehat{\cup}_i(S)$'s in terms of the annular basis $\cup_i(S)$. (For annular multiplicities $*10$, $S \circ S$ lies in annular consequences, so taking inner products is easy.)

Proposition 4.18 *If $P_\bullet$ has annular multiplicities $*10$, then there is no set of 1-strand jellyfish generators for $P_\bullet$ in $P_{n,+}$. Moreover, n is even, and*

$$r + \frac{1}{r} = 2 + \frac{2 + \omega_S + \omega_S^{-1}}{[n+2][n]}.$$

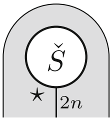
Remark 4.19 Before we prove Proposition 4.18, we will briefly explain the idea of the argument. Since $P_\bullet$ has annular multiplicities $*10$, we can write $S \circ S$ as a linear combination of the $\cup_i(S)$. If the coefficient of $\cup_0(S) = j(\check{S})$ is non-zero, then

$$j(\check{S}) \in \operatorname{span}(\text{trains}_{n+1,+}(\{S\}),$$

and similarly swapping + with − and S with $\check{S}$. A priori, one would not expect either of these coefficients to vanish, so one might expect that $\{S\} \cup \{\check{S}\}$ is a set of 1-strand jellyfish generators for $P_\bullet$. However, since Ocneanu's triple point obstruction tells us Γ_- is not a spoke graph, we cannot have 1-strand jellyfish generators, so at least one of these coefficients must vanish, giving some constraint.

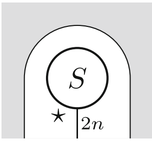
Proof of Proposition 4.18 Since Γ_- is not a spoke graph, the first claim follows from Theorem 4.9.

To prove the quadratic tangles constraint, note that since Γ_+ is stable at depth n, by Theorem 3.6,



$$\in P_{n+1,+} = \operatorname{span}(\text{trains}_{n+1,+}(\{S\})),$$

and since Γ_- is not a spoke graph, by Theorems 3.6 and 4.9,



$$\notin \operatorname{span}(\text{trains}_{n+1,-}(\{\check{S}\})).$$

Hence the coefficient of $j(S) = \cup_0(\check{S})$ in $\check{S} \circ \check{S}$ must be zero. Since $\check{S}$ is uncappable, $\langle \check{S} \circ \check{S}, \cup_i(\check{S})\rangle = 0$ for $i \neq 0, n+1$, so by [12, Proposition 4.2.9 (ii)],

$$\begin{aligned}
0 &= \langle \check{S} \circ \check{S}, \widehat{\cup}_0(\check{S})\rangle \\
&= \left\langle \check{S} \circ \check{S}, \frac{[2n+2]}{W_{2n+2,\omega_S}} \cup_0(\check{S}) + \frac{[n+1]}{W_{2n+2,\omega_S}}((-\sigma_S)^{n+1} + (-\sigma_S)^{-n-1})\cup_{n+1}(\check{S})\right\rangle \\
&= \frac{[2n+2]}{W_{2n+2,\omega_S}} \operatorname{Tr}(S^3) + \sigma_S^n \frac{[n+1]}{W_{2n+2,\omega_S}}((-\sigma_S)^{n+1} + (-\sigma_S)^{-n-1}) \operatorname{Tr}(\check{S}^3). \qquad (1)
\end{aligned}$$

If n is odd, then $\operatorname{Tr}(S^3) = \pm \operatorname{Tr}(\check{S}^3)$ (by sphericality), and thus

$$q^{n+1} + q^{-n-1} = \frac{[2n+2]}{[n+1]} = \pm(\sigma_S + \sigma_S^{-1}) \leq 2,$$

which is impossible if $q > 1$. Now substituting

$$\operatorname{Tr}(S^3) = \frac{r^{1/2} - r^{-1/2}}{[n+1]^{1/2}}, \ \operatorname{Tr}(\check{S}^3) = \frac{\check{r}^{1/2} - \check{r}^{-1/2}}{[n+1]^{1/2}}, \quad \text{and} \quad \check{r} = \frac{[n+2]}{[n]}$$

into Eq. (1), it simplifies to

$$(r^{1/2}-r^{-1/2})[2n+2]-(\sigma_S+\sigma_S^{-1})\left(\left(\frac{[n+2]}{[n]}\right)^{1/2}-\left(\frac{[n+2]}{[n]}\right)^{-1/2}\right)[n+1]=0.$$

