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The pop-switch planar algebra
and the Jones–Wenzl idempotents

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ABSTRACT

The Jones–Wenzl idempotents are elements of the Temperley–Lieb (TL) planar algebra that are important, but complicated to write down. We will present a new planar algebra, the pop-switch planar algebra (PSPA), which contains the TL planar algebra. It is motivated by Jones' idea of the graph planar algebra (GPA) of type A_n. In the tensor category of idempotents of the PSPA, the nth Jones–Wenzl idempotent is isomorphic to a direct sum of $n + 1$ diagrams consisting of only vertical strands.

Keywords: Planar algebra; Jones–Wenzl; Temperley–Lieb.

Mathematics Subject Classification 2010: 57M27

1. Introduction

The Temperley–Lieb (TL) algebras were first introduced by Temperley and Lieb [7] in their work on transfer matrices in statistical mechanics. Vaughn F. R. Jones independently rediscovered TL algebras in his work on von Neumann algebras [4]. He assembled these algebras together to form the TL planar algebra, the simplest example of a subfactor planar algebra.

The Jones–Wenzl idempotents, first introduced in [11], are elements of the TL algebras. One way they arise naturally is in representation theory. The TL algebras encode the category of representations of $U_q(\mathfrak{sl}_2)$, and the Jones–Wenzl idempotents represent the irreducible representations. Chapters in books have been devoted to them [6]. They have been categorified by [1] and [3], and generalized [10].

While important, the Jones–Wenzl idempotents are difficult to write down explicitly. The nth Jones–Wenzl idempotent is a linear combination of every

diagram with n non-intersecting strands. The number of these diagrams is the nth Catalan number. To find the coefficient of a given diagram requires a complicated algorithm originally given by Frankel and Khovanov [2] and later written down by Morrison [8].

In this paper, we define the PSPA, a new planar algebra that contains the TL planar algebra. Our original motivation was a diagrammatic treatment of the graph planar algebra (GPA) introduced by Jones [5]. The PSPA captures with simple diagrams the complicated calculations involved in working with objects in the GPA.

The main theorem of this paper shows that each Jones–Wenzl idempotent is isomorphic to a direct sum of diagrams with only vertical strands. It is to be hoped that this makes them easier to work with, and gives a new approach to some open problems.

2. Background

For convenience, we work over the field $\mathbb{C}$ and let q be a nonzero complex number that is not a root of unity. Many of the results hold over other fields, but if q is a root of unity the proofs fail due to division by zero.

Definition 2.1. The nth *quantum number* is defined as

$$[n] = [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

and the *quantum binomial* is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n][n-1]\cdots[n-k+1]}{[k][k-1]\cdots[1]},$$

where $0 \leq k \leq n$ are natural numbers.

We have the following identities.

Lemma 2.2. $[k+l] = [k][l+1] - [k-1][l]$.

Proof. This follows from the definition and a simple computation. $\square$

Corollary 2.3. $\begin{bmatrix} k+l \\ l \end{bmatrix} = [l+1]\begin{bmatrix} k+l-1 \\ l \end{bmatrix} - [k-1]\begin{bmatrix} k+l-1 \\ l-1 \end{bmatrix}$.

Proof. After taking a common denominator and cancelling common terms, this reduces to the previous lemma. $\square$

2.1. *Planar algebras*

We won't define planar algebras in great detail. See Jones' original paper [4] for a formal definition. See [9] for a helpful introduction.

We will use what are sometimes called *vanilla* planar algebras. These lack any of the optional extra features or properties that are often included in the definition.

A planar tangle T consists of:

- a disk D called the *output disk*,
- a finite set of disjoint disks D_i called the *input disks* in the interior of D,
- a point called a *basepoint* of ∂D and of each ∂D_i, and
- a collection of disjoint curves called *strands* in D.

The strands can be closed curves, or can have endpoints on ∂D or ∂D_i or both. Apart from the endpoints, the strands lie in the interior of D and do not intersect D_i. The basepoints do not coincide with endpoints of strands. Planar tangles are considered up to isotopy in the plane.

