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Abstract:	We give a diagrammatic definition of $U_q(sl_2)$ when q is not a root of unity, including the Hopf algebra structure and relationship with the Temperley-Lieb category.
Response to Reviewers:	1) There are not a lot of references in the paper... I extended the introduction and added some references. 2) ... a clear explanation of the work of this paper and the work of Frenkel and Khovanov in [FK97] should be given... I did my best. My introduction of a pole is, I think, the most important difference. 3) Can you give some intuition into why one uses the pole in the definition of TL^dot? I added a couple of sentences about this. 4) The definition of the map eta is missing in the definition of TL^dot. Fixed. Editor's question: Corollary 6.6 says that U_q(sl2) (as defined in Kassel) maps onto a certain subalgebra of H' but does not say this map is injective... This was indeed sloppy. It's injective when q is not a root of unity. I added section 7 and gave a proof of this. I changed claims from earlier and in the abstract to make them more accurate.

A DIAGRAMMATIC DEFINITION OF $U_q(\mathfrak{sl}_2)$

STEPHEN J. BIGELOW

ABSTRACT. We give a diagrammatic definition of $U_q(\mathfrak{sl}_2)$ when q is not a root of unity, including the Hopf algebra structure and relationship with the Temperley-Lieb category.

Quantum groups, knot diagrams, skein theory.

1. INTRODUCTION

This paper is about $U_q(\mathfrak{sl}_2)$, one of the simplest examples of a quantum group. For an account of the early history of quantum groups and some of their applications, see [3]. Our goal is to give a definition of $U_q(\mathfrak{sl}_2)$ and its representation theory using formal linear combinations of certain diagrams in the plane. This diagrammatic approach to algebra has origins that go back to the use of Feynman diagrams in physics. For a survey of this and some of its varied applications, see [1].

The Temperley-Lieb category **TL** is a category of certain representations of $U_q(\mathfrak{sl}_2)$. The morphisms are represented by linear combinations of Temperley-Lieb diagrams. We will define a category $\mathbf{TL}^\bullet$ that contains **TL**, but also allows diagrams with interior endpoints and orientations.

Next we will define a Hopf algebra H, whose diagrams include a vertical pole. If we work over $\mathbb{C}$ and q is not a root of unity then we find $U_q(\mathfrak{sl}_2)$ as a subalgebra of a quotient of H. The relationship between $U_q(\mathfrak{sl}_2)$ and **TL** is described by the process of threading a Temperley-Lieb diagram in place of the pole in a diagram in H.

Orientations appeared in the earliest applications of Temperley-Lieb diagrams to ice-type models in statistical mechanics, such as [9]. The orientations in $\mathbf{TL}^\bullet$ are very similar, and also satisfy the *ice rule*, which says that every crossing has two arrows pointing in and two pointing out.

Orientations again appeared in work of Frenkel and Khovanov [2]. Their idea is that, whereas a Temperley-Lieb diagram represents a linear map between representations of $U_q(\mathfrak{sl}_2)$, an oriented Temperley-Lieb diagram represents a single matrix entry of that linear map. Such diagrams form a category that is basically the same as our $\mathbf{TL}^\bullet$.

Although it does not use the same oriented diagrams, similar ideas are covered in the "Kyoto path model", pioneered by Kashiwara [4] and others.

More recently, Lauda [8] used diagrammatic methods to define a categorified $U_q(\mathfrak{sl}_2)$. It is not clear whether our definitions can also be categorified, or how they relate to Lauda's.

The main feature of our work that seems to be new is the pole, which will let us describe both $U_q(\mathfrak{sl}_2)$ and its representation theory in the same picture. One advantage of this is that the key "intertwining" relationship becomes visually

obvious. It is proved in Theorem 4.1 by physically sliding one action through the other. This is reminiscent of Morton's diagrammatic proofs that certain elements of the Temperley-Lieb algebra commute [10]. I hope our approach makes the algebra more accessible to others like me who are more comfortable with tangle diagrams and skein relations.

Throughout the paper, we work over a field $\mathbb{F}$ containing an element q that is neither 0 nor ± 1. We will also need square roots $\sqrt{q}$ and $\sqrt{-q}$.

2. The category $\mathbf{TL}^\bullet$

In this section, we define a monoidal category $\mathbf{TL}^\bullet$.

