Alexander Representation of Tangles

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Abstract A tangle is an oriented 1-submanifold of the cylinder whose endpoints lie on the
two disks in the boundary of the cylinder. Using an algebraic tool developed by Lescop, we
extend the Burau representation of braids to a functor from the category of oriented tangles
to the category of $\mathbb{Z}[t, t^{-1}]$-modules. For $(1, 1)$-tangles (i.e., tangles with one endpoint on
each disk), this invariant coincides with the Alexander polynomial of the link obtained by
taking the closure of the tangle. We use the notion of plat position of a tangle to give a
constructive proof of invariance in this case.

1 Introduction

The Burau representation was the first non trivial representation of the braid groups. It was
defined by writing an explicit matrix in $GL_n(\mathbb{Z}[t, t^{-1}])$ for every generator of B_n, and

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verifying that these matrices satisfy the braid relations. Later, it was reinterpreted with the point of view of mapping class group and action on *Burau modules*, which are homology modules of the punctured disk with twisted coefficients in $\mathbb{Z}[t, t^{-1}]$, as a particular case of the Magnus representation [4]. This representation was also the first endowed with a unitary structure, and extensively studied for its (non)-faithfulness.

Several extensions of the Burau representation have been considered. The first was due to Le Dimet [14] for string-links, and studied in more detail by Kirk and Livingston [11]. An extension for the tangle category was defined by Cimasoni and Turaev [8, 9]; it takes value in the Lagrangian category, which contains that of $\mathbb{Z}[t, t^{-1}]$ -modules.

An important aspect of braid group representations is their connection with polynomial invariants of knots. One reconstructs the Alexander polynomial of a closed braid from the Burau representation as $\det(id - \rho)$. The other works on the extension of the representation to string links or tangles contain some relations with the Alexander polynomial, but no direct formula.

In this paper, we use an algebraic tool introduced by Lescop for link complements to construct an isotopy invariant for oriented tangles. This invariant is related to the Alexander module of the tangle exterior in the cylinder. This approach is functorial: the invariant provides a functor from the category of oriented tangles to the category of $\mathbb{Z}[t, t^{-1}]$-modules and homomorphisms up to multiplication by a unit and for $(1, 1)$-tangles, it coincides with the Alexander polynomial of the closure of the tangle. Our construction is a particular case of a more general construction developed in [10].

Section 2 is devoted to the construction of the invariant. In Section 3, we compute the value of the invariant for the generators of the category. In Section 4, the invariant is computed explicitly when the tangle is written as the plat closure of an oriented braid. This illustrates the similarities with the approach of Lawrence [12] for the Jones polynomial and with the model of Bigelow for a link presented as a plat closure. In the last section, we prove the functoriality of the invariant.

It should be noted that the homology of symmetric spaces of the punctured disk (or configuration spaces), with some twisted coefficients, coincides with the exterior algebra of the Burau modules. The modules can also be endowed with a Hermitian structure coming from the exterior power of the twisted intersection form. In particular, for $(1, 1)$-tangles, the calculation could be interpreted as an intersection (with twisted coefficients) of a set of curves in the 2-disk.

It is worth mentioning that the model of Bigelow for the Jones polynomial inspired some construction of categorification. Our approach here should similarly give rise to a categorification of the Alexander polynomial-presumably some form of Heegaard Floer homology.

2 The Main Construction

2.1 The Alexander Functions of a Module

Let $R = \mathbb{Z}[t, t^{-1}]$ be the Laurent polynomial ring. Let H be an R-module of finite type. Consider a free resolution of H of the form

$$R^p \xrightarrow{A} R^{p+k} \longrightarrow H \longrightarrow 0. \tag{1}$$

In other words, (1) determines a k-deficiency presentation of H over R with $(p + k)$ generators $\gamma_1, \ldots, \gamma_{p+k}$ and p relators $\rho_1, \ldots, \rho_p$. Given such a presentation, it is possible to define an R-linear map $\wedge^k H \longrightarrow R$ as follows (see [13, Section 3.1]).

Let $\hat{\rho} = \rho_1 \wedge \cdots \wedge \rho_p$ and $\hat{\gamma} = \gamma_1 \wedge \cdots \wedge \gamma_{p+k}$. The *Alexander function of H relative to k* is the R-linear map $\varphi = \varphi(H, k) \colon \wedge^k H \to R$ defined by

$$u \wedge \hat{\rho} = \varphi(u) \cdot \hat{\gamma} \tag{2}$$

for each $u \in \wedge^k H$. For fixed k, different k-deficient presentations will give rise to Alexander functions that differ only by multiplication by a unit in R.

Note that if $\hat{u} = u_1 \wedge \cdots \wedge u_k$ and $A \in \mathbf{M}_{p \times (p+k)}(R)$ is the matrix associated to the presentation (1), then $\varphi(\hat{u})$ coincides with the determinant of the matrix constructed adding to A the k columns $u_1, \ldots, u_k$ expressed in terms of the generators γ_i.

Remark 2.1 From (1), by tensoring with $\mathbb{Q}(t)$, we obtain a presentation of deficiency k for $H \otimes \mathbb{Q}(t)$. So, if $k < \mathrm{rank} H = \dim_{\mathbb{Q}(t)} H \otimes \mathbb{Q}(t)$, then φ is the zero map. Moreover, if H is free of rank k, then φ coincides with the volume form

$$\wedge^k H \xrightarrow{\cong} R$$

induced by the choice of $\hat{\gamma}$ as a basis of $\wedge^k H$, i.e., $\varphi(\gamma_1 \wedge \cdots \wedge \gamma_k) = 1$.