Solving for $r^{1/2}-r^{-1/2}$ and squaring gives the desired equation after using the identity

$$[2n+2]^2-[n+1]^2([n+2]^2+[n]^2-2[n+2][n])=0.$$

□

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References

1. Asaeda, M., Haagerup, U.: Exotic subfactors of finite depth with Jones indices $(5+\sqrt{13})/2$ and $(5+\sqrt{17})/2$. Comm. Math. Phys. **202**(1), 1–63 (1999). arXiv:math.OA/9803044. doi:10.1007/s002200050574
2. Brouwer, A.E., Haemers, W.H.: Spectra of graphs (2012). http://www.win.tue.nl/~aeb/2WF02/spectra.pdf
3. Bigelow, S., Morrison, S., Peters, E., Snyder, N.: Constructing the extended Haagerup planar algebra. Acta Math. **209**(1), 29–82 (2012). arXiv:0909.4099. doi:10.1007/s11511-012-0081-7
4. Goodman, F.M., de la Harpe, P., Jones, V.F.R.: Coxeter graphs and towers of algebras. Mathematical Sciences Research Institute Publications, vol. 14, Springer, New York (1989)
5. Haagerup, U.: Principal graphs of subfactors in the index range $4<[M:N]<3+\sqrt{2}$. Subfactors (Kyuzeso, 1993), pp. 1–38. World Science Publication, River Edge, NJ (1994)
6. Han, R.: A construction of the “2221” planar algebra. Ph.D. thesis, University of California, Riverside (2010). arXiv:1102.2052
7. Izumi, M., Jones, V.F.R., Morrison, S., Snyder, N.: Subfactors of index less than 5, Part 3: Quadruple points. Comm. Math. Phys. **316**(2), 531–554 (2012). arXiv:1109.3190. doi:10.1007/s00220-012-1472-5
8. Izumi, M., Kawahigashi, Y.: Classification of subfactors with the principal graph $D_n^{(1)}$. J. Funct. Anal. **112**(2), 257–286 (1993). doi:10.1006/jfan.1993.1033
9. Jones, V.F.R.: Index for subfactors. Invent. Math. **72**(1), 1–25 (1983). doi:10.1007/BF01389127
10. Jones, V.F.R.: Subfactors of II_1 factors and related topics. Proceedings of the International Congress of Mathematicians (Berkeley), pp. 939–947 (1986)
11. Jones, V.F.R.: The annular structure of subfactors. Essays on geometry and related topics, vol. 1, 2, Monogr. Enseign. Math., vol. 38, Enseignement Math., Geneva, pp. 401–463 (2001)
12. Jones, V.F.R.: Quadratic tangles in planar algebras. Duke. Math. J. **161**(12), 2257–2295. doi:10.1215/00127094-1723608
13. Jones, V.F.R.: Notes on planar algebras. http://math.berkeley.edu/~vfr/VANDERBILT/pl21.pdf (2011)
14. Jones, V.F.R., Penneys, D.: The embedding theorem for finite depth subfactor planar algebras. Quantum Topol. **2**(3), 301–337 (2011). arXiv:1007.3173. doi:10.4171/QT/23
15. Morrison, S., Penneys, D.: The affine A and D planar algebras (2012) In preparation
16. Morrison, S., Penneys, D.: Constructing spoke subfactors using the jellyfish algorithm (2012). arXiv:1208.3637, to appear in Trans. Am. Math. Soc.
17. Morrison, S., Peters, E.: The little desert? Some subfactors with index in the interval $(5, 3+\sqrt{5})$. (2012). arXiv:1205.2742
18. Morrison, S., Penneys, D., Peters, E., Snyder, N.: Classification of subfactors of index less than 5, part 2: triple points. Int. J. Math. **23**(3), 1250016 (2012). arXiv:1007.2240. doi:10.1142/S0129167X11007586

19. Morrison, Scott, Peters, E., Snyder, N.: Skein theory for the $\mathcal{D}_{2n}$ planar algebras. J. Pure Appl. Algebra **214**(2), 117–139 (2010). arXiv:math/0808.0764. doi:10.1016/j.jpaa.2009.04.010
20. Morrison, S., Snyder, N.: Subfactors of index less than 5, part 1: the principal graph odometer. Commun. Math. Phys. **312**(1), 1–35 (2012). arXiv:1007.1730. doi:10.1007/s00220-012-1426-y
21. Morrison, S., Walker, K.: Planar algebras, connections, and Turaev-Viro theory. preprint available at http://tqft.net/tvc
22. Mohar, B., Woess, W.: A survey on spectra of infinite graphs. Bull. Lond. Math. Soc. **21**(3), 209–234 (1989)
23. Peters, E.: A planar algebra construction of the Haagerup subfactor. Int. J. Math. **21**(8), 987–1045 (2010). arXiv:0902.1294. doi:10.1142/S0129167X10006380
24. Popa, S.: Classification of amenable subfactors of type II. Acta Math. **172**(2), 163–255 (1994). doi:10.1007/BF02392646
25. Popa, S.: An axiomatization of the lattice of higher relative commutants of a subfactor. Invent. Math. **120**(3), 427–445 (1995). doi:10.1007/BF01241137
26. Popa, S.: Some ergodic properties for infinite graphs associated with subfactors. Ergodic Theory Dynam. Syst. **15**(5), 993–1003 (1995). doi:10.1017/S0143385700009731
27. Penneys, D., Tener, J.: Classification of subfactors of index less than 5, part 4: vines. Int. J. Math. **23**(3), 1250017 (18 pages) (2012). arXiv:1010.3797. doi:10.1142/S0129167X11007641