We start with a quick review of the Temperley-Lieb category $\mathbf{TL}$. The objects are the non-negative integers. The morphisms from n to m are formal linear combinations of Temperley-Lieb diagrams that have n endpoints at the bottom and m at the top. Composition is by stacking. A closed loop can be deleted in exchange for the scalar $q + q^{-1}$. The tensor product of objects is given by $n \otimes m = n + m$. The tensor product $f \otimes g$ of two diagrams f and g is obtained by placing f to the left of g.

We also allow diagrams with crossings, which are defined as follows:

$$\times = \sqrt{-q} \, \Big) \ \Big(+ \frac{1}{\sqrt{-q}} \, \asymp \, .$$

Crossings satisfy Reidemeister moves two and three (as proved in [6]).

We extend $\mathbf{TL}$ to $\mathbf{TL}^\bullet$ by introducing diagrams with univalent vertices. A vertex is the endpoint of a strand, lying in the interior of the diagram. We require that, at every vertex, the strand must have a horizontal tangent vector, and must be given an orientation either into or out of the vertex. Unlike ordinary Temperley-Lieb diagrams, a diagram with vertices is not considered up to planar isotopy. Instead, we only allow planar isotopies that preserve the horizontal tangent vector at each vertex. We also impose the following *turning*, *confetti*, and *cutting* relations.

The *turning relations* let us rotate a vertex at the expense of a power of $\sqrt{q}$.

$$\sqrt{q} \; \overset{\bullet}{\subset} \; = \; \bullet\!\!-\!\!\bullet \; = \; \frac{1}{\sqrt{q}} \; \underset{\bullet}{\subset} \; , \qquad \frac{1}{\sqrt{q}} \; \overset{\bullet}{\subset} \; = \; \bullet\!\!-\!\!\bullet \; = \; \sqrt{q} \; \underset{\bullet}{\subset} \; .$$

The *confetti relations* let us eliminate any straight strand that has univalent vertices at both ends.

$$\bullet\!\!-\!\!\!\to\!\!\!-\!\!\bullet \; = \; \bullet\!\!-\!\!\!\leftarrow\!\!\!-\!\!\bullet \; = \; 0, \qquad \bullet\!\!\!\to\!\!\!-\!\!\bullet \; = \; 1.$$

The *cutting relation* lets us replace a strand with a sum of "cut" strands with the two possible orientations.

$$-\!\!\!-\!\!\!- \; = \; \to\!\!\bullet \; \bullet\!\!\to\!\!- \; + \; \to\!\!\!-\!\!\bullet \; \bullet\!\!\leftarrow \; .$$

Note that univalent vertices do not interact particularly well with crossings. There is no relation to let you pass a strand over or under a vertex, and the orientation on a strand may change when it goes through a crossing.

3. The Hopf algebra H

In this section, we define a Hopf algebra H consisting of formal linear combinations of certain diagrams. The diagrams in H are similar to those in $\mathbf{TL}^\bullet$, but with a special straight vertical edge called the *pole*. No other strands are allowed to have endpoints on the top or bottom of the diagram. Strands are allowed to cross over or under the pole. You can think of the pole as a kind of place holder. In the next section we will replace it with arbitrary numbers of parallel strands.

The turning, confetti, and cutting relations from $\mathbf{TL}^\bullet$ still hold in H. We also allow Reidemeister moves involving the pole. That is, we impose the relations

and their horizontal reflections. (Other versions of Reidemeister three follow from Reidemeister two and the definition of a crossing.)

The product $\nabla\colon H \otimes H \to H$ is such that, for diagrams x and y, $\nabla(x \otimes y)$ is obtained by stacking x on top of y. We write xy for $\nabla(x \otimes y)$.

The unit $\eta\colon \mathbb{F} \to H$ is such that $\eta(1)$ is the diagram that is empty except for the pole.

The tensor product $x \otimes y$ of two diagrams x and y in H is obtained by placing x to the left of y, resulting in a diagram with two poles. In general, any diagram z with two poles represents an element of $H \otimes H$. If z contains strands that go from one pole to the other, then use the cutting relation to write z as a sum of tensor products of diagrams from H.