2.2 The Alexander Invariant of a Tangle

Let D be the closed unitary disk in $\mathbb{C}$. Consider two copies of the disk D with fixed ordered finite sequences of points $p_0, \ldots, p_{k_-}$ (respectively $p_0, \ldots, p_{k_+}$) on the real line. Let ε^- and ε^+ be sequences of signs ± 1 of length $k_- + 1$ and $k_+ + 1$, respectively. An $(\varepsilon^-, \varepsilon^+)$-*tangle* is an oriented 1-submanifold τ of $D \times I$ whose oriented boundary $\partial \tau$ is

$$\sum_{i = 0}^{k_+} \varepsilon_i^+ (p_i, 1) - \sum_{i = 0}^{k_-} \varepsilon_i^- (p_i, 0). \tag{3}$$

Note that if such a tangle exists then $\sum_{i=1}^{k_+} \varepsilon_i^+ = \sum_{i=1}^{k_-} \varepsilon_i^-$.

Let D_- and D_+ be the punctured disks $(D \times \{0\}) \setminus \tau$ and $(D \times \{1\}) \setminus \tau$, respectively. Any sequences of signs $\varepsilon^\pm$ determine epimorphisms $\chi_\pm \colon \pi_1(D_\pm, *) \to \mathbb{Z} = \langle t \rangle$ sending a simple loop x_i turning once around p_i in the counterclockwise direction to $t^{\varepsilon_i^\pm}$. Let $H_\pm$ be the R-module $H_1^{\chi_\pm}(D_\pm; R)$, where the coefficients are twisted by $\chi_\pm$. Equivalently, $H_\pm$ is the first homology of the infinite cyclic covering induced by $\chi_\pm$.

Remark 2.2 The disk $D_\pm$ deformation retracts to the wedge of circles $x_0 \ldots x_{k_\pm}$. So the R-module $H_\pm$ is freely generated by $u_i^\pm = \tilde{x}_i - \tilde{x}_{i+1}$, for $i = 0, \ldots, k_\pm - 1$, where $\tilde{x}_i$ is a lift of x_i. More precisely, $u_i^\pm$ will be a loop around both p_i and p_{i+1} if these punctures have opposite signs, or a figure eight around them if they have the same sign.

Let B_τ be the exterior of τ in $D \times I$. For each connected component τ_j of τ, let m_j be a meridian around τ_j oriented so that its linking number with τ_j is one.

Since $H_1(B_\tau) = \oplus_{j=1}^n \mathbb{Z} m_j$ (see [8, Section 3.3]), the composition of the Hurewicz homomorphism with the homomorphism $H_1(B_\tau) \to \mathbb{Z}\langle t \rangle$ sending m_i to t, gives an epimorphism $\chi \colon \pi_1(B_\tau) \to \mathbb{Z}$ extending both χ_+ and χ_-. We set $H = H_1^\chi(B_\tau; R)$ and denote by $i_\pm$ the maps from $H_\pm$ to H, induced by the inclusion.

Let $\delta k = \frac{k_+ - k_-}{2}$. Our aim is to construct linear maps $\rho_{\tau, \star}$, of degree δk, between the R-modules $\wedge^\star H_-$ and $\wedge^\star H_+$.

For any generator w_+ of $\wedge^{k_+} H_+ \simeq R w_+$, and r with $0 \leq r \leq k_+$, consider the isomorphism

$$d_r : \wedge^r H_+ \longrightarrow \mathrm{Hom}(\wedge^{k_+ - r} H_+; R),$$

defined by the formula $x \wedge y = (d_r(x)(y)) w_+$, for any $x \in \wedge^r H_+$ and $y \in \wedge^{k_+ - r} H_+$.

Let $k = \frac{k_- + k_+}{2}$. We will show in Section 4 that there exists a free deficiency k resolution for H (see Lemma 1). For any i with $0 \leq i \leq k$, the choice of generators $\hat{\rho}$ for $\wedge^p R^p$ and $\hat{\gamma}$ for $\wedge^{k+p} R^{k+p}$ similarly induces isomorphisms

$$\varphi_i : \wedge^i H \longrightarrow \mathrm{Hom}(\wedge^{k-i} H; R),$$

defined by the formula $x \wedge y \wedge \hat{\rho} = (\varphi_i(x)(y)) \hat{\gamma}$ in $\wedge^{k+p} R^{k+p}$ for any $x \in \wedge^i H$ and $y \in \wedge^{k-i} H$ (see Section 2.1).

Definition 1 For all $i \in \{0, \ldots, k\}$, let the homomorphism $\rho_{\tau, i} : \wedge^i H_- \to \wedge^{i+\delta k} H_+$ be defined by the following composition

$$\wedge^i H_- \xrightarrow{\wedge^i i_-} \wedge^i H \xrightarrow{\varphi_i} \mathrm{Hom}(\wedge^{k-i} H; R) \xrightarrow{(\wedge^{k-i} i_+)^*} \mathrm{Hom}(\wedge^{k-i} H_+; R) \xrightarrow{d_{i+\delta k}^{-1}} \wedge^{i+\delta k} H_+.$$

The map $\rho_\tau = \oplus_i \rho_{\tau, i}$ is an isotopy invariant of τ, defined up to a global multiplication by a unit of R. We call it the *Alexander invariant* of τ.

The special case of $(1, 1)$-tangles, corresponding to $k_- = k_+ = 0$, illustrates the relation with the Alexander polynomial.

Theorem 1 *Let τ be a $(1, 1)$-tangle and let $\hat{\tau}$ be the link obtained by closing up together the two free strands of τ. Then ρ_τ is equal to $m_{\Delta(\hat{\tau})}$ where $\Delta(\hat{\tau})$ is the Alexander polynomial of $\hat{\tau}$ and $m_u : \wedge^k H_- \to \wedge^k H_+$ denotes the multiplication by u.*

Proof Since $k_- = k_+ = 0$, it follows that $H_\pm = \{0\}$ and $\wedge^0 \{0\} = R$. Hence ρ_τ is a homomorphism $R \to R$ that is the multiplication by the determinant of a square presentation matrix of H. By definition, this determinant is the Alexander polynomial of $\hat{\tau}$. $\square$

3 The Tangle Category

Let $\mathcal{T}$ denote the oriented tangle category, that is, the category whose objects are sequences $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)$ of signs $\pm$ attached to the punctures of a punctured disk, and whose morphisms are oriented tangles.

Let $\mathcal{M}$ be the category whose objects are R-modules and whose morphisms are classes of homomorphisms modulo the equivalence relations defined by $f \sim g$ if there exists a unit $u \in R$ such that $f = m_u \circ g$ where m_u denotes the multiplication by u. There is a functor from $\mathcal{T}$ to $\mathcal{M}$ taking $\varepsilon^- \xrightarrow{\tau} \varepsilon^+$ to the class of modules homomorphisms $H_-(D_{\varepsilon^-}) \xrightarrow{\rho_\tau} H_+(D_{\varepsilon^+})$. We postpone the proof of this statement to the last section, and, in the following, we compute ρ_τ on the generators of the tangle category.

3.1 Braids

Let us start with an oriented identity braid τ with $k+1$ strands. Thus H is free of rank k, and $\varphi \colon \wedge^k H \to R$ is an isomorphism and coincides with the volume form induced by the choice of a basis. Moreover, i_- and i_+ are isomorphisms, and the choice of $\det_+$ induces a canonical choice of φ, such that the following diagram commutes

$$\wedge^k H_+ \xrightarrow{\ i_+\ } \wedge^k H$$

$$\det_+ \searrow \qquad \swarrow \varphi$$

$$R$$

Let u_- be in $\wedge^i H_-$. From the diagram, for all $w_+ \in \wedge^{k-i} H_+$, we get

$$\det{}_+(i_+^{-1}(i_-(u_-) \wedge i_+(w_+))) = \varphi(i_-(u_-) \wedge i_+(w_+)).$$

By definition,

$$\varphi(i_-(u_-) \wedge i_+(w_+)) = \det{}_+(\rho_\tau(u_-) \wedge w_+).$$

Since $\det_+$ is non singular, we get

$$\rho_\tau(u_-) = i_+^{-1}(i_-(u_-)).$$

Now consider the general case of an oriented braid σ in B_{k+1}, i.e., σ is the isotopy class of a homeomorphism of the $(k+1)$-punctured disk (that we still denote by σ). The geometric realisation of σ as a braid with $(k+1)$-strings in $D \times I$ can be viewed as the mapping cylinder of σ. Let $\{u_1^\pm, \ldots, u_{k_\pm}^\pm\}$ be the basis of $H_\pm$ described in Remark 2.2. Since $i_- \colon H_- \to H$ is still an isomorphism, we can choose $\{i_-(u_1^-), \ldots, i_-(u_{k_-}^-)\}$ as a free basis for H. With respect to these bases, the matrix associated to i_+ is $b(\sigma)^{-1}$, where $b(\sigma)$ is the image of σ by the oriented Burau representation. It follows that, with respect to these bases the matrix associated to $(\rho_\tau)_i \colon \wedge^i H \to \wedge^i H_+$ is $\det(b(\sigma))(\wedge^i(b(\sigma)^{-1}))$. Moreover, the matrices of the oriented Burau representation of the Artin generators $\sigma_1, \ldots, \sigma_{k-1}$ of B_{k+1} in the bases $u_1^\pm, \ldots, u_{k_\pm}^\pm$ are

$$M_{b(\sigma_1)} = \begin{pmatrix} -t^{\varepsilon_2} & 1 \\ 0 & 1 \end{pmatrix} \oplus Id_{k-3}, \ M_{b(\sigma_{k-1})} = Id_{k-3} \oplus \begin{pmatrix} 1 & 0 \\ t^{\varepsilon_k} & -t^{\varepsilon_k} \end{pmatrix}$$

$$M_{b(\sigma_i)} = Id_{i-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ t^{\varepsilon_{i+1}} & -t^{\varepsilon_{i+1}} & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus Id_{k-i-2}.$$

3.2 A Cup

Let *cup* be the tangle in Fig. 1. We have $k_- = k_+ - 2$, $k = k_+ - 1$, $\delta k = 1$ and H free. Note that i_- is a monomorphism and we can choose the bases of H_- and H, such that the matrix for i_- is $I_{k_-} \oplus 0$. On the other hand, i_+ is an epimorphism and the kernel of i_+ has rank one. Using Mayer-Vietoris arguments on a small disk around the points p_{k+1} and p_{k+2} in D_+, we can identify the basis of H with a sub-basis of H_+ so that i_+ is simply a projection.