The coproduct $\Delta\colon H \to H \otimes H$ acts on any diagram by splitting the pole into two parallel poles. Every crossing where a strand passes over (or under) the pole becomes a pair of crossings where the strand passes over (or under) both poles.

The counit ϵ acts on any diagram x by deleting the pole. The result is a scalar multiple of the empty diagram in $\mathbf{TL}^\bullet$, and $\epsilon(x)$ is defined to be that scalar.

The antipode $S\colon H \to H$ acts on any diagram by a planar isotopy that rotates the pole clockwise through an angle of 180 degrees. Throughout the isotopy, we must preserve the horizontal tangent vectors at every vertex. The result is the same as rigidly rotating the diagram and then multiplying the result by $\sqrt{q}$ to the power of the number of inward oriented vertices minus the number of outward oriented vertices.

Proposition 3.1. *H satisfies the axioms of a Hopf algebra.*

Proof. It is easy to check that H satisfies the axioms of a bialgebra. It remains to check that the antipode satisfies:

$$\nabla \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = \nabla \circ (\mathrm{id} \otimes S) \circ \Delta.$$

Here is a schematic representation of the effect of $\nabla \circ (S \otimes \mathrm{id}) \circ \Delta$ on a diagram:

First Δ splits the pole into two. Then $S \otimes \mathrm{id}$ rotates the left pole clockwise, bringing it above the other pole. As usual, the vertices do not rotate throughout this isotopy. Although the resulting diagram is oddly shaped and has one pole on

top of the other, it still represents an element of $H \otimes H$ by the same construction as when the poles are side by side. Finally, ∇ joins the two poles so that the tensor product becomes multiplication by stacking.

Consider the last of the above sequence of four diagrams. The curved part of the rectangle represents a collection of parallel strands that can be moved off the pole, one by one, using Reidemeister two. Thus the entire collection of strands can be slid off the pole to the right. We can then use an isotopy to straighten out the rectangle again. The overall effect is to delete the pole from the original diagram and insert a new pole some distance to the left. But this exactly describes the action of $\eta \circ \epsilon$. Thus

$$\nabla \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon.$$

An upside-down version of this argument works for $\nabla \circ (\mathrm{id} \otimes S) \circ \Delta$. $\square$

4. Representations of H

If h is a diagram in H, let $\rho(h)$ be the diagram in $\mathbf{TL}^\bullet$ given by replacing the pole in h with an ordinary strand. Extend ρ by linearity to an algebra morphism from H to the algebra of automorphisms of the object 1 in $\mathbf{TL}^\bullet$. Note that ρ is a two-dimensional representation of H, since its codomain is isomorphic to the algebra of two-by-two matrices over $\mathbb{F}$.

Using the coproduct on H, if h is a diagram in H then $\rho^{\otimes n}(h)$ is the diagram in $\mathbf{TL}^\bullet$ given by threading n parallel strands in place of the pole.

The most important relationship between H and $\mathbf{TL}$ is that their actions "intertwine" as follows.

Theorem 4.1. *Suppose $h \in H$ and f is a morphism in $\mathbf{TL}$ from n to m. Then*

$$\rho^{\otimes m}(h) \circ f = f \circ \rho^{\otimes n}(h)$$

in the category $\mathbf{TL}^\bullet$.

Proof. We can assume h and f are diagrams. To obtain $\rho^{\otimes m}(h) \circ f$, replace the pole in h with m parallel strands and attach f to the bottom. To obtain $f \circ \rho^{\otimes n}(h)$, replace the pole in h with n parallel strands and attach f to the top. The resulting diagrams represent the same element of $\mathbf{TL}^\bullet$, since we can use Reidemeister moves to slide f through h. $\square$

5. Generators and relations in H

We define the following elements of H.

$$e = \text{⬦─│─⬦}, \qquad e_0 = \text{⬦─│─⬦}, \quad k = \text{⬦─│─⬦}, \qquad k' = \text{⬦─│─⬦},$$

$$f = \text{⬦─│─⬦}, \qquad f_0 = \text{⬦─│─⬦}, \quad \ell = \text{⬦─│─⬦}, \qquad \ell' = \text{⬦─│─⬦}.$$

Lemma 5.1. *H is generated by the above eight elements.*

Proof. Start with an arbitrary diagram in H. Apply the definition of a crossing to eliminate any crossings that do not involve the pole. Use the cutting relation to cut all strands into segments that cross the pole at most once. Use the turning relations to straighten out all of the strands. Finally, use the confetti relations to eliminate any strands that do not cross the pole. We are left with only horizontal

segments that cross the pole exactly once. The eight generators consist of every pair of orientations for either type of crossing. $\square$

We now give the Hopf algebra structure of H. To save space, we only list the four generators in which the strand passes under the pole. These calculations remain the same if we switch the crossing.