Let $\alpha \in H_+$ be a generator of the kernel of i_+. Consider the *contraction along* α, that is

$$i_\alpha \colon \mathrm{Hom}(\wedge^k H_+, R) \to \mathrm{Hom}(\wedge^{k+-1} H_+, R)$$
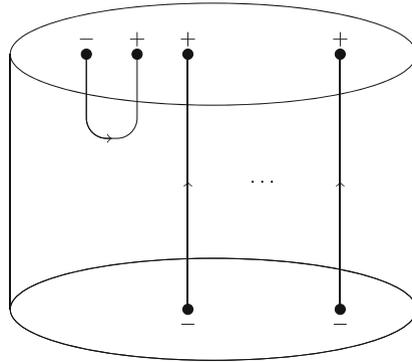
Fig. 1 A cup

such that $i_\alpha(\det_+)(v_+) = \det_+(v_+ \wedge \alpha)$ for any $v_+ \in \wedge^{k_+-1} H_+$. In other words, the following diagram commutes

$$\wedge^{k_+-1} H_+ \xrightarrow{\wedge \alpha} \wedge^{k_+} H$$
$$i_\alpha(\det_+) \searrow \qquad \swarrow \det_+$$
$$R$$

Since $\alpha \in \ker i_+$, one has the following

$$\wedge^{k_+-1} H_+ \xrightarrow{i_+} \wedge^{k_+-1} H$$
$$i_\alpha(\det_+) \searrow \qquad \swarrow \varphi$$
$$R$$

Let u_- be in $\wedge^i H_-$ for $i = 1, \ldots, k$. By definition, for all $w_+ \in \wedge^{k-i} H_+$,

$$\det_+(\rho_\tau(u_-) \wedge w_+) = \varphi(i_-(u_-) \wedge i_+(w_+)).$$

By the previous diagrams,

$$\varphi(i_-(u_-) \wedge i_+(w_+)) = \det_+(i_+^{-1} i_-(u_-) \wedge w_+ \wedge \alpha),$$

where by $i_+^{-1} i_-(u_-)$ we mean any preimage of $i_-(u_-)$, since two of them differ by addition of a multiple of α. From above, since $\det_+$ is non-singular, we get

$$\rho_\tau(u_-) = (-1)^{k-i} i_+^{-1} i_-(u_-) \wedge \alpha.$$

3.3 A Cap

Let *cap* be the tangle in Fig. 2. We have $k_- = k_+ + 2$, $k = k_+ + 1$, $\delta k = -1$ and H free. Here, i_+ is a monomorphism. We write $H = i_+(H_+) \oplus \beta$. Similarly to the previous
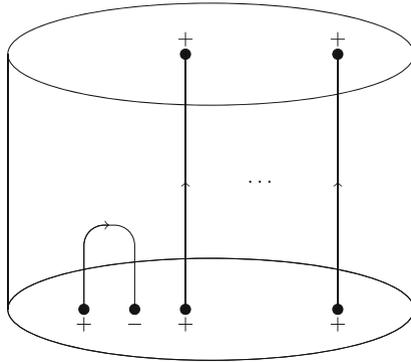
Fig. 2 A cap

section, we can define i_β the contraction along β and we have the following commutative diagrams

$$\wedge^{k_+} H \xrightarrow{\wedge\beta} \wedge^{k_++1} H$$

$$i_\beta(\varphi) \searrow \quad \swarrow \varphi$$

$$R$$

$$\wedge^{k_+} H \xrightarrow{i_\alpha} \wedge^{k_+} H$$

$$\delta_+ \searrow \quad \swarrow i_\beta(\varphi)$$

$$R$$

Let u_- be in $\wedge^i H_-$. For all $w_+ \in \wedge^{k_+-i+1} H_+$, one has, using the definition and the commutativity of the diagrams

$$\varphi(i_-(u_-) \wedge i_+(w_+)) = \det{}_+(\rho_\tau(u_-) \wedge w_+)$$
$$= i_\beta(\varphi)\big(i_+(\rho_\tau(u_-)) \wedge i_+(w_+)\big)$$
$$= \varphi\big(i_+(\rho_\tau(u_-)) \wedge i_+(w_+) \wedge \beta\big).$$

So we get

$$i_-(u_-) = (-1)^{k_+-i+1} i_+(\rho_\tau(u_-)) \wedge \beta.$$

It follows that $\rho_\tau(u_-)$ is zero unless u_- is of the form $u'_- \wedge i_-^{-1}(\beta)$, in which case $\rho_\tau(u_-) = (-1)^{k_+-i+1} i_+^{-1} i_-(u'_-)$. Here, $i_-^{-1}(\beta)$ and $i_+^{-1} i_-(u'_-)$ denote arbitrary elements of the preimage.

4 Plat Position

A system of n disjoint arcs $A_1, \ldots, A_n$ properly embedded in $D \times I$ is called *trivial* if they are not linked and are boundary parallel; more precisely, if there exist n disjoint disks $D_1, \ldots, D_n$ such that $A_i \cap D_i = A_i \subset \partial D_i$, $\partial D_i \setminus A_i \subset \partial(D \times I)$, and $A_j \cap D_i = \emptyset$, for each $i, j = 1, \ldots, n$ with $i \neq j$. A *trivial tangle of type* (k, n) is a tangle whose component are k vertical strands connecting D_+ with D_- and a system of n trivial arcs such that if there exists an arc of τ connecting two points in D_+, then no arc of τ connects two points in D_-.

A *Heegaard surface* for a tangle $\tau \subset D \times I$ is a disk that intersects the tangle transversally and cuts τ into two trivial tangles.

To see the existence of a Heegaard surface for a given tangle, start with a real valued function f on $D \times I$ that is of Morse type on the tangle: typically, we can imagine f as the

height function on $D \times I$. By perturbing the function, we can assume that minima occur before maxima; so if $y = 1/2$ is a regular value separating the minima from the maxima, then $f^{-1}(1/2)$ is the required Heegaard disk.

If we choose a standard model $\tau(k, n)$ for a trivial tangle of type (k, n), the existence of a Heegaard surface for each tangle can be rephrased in the following way: each tangle $(D \times I, \tau)$ can be obtained as

$$(D \times [0, 1/2], \tau(k_-, n_-)) \cup_\sigma (D \times [1/2, 1], \tau(k_+, n_+)),$$

where σ is an automorphism of the $k_- + 2n_- = k_+ + 2n_+$ punctured disk $D \times \{1/2\} \cap \tau$. Since everything can be done up to isotopy, we can think of σ as an element of the braid group $B_{k_-+2n_-}$. We call such a decomposition of a given tangle a *Heegaard splitting* of the tangle.

Lemma 1 *Let τ be a $(\varepsilon^-, \varepsilon^+)$-tangle where the length of $\varepsilon^\pm$ equals $k_\pm + 1$. Then any Heegaard splitting of τ induces a presentation of H with deficiency $k = \frac{k_- + k_+}{2}$.*

Proof Let

$$(D \times I, \tau) = (D \times [0, 1/2], \tau(k_-, n_-)) \cup_\sigma (D \times [1/2, 1], \tau(k_+, n_+))$$

be a Heegaard splitting for τ, with $\sigma \in B_{k_-+2n_-}$. Set

$$X_1 = \left(D \times \left[0, \frac{1}{2}\right]\right) \setminus \tau(k_-, n_-),$$

$$X_2 = \left(D \times \left[\frac{1}{2}, 1\right]\right) \setminus \tau(k_+, n_+),$$

and

$$S = \left(D \times \left\{\frac{1}{2}\right\}\right) \setminus \tau(k_-, n_-) = D \setminus \{p_0, \ldots, p_{k_-+2n_-}\}.$$

Thus, $B_\tau = (D \times I) \setminus \tau = X_1 \cup_\sigma X_2$. By applying the Mayer-Vietoris exact sequence to this decomposition, we obtain the exact sequence of reduced homology groups

$$\ldots \longrightarrow H_1(S) \xrightarrow{i_{S_-} \oplus i_{S_+}} H_1(X_1) \oplus H_1(X_2) \longrightarrow H_1(B_\tau) \longrightarrow 0, \qquad (4)$$

where i_{S_-} is induced by the inclusion $S \hookrightarrow X_1$ while i_{S_+} is induced by $S \xrightarrow{\sigma} \sigma(S) \hookrightarrow X_2$. Let $\chi : \pi_1(B_\tau) \to \mathbb{Z}$ be the epimorphism defining $H = H_1^\chi(B_\tau; R)$. This exact sequence lifts to the following exact sequence of homology with coefficients in R

$$\ldots \longrightarrow H_1^\chi(S) \xrightarrow{i_{S_-} \oplus i_{S_+}} H_1^\chi(X_1) \oplus H_1^\chi(X_2) \longrightarrow H \longrightarrow 0. \qquad (5)$$

This is a presentation of H as an R-module, since $H_1^\chi(S)$, $H_1^\chi(X_1)$, and $H_1^\chi(X_2)$ are free R-modules.

Since X_1, X_2, and S have the homotopy type of a wedge of circles, the homology modules have, respectively, ranks $k_- + n_-$, $k_+ + n_+$, and $k_- + 2n_-$. It follows that the deficiency of the presentation is k. $\qquad \square$

By construction, ρ_τ is well-defined up multiplication by units of R. This implies that changing the Heegaard splitting of τ will change ρ_τ up to multiplication by units of R. Nevertheless, we give a brief constructive proof of this fact, since it illustrates the similarities with the approach of Bigelow for the Jones polynomial [3]. In order to do this, we need to introduce some new definitions.

Given an oriented braid $\sigma \in B_r$, and integers $k_+, k_- \leq r$ of the same parity as r, the (k_+, k_-)-plat closure of σ is the tangle obtained from σ as follows (Fig. 3). Place caps to connect adjacent pairs of endpoints at the top right, so as to leave only the k_+ endpoints at the top left. Similarly, place cups to connect adjacent pairs of endpoints at the bottom right, so as to leave only the k_- endpoints at the bottom left (see Fig. 4). Let $\hat{\sigma}(k_+, k_-)$ denote the (k_+, k_-)-plat closure of σ. Clearly

$$(D \times I, \hat{\sigma}(k_+, k_-)) = (D \times [0, 1/2], \tau(k_-, n_-) \cup_\sigma (D \times [1/2, 1], \tau(k_+, n_+)).$$

In order to relate two braids having the same (k_+, k_-)-plat closure, we define particular subgroups of the braid groups: let Hil_{k+2n} be the subgroup of B_{k+2n} generated by the following braids

- σ_{k+1}
- $\sigma_{k+2}\sigma_{k+1}^2\sigma_{k+2}$
- $\sigma_{k+2i}\sigma_{k+2i-1}\sigma_{k+2i+1}\sigma_{k+2i}$ for $i = 1, \ldots n - 1$,

where $\sigma_1, \ldots \sigma_{k+2n-1}$ are the standard generators of B_{k+2n}. When $k = 0$ the above subgroups are well known and are called Hilden braid groups on $2n$ strings (see [2]).

We have the following theorem that extends to our setting a result of Birman for classical plat closure of braids (see [5]) .

Theorem 2 *Two oriented braids have isotopic (k_+, k_-)-plat closures if and only if they are related by a finite sequence the following moves:*

- $\sigma \to \sigma k$, where $\sigma \in B_{k_-+2n}$ and $k \in \mathrm{Hil}_{k_-+2n}$
- $\sigma \to h\sigma$, where $\sigma \in B_{k_++2n}$ and $h \in \mathrm{Hil}_{k_++2n}$
- $\sigma \leftrightarrow \sigma\sigma_s$, where $\sigma \in B_s$ and $\sigma_s\sigma \in B_{s+2}$.