Lemma 5.2. *In H, the coproduct satisfies:*
$$\Delta(e) = e \otimes k + k' \otimes e, \quad \Delta(e_0) = e_0 \otimes k' + k \otimes e_0,$$
$$\Delta(k) = k \otimes k + e_0 \otimes e, \quad \Delta(k') = k' \otimes k' + e \otimes e_0,$$
the counit satisfies:
$$\epsilon(e) = \epsilon(e_0) = 0, \quad \epsilon(k) = \epsilon(k') = 1,$$
and the antipode satisfies:
$$S(e) = qe, \quad S(e_0) = q^{-1}e_0, \quad S(k) = k', \quad S(k') = k.$$

Proof. These follow immediately from the definitions. $\square$

We list some relations satisfied by the generators of H. We do not attempt a complete presentation of H, since we will soon be taking a quotient anyway.

Lemma 5.3. *H satisfies the relations*
- $k'k + q^{-1}ee_0 = 1$,
- $kk' + qe_0e = 1$,
- $ek' + qk'e = 0$,
- $ef - fe = (q - q^{-1})(\ell k - k'\ell')$.

Proof. The first three relations follow from Reidemeister two:
$$\vcenter{\hbox{\includegraphics}} = \sqrt{q}, \quad \vcenter{\hbox{\includegraphics}} = \frac{1}{\sqrt{q}}, \quad \vcenter{\hbox{\includegraphics}} = 0.$$

The fourth relation follows from Reidemeister three:
$$\vcenter{\hbox{\includegraphics}} = \vcenter{\hbox{\includegraphics}}.$$

In each case, we can express the equation of diagrams in terms of the generators of H, using the method described in the proof of Lemma 5.1. After some algebraic manipulation, we obtain the desired relations. $\square$

6. A QUOTIENT OF H

Let H' be the quotient of H by the intersection of the kernels of all $\rho^{\otimes n}$. The aim of this section is to prove the following.

Theorem 6.1. *H' has generators e, f, k and k^{-1}, which satisfy the relations:*
$$kk^{-1} = k^{-1}k = 1$$
and
$$ek = -q^{-1}ke, \quad fk = -qkf, \quad ef - fe = (q - q^{-1})(k^2 - k^{-2}).$$
The Hopf algebra structure on H' is given by the coproduct:
$$\Delta(e) = e \otimes k + k^{-1} \otimes e, \quad \Delta(f) = f \otimes k + k^{-1} \otimes f, \quad \Delta(k^{\pm 1}) = k^{\pm 1} \otimes k^{\pm 1},$$

the counit:

$$\epsilon(e) = \epsilon(f) = 0, \quad \epsilon(k^{\pm 1}) = 1,$$

and the antipode:

$$S(e) = qe, \quad S(f) = q^{-1}f, \quad S(k^{\pm 1}) = k^{\mp 1}.$$

To save us some work, we use the following symmetry of H.

Definition 6.2. The *Cartan involution* of H is the linear map $\theta \colon H \to H$ that acts on a diagram by rotating it 180 degrees around the pole, and reversing the direction of all arrows.

Thus θ permutes the generators of H as follows.

$$e \leftrightarrow f, \quad e_0 \leftrightarrow f_0, \quad k \leftrightarrow \ell', \quad k' \leftrightarrow \ell.$$

Lemma 6.3. *The Cartan involution is a bialgebra automorphism of H, preserves the kernel of $\rho^{\otimes n}$ for all n, and satisfies $\theta \circ S = S^{-1} \circ \theta$.*

Proof. The bialgebra operations on H have diagrammatic descriptions that commute with θ. We can also say $\rho^{\otimes n}$ commutes with H, if we interpret θ as acting on **TL**$^\bullet$ in the obvious way. Finally, θ does not commute with S, but instead reverses the direction of rotation of the pole in the definition of S. $\square$