Proof The proofs of [5, Theorems 1 and 1′] extend to our case. □

Now, we are ready to state our result.

Theorem 3 *Let $\mathcal{H}$ and $\mathcal{H}'$ be two Heegaard splittings of the same $(\varepsilon^-, \varepsilon^+)$ tangle τ. Specifically, suppose $\mathcal{H}$ and $\mathcal{H}'$ are, respectively,*

$$(D \times I, \tau) = (D \times [0, 1/2], \tau(k_-, n_-)) \cup_\sigma (D \times [1/2, 1], \tau(k_+, n_+)),$$
$$(D \times I, \tau) = (D \times [0, 1/2], \tau(k_-, m_-)) \cup_\varsigma (D \times [1/2, 1], \tau(k_+, m_+)).$$
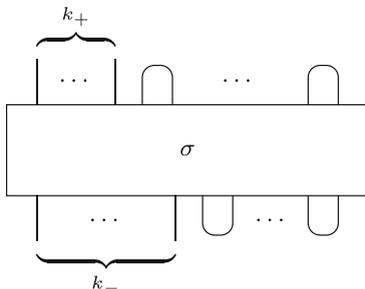
Fig. 3 The (k_+, k_-)-plat closure of σ

Moreover, let ρ_τ and ρ'_τ be the module homomorphisms computed as defined in Section 2 *using the presentations of H associated to the two Heegaard splittings (see Lemma 1). Then* $\rho'_\tau = m_u \circ \rho_\tau$*, where m_u denotes the multiplication by a unit of R.*

Proof We keep the notations of the proof of Lemma 1.

Let $\gamma_1^-, \ldots, \gamma_{k_-}^-, \beta_1^-, \alpha_1^-, \ldots, \beta_{n_-}^-, \alpha_{n_-}^-$ be the basis for $H_1^\chi(S)$ as described in Remark 2.2. Thus, each of these is either a loop or figure eight around two adjacent punctures in S. The γ_i^- involve two punctures connected to strands that connect to the bottom of the tangle. The α_i^- are loops around the two endpoints of a cup, and become zero in the homology of the tangle complement since they could slide off the cup. The β_i^- involve punctures from adjacent cups, or in the case of β_1^-, a cup and a strand. See Fig. 4, but there we assumed the punctures alternate in sign so as to avoid having to draw any figure eight.

Now $\gamma_1^-, \ldots, \gamma_{k_-}^-, \beta_1^-, \ldots, \beta_{n_-}^-$ is a basis for the free R-module $H_1^\chi(X_1)$, and ι_{S_-} is a projection with kernel generated by the α_i^-. Moreover, $\gamma_1^-, \ldots, \gamma_{k_-}^-$ is a basis for the free R-module $H_- = H_1^{\chi_-}(D_-; R)$.

Let $\gamma_1^+, \ldots, \gamma_{k_+}^+, \beta_1^+, \ldots, \beta_{n_+}^+, \alpha_1^+, \ldots, \alpha_{n_+}^+$ be the lifts of the loops of Fig. 5. Then $\gamma_1^+, \ldots, \gamma_{k_+}^+, \beta_1^+, \ldots, \beta_{n_+}^+$ is a basis for the free R-module $H_1^\chi(X_2)$. Note that $k_- + 2n_- = k_+ + 2n_+$.

Let $(f_1, \ldots, f_{k_+})$ be an ordered basis of H_+ as a free R-module and let

$$\det{}_+ : \wedge^{k_+} H_+ \to R$$

be the corresponding determinant. For each sequence I:

$$1 \le i_1 < \cdots < i_n \le k_+ - 1$$

we set $\hat{f}_I = f_{i_1} \wedge \cdots \wedge f_{i_n}$. Moreover, we denote by $\overline{I}$ the sequence complementary to I with respect to $1 < 2 < \cdots < k_+$. If $u_- \in \wedge^i H_-$, then $\rho_\tau(u_-) = \sum_I a_I \hat{f}_I \in \wedge^{i+\delta k} H_+$, where

$$a_I = \det{}_+^{-1}\left(\varphi(H, k)(i_-(u_-) \wedge i_+(\hat{f}_{\overline{I}}))\right)$$

and $\varphi(H, k)$ is the Alexander function associated to the presentation of H induced by $\mathcal{H}$ as in the proof of Lemma 1.
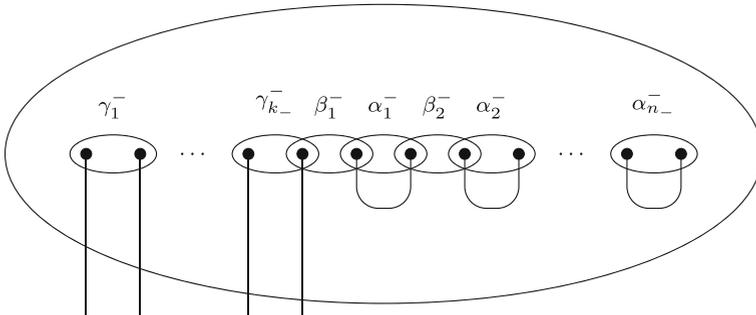
Fig. 4 Curves on S, and the bottom half of the tangle
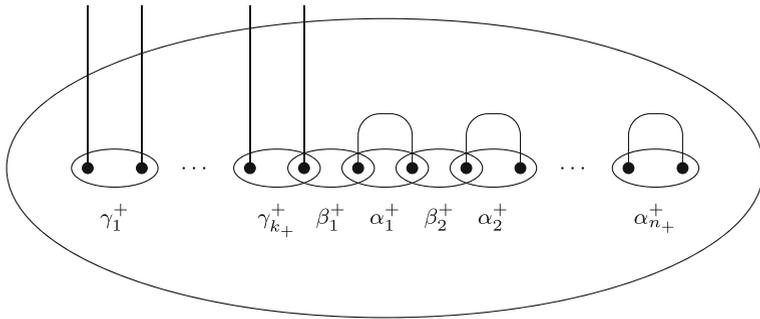
Fig. 5 Curves on S, and the top half of the tangle