Lemma 6.4. *For all $n \geq 0$, we have:*

$$\rho^{\otimes n}(e_0) = \rho^{\otimes n}(f_0) = 0, \quad \rho^{\otimes n}(k) = \rho^{\otimes n}(\ell), \quad \rho^{\otimes n}(k') = \rho^{\otimes n}(\ell').$$

Proof. The proof is by induction on n. The case $n = 0$ is easy. The case $n = 1$ is a simple computation involving diagrams with a single crossing. For $n > 1$, use the formulae for the coproduct taken from Lemma 5.2 and Lemma 6.3. $\square$

To prove Theorem 6.1, let $k^{-1} = k'$ and combine Lemmas 5.2, 5.3, 6.3 and 6.4.

7. Connection to $U_q(\mathfrak{sl}_2)$

In this section we make the connection between $U_q(\mathfrak{sl}_2)$ and H'.

We use the definition of $U_q(\mathfrak{sl}_2)$ given in [5] and [7]. The most interesting relations are:

$$EK = q^{-2}KE, \quad FK = q^2 KF, \quad EF - FE = (K - K^{-1})/(q - q^{-1}),$$
$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1,$$
$$S(E) = -EK^{-1}, \quad S(F) = -KF.$$

The other relations are the definition of ϵ and some obvious relations involving only $K^{\pm 1}$.

Define $\phi \colon U_q(\mathfrak{sl}_2) \to H'$ by

$$\phi(K^{\pm 1}) = k^{\pm 2}, \quad \phi(E) = \frac{1}{q - q^{-1}}ek, \quad \phi(F) = \frac{1}{q - q^{-1}}k^{-1}f.$$

Theorem 7.1. *The above ϕ is a well-defined morphism of Hopf algebras. The image of ϕ is the algebra of words of even length in the generators e, f and $k^{\pm 1}$. The representation $\rho \circ \phi$ of $U_q(\mathfrak{sl}_2)$ is isomorphic to $V_{-1,1}$. The kernel of ϕ is the intersection of the kernels of the representations $V_{-1,1}^{\otimes n}$.*

Proof. To check that ϕ is well defined, simply check all of the defining relations of $U_q(\mathfrak{sl}_2)$. To see that the image of ϕ is as claimed, note that it is easy to convert any word of even length to a power of $-q$ times a word in the image of ϕ.

We can compute $\rho \circ \phi$ completely, or just enough to identify it by a process of elimination. We know that $\rho \circ \phi$ is a two-dimensional representation, so it is either trivial, $V_{1,1}$, or $V_{-1,1}$. But

$$\rho \circ \phi(E) \neq 0,$$

so it is not trivial. Also,

$$\rho \circ \phi(KE) = (-q)\rho \circ \phi(E)$$

so K has an eigenvalue $-q$, and the representation must be $V_{-1,1}$.

The statement about the kernel of ϕ follows immediately from the definition of H'. $\qquad\square$

Corollary 7.2. *If $\mathbb{F} = \mathbb{C}$ and q is not a root of unity then $U_q(\mathfrak{sl}_2)$ is isomorphic to the algebra of words of even length in the generators e, f and $k^{\pm 1}$ of H'.*

Proof. We must show that ϕ is injective. We use basic properties of representations of $U_q(\mathfrak{sl}_2)$ when q is not a root of unity.

By [7, Theorem 7.13], the intersection of the kernels of the representations $V_{1,n}$. is trivial. But $V_{1,n}$ is a summand of $V_{1,1}^{\otimes n}$, so the intersection of the kernels of tensor powers of $V_{1,1}$ is trivial. There is an isomorphism from $U_q(\mathfrak{sl}_2)$ to $U_{-q}(\mathfrak{sl}_2)$ that switches $V_{-1,1}$ and $V_{1,1}$, so the intersection of the kernels of tensor powers of $V_{-1,1}$ is also trivial. Thus the kernel of ϕ is trivial. $\qquad\square$

Even if q is a root of unity, Theorem 4.1 shows that **TL** is a category of all tensor powers of the representation $V_{-1,1}$ of $U_q(\mathfrak{sl}_2)$, and some of the morphisms between them. If q is not a root of unity then **TL** includes *all* such morphisms, but I do not know a diagrammatic proof of this fact.

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