Let $b(\sigma)$ denote the Burau matrix of σ, using the above bases. The presentation matrix of H with respect to the basis $(\gamma_i^-, \beta_j^-, \alpha_j^-)$ of $H_1^\chi(S)$ and $(\gamma_i^-, \beta_j^-, \gamma_h^+, \beta_l^+)$ of $H_1^\chi(X_1) \oplus H_1^\chi(X_2)$ is

$$
\begin{pmatrix}
I_{k_-} & 0 & 0 \\
0 & I_{n_-} & 0 \\
b(\sigma)(\gamma^-)_{|\gamma+} & b(\sigma)(\beta^-)_{|\gamma+} & b(\sigma)(\alpha^-)_{|\gamma+} \\
b(\sigma)(\gamma^-)_{|\beta+} & b(\sigma)(\beta^-)_{|\beta+} & b(\sigma)(\alpha^-)_{|\beta+}
\end{pmatrix},
$$

where the terms of the matrix are blocks.

As a basis of H_- (respectively H_+), we can choose the lifts of the loops $(\gamma_1^-, \ldots, \gamma_{k_+}^-)$ (respectively $(\gamma_1^+, \ldots, \gamma_{k_+}^+)$) depicted in Figs. 4 and 5, so we just have to check how the matrix used to compute $\varphi(H, k)(i_-(\hat{\gamma}_I^-) \wedge i_+(\hat{\gamma}_J^+))$ changes under the moves of Theorem 2. Following the definition of Alexander function and the computation of Lemma 1, we have that $\varphi(H, k)(i_-(\hat{\gamma}_I^-) \wedge i_+(\hat{\gamma}_J^+))$ is the determinant of the following matrix

$$
M = \begin{pmatrix}
I_{k_-} & 0 & 0 & E_I^- & 0 \\
0 & I_{n_-} & 0 & 0 & 0 \\
b(\sigma)(\gamma^-)_{|\gamma+} & b(\sigma)(\beta^-)_{|\gamma+} & b(\sigma)(\alpha^-)_{|\gamma+} & 0 & E_J^+ \\
b(\sigma)(\gamma^-)_{|\beta+} & b(\sigma)(\beta^-)_{|\beta+} & b(\sigma)(\alpha^-)_{|\beta+} & 0 & 0
\end{pmatrix},
$$

where

$$
(E_I^-)_{hk} = \begin{cases} 1 & \text{if } k = i_h \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (E_J^+)_{hk} = \begin{cases} 1 & \text{if } k = j_h \\ 0 & \text{otherwise.} \end{cases}
$$

So we want to check how the determinant of the matrix M changes under the moves of Theorem 2.

The last move changes the braid index by two. Let $N = k_- + 2n_-$ and suppose that $\sigma \in B_N$. Then $\varsigma = \sigma\sigma_N \in B_{N+2}$. Moreover, by the functoriality of the Burau

representation, $b(\sigma \sigma_N) = b(\sigma)b(\sigma_N)$. An explicit computation gives the presentation matrix of H, relatively to the Heegaard decomposition $\mathcal{H}'$:

$$
P' = \begin{pmatrix}
I_{k_-} & 0 & 0 & 0 & 0 \\
0 & I_{n_-} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
b(\sigma)(\gamma^-)_{|\gamma^+} & b(\sigma)(\beta^-)_{|\gamma^+} & 0 & b(\sigma)(\alpha^-)_{|\gamma^+} & \\
b(\sigma)(\gamma^-)_{|\beta^+} & b(\sigma)(\beta^-)_{|\beta^+} & 0 & b(\sigma)(\alpha^-)_{|\beta^+} & 0 \\
0 & 0 & -t^{\varepsilon+1} & 0 \ldots 0\, t^{\varepsilon+1} & 1
\end{pmatrix},
$$

with respect to the bases

$$
(\gamma_i^-, \beta_j^-, \beta_{n_-+1}^-, \alpha_j^-, \alpha_{n_-+1}),
$$
$$
(\gamma_i^-, \beta_j^-, \beta_{n_-+1}^-, \gamma_h^+, \beta_l^+, \beta_{n_++1}^+).
$$

It is straightforward to check that the Alexander function associated to this presentation matrix is $\pm \det M$.

The first two moves of Theorem 2 do not change the braid index of σ and so the size of M. So it is enough to check how the matrix M changes by composing σ (on the left or on the right) with a generator of Hil_{k+2n}. We leave the detail of the computations to the reader. □

5 Functoriality

In this section, we prove that the Alexander invariant defined in Section 2 and computed on cups, caps and braids in Section 3, is a functorial invariant of the tangle category.

Lemma 2 *Let τ be a tangle in plat position. The Alexander invariant of τ from Section 4 is the same as the Alexander invariant computed as a product of cups, a braid and caps from Section 3.*

Proof Fix a tangle τ, which consists of a collection of cups at the bottom, a collection of caps at the top, and a braid σ in the middle.

Let γ_i^- be the generators of the homology of the bottom disk of τ, as shown in Fig. 4. Fix γ_I^-, a wedge product of a sequence of these basis vectors.

Let γ_j^+ be the generators of the homology of the top disk of τ, as shown in Fig. 5. Fix $\gamma_{\overline{J}}^-$, a wedge product of a sequence of these basis vectors, where $\overline{J}$ is the complement of J. We will compute the coefficient of $\gamma_{\overline{J}}^-$ in $\rho(\tau)(\gamma_I^-)$.

Let α_i^- be the loops around the cups, as shown in Fig. 4. Let α^- be the wedge product of all of these elements in order. If we apply just the cups from τ to γ_I^-, we obtain $\gamma_I^- \wedge \alpha^-$. Now apply the braid σ, to obtain $b(\sigma)((\gamma_I^-) \wedge (\alpha^-))$.

Let β_i^+ be the loops that go between adjacent caps of τ, as shown in Fig. 5. Let β^+ be the wedge product of all of these elements in order. The coefficient of $\gamma_{\overline{J}}^-$ in $\rho(\tau)(\gamma_I^-)$ is the coefficient of $\gamma_{\overline{J}}^- \wedge \beta^+$ in $b(\sigma)(\gamma_I^-) \wedge b(\sigma)(\alpha^-)$.

Now, we recompute this same coefficient. Using notation from Section 4, the coefficient of $\gamma_{\overline{J}}^-$ in $\rho(\tau)(\gamma_I^-)$ is the determinant of

$$
\begin{pmatrix} E_{\overline{J}}^+ & E_J^+ \end{pmatrix}
$$

times the determinant of

$$
\begin{pmatrix}
I_{k_-} & 0 & 0 & E_I^- & 0 \\
0 & I_{n_-} & 0 & 0 & 0 \\
b(\sigma)(\gamma^-)_{|\gamma+} & b(\sigma)(\beta^-)_{|\gamma+} & b(\sigma)(\alpha^-)_{|\gamma+} & 0 & E_J^+ \\
b(\sigma)(\gamma^-)_{|\beta+} & b(\sigma)(\beta^-)_{|\beta+} & b(\sigma)(\alpha^-)_{|\beta+} & 0 & 0
\end{pmatrix}.
$$

The first of these determinants is ± 1, which we will deal with later. The second matrix will be simplified using column operations that do not change the determinant. Specifically, use the left column of blocks to cancel out the E_I^- in the fourth column of blocks. This gives the following

$$
\begin{pmatrix}
I_{k_-} & 0 & 0 & 0 & 0 \\
0 & I_{n_-} & 0 & 0 & 0 \\
b(\sigma)(\gamma^-)_{|\gamma+} & b(\sigma)(\beta^-)_{|\gamma+} & b(\sigma)(\alpha^-)_{|\gamma+} & -b(\sigma)(\gamma_I^-)_{|\gamma+} & E_J^+ \\
b(\sigma)(\gamma^-)_{|\beta+} & b(\sigma)(\beta^-)_{|\beta+} & b(\sigma)(\alpha^-)_{|\beta+} & -b(\sigma)(\gamma_I^-)_{|\beta+} & 0
\end{pmatrix}.
$$

This has the same determinant as the smaller matrix from its bottom right

$$
\begin{pmatrix}
b(\sigma)(\alpha^-)_{|\gamma+} & -b(\sigma)(\gamma_I^-)_{|\gamma+} & E_J^+ \\
b(\sigma)(\alpha^-)_{|\beta+} & -b(\sigma)(\gamma_I^-)_{|\beta+} & 0
\end{pmatrix}.
$$

Up to sign, this is the same as the determinant of

$$
\begin{pmatrix}
b(\sigma)(\gamma_I^-)_{|\gamma+} & E_J^+ & b(\sigma)(\alpha^-)_{|\gamma+} \\
b(\sigma)(\gamma_I^-)_{|\beta+} & 0 & b(\sigma)(\alpha^-)_{|\beta+}
\end{pmatrix}.
$$

Here, we switched two block columns and changed the sign of one. This may change the sign of the determinant, depending on the parity of the number of basis vectors in γ_I^-. We can ignore this, since we are free to consistently change the sign of every entry of the matrix $\rho(\tau)$.

Finally, observe that the left and right columns of the above matrix represent the terms of $b(\sigma)(\gamma_I^-) \wedge b(\sigma)(\alpha^-)$ that are wedge products of vectors from γ^+ and β^+. The determinant picks out the terms that involve precisely the vectors γ_J^+ and β^+. Such terms are, up to sign, $\gamma_J^+ \wedge \beta^+$. The sign correction is the determinant of

$$
\begin{pmatrix}
E_{\bar{J}}^+ & E_J^+ & 0 \\
0 & 0 & I
\end{pmatrix}.
$$

We conclude that this direct calculation using the methods of Section 4 gave the same answer as composing the operations of cup, braid, and cap. $\qquad\square$

Theorem 4 *The Alexander invariant as computed in Section 3 on cups, braid, and caps determines a functorial invariant of the tangle category.*

Proof A presentation of the tangle category by generators and relations can be found in [15]. Other presentations appear in the literature, sometimes with minor differences. Certainly, cups, caps, and generators of the braid groups are enough to generate the tangle category. By the previous lemma, we have every relation that equates different plat presentations for the same tangle. These, together with some simple commutativity relations, are enough to give defining relations for the tangle category. $\qquad\square$

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