It is sometimes possible to insert a planar tangle T_1 into one of the input disks of another planar tangle T_2 to obtain a new planar tangle. Specifically, this is possible if the number of endpoints on the output disk of T_1 is the same as the number of endpoints on the chosen input disk of T_2. Then we can use an isotopy to make the endpoints match up. This still leaves an ambiguity of how to rotate T_1. The basepoints remove this ambiguity: we require the basepoint of the output disk of T_1 to coincide with the basepoint of the chosen input disk of T_2.

The planar tangles, together with this operation of inserting one planar tangle into an input disk of another, form a rather general type of algebraic gadget called an *operad*. Briefly, a planar algebra is a representation of the operad of planar tangles.

More concretely, a planar algebra $\mathcal{P}$ is a sequence of vector spaces $\mathcal{P}_i$ for $i \geq 0$. Suppose T is a planar tangle with input disks $D_1, \ldots, D_n$. Let d_i be the number of endpoints on ∂D_i and let d be the number of endpoints on ∂D. Suppose $v_i \in \mathcal{P}_{d_i}$ for all i. Then there is an action of T

$$T(v_1, \ldots, v_n) \in \mathcal{P}_d.$$

The action of planar tangles must be multilinear, and it must be compatible with the operad structure in a natural sense.

The definition of a planar algebra may seem complicated. However it formalizes a fairly simple idea, familiar to knot theorists, of tangle-like diagrams that can be glued together in arbitrary planar ways. Perhaps the main novelty is that we allow formal linear combinations of diagrams, which glue together in a multilinear way.

An example might help.

2.2. *The Temperley–Lieb planar algebra*

The simplest planar algebra is the Temperley–Leib planar algebra $\mathcal{TL}$. The vector space $\mathcal{TL}_i$ is spanned by tangle diagrams that have no input disks and i endpoints on the output disk.

There is one relation. A closed loop in a diagram may be deleted at the expense of multiplying by the scalar $q + q^{-1}$. We call this the *bubble-bursting relation*.

If i is odd then $\mathcal{T}_i$ is zero. A basis for $\mathcal{T}_{2n}$ is given by tangle diagrams that have n strands and no closed loops.

In practice, most planar algebras can be thought of as formal linear combinations of diagrams that are similar to TL diagrams, but with optional extra features, like crossings, orientations, colors, or vertices.

2.3. *The category corresponding to a planar algebra*

Suppose $\mathcal{P}$ is a planar algebra. We now describe how $\mathcal{P}$ can be thought of as a category. In this context, the input and output disks in the definition of $\mathcal{P}$ should be thought of as rectangles instead of round disks.

The category **C** corresponding to $\mathcal{P}$ is as follows:

- The objects are the non-negative integers.
- The morphisms from i to j are the elements of $\mathcal{P}_{i+j}$, thought of as having i endpoints on the bottom of the rectangle and j on the top.
- The composition $f \circ g$ is given by stacking f on top of g.

  Let $\mathcal{P}_i^j$ denote $\mathcal{P}_{i+j}$ with the elements treated as morphisms from i to j.
  An idempotent is an element p of $\mathcal{P}_n^n$ such that $p^2 = p$.
  We can expand the objects in the category by a construction known as the *Karoubi envelope*. This new category **C**′ is defined as follows:

- The objects of **C**′ are the idempotents of **C**.
- The morphisms from p to q are morphisms in **C** of the form qxp.

Next, note that **C** and **C**′ are also tensor categories, where $x \otimes y$ is obtained by placing x to the left of y.

Finally, we can define a **matrix category** of **C**′. The objects are formal direct sums of objects of **C**′ and the morphisms are formal matrices. Instead of this abstract definition, all we need is the following lemma.

Lemma 2.4. *Suppose p and $q_1, \ldots, q_n$ are idempotents such that*

$$p = q_1 + \cdots + q_n,$$

and $q_i q_j = 0$ whenever $i \neq j$. Then

$$p \simeq q_1 \oplus \cdots \oplus q_n.$$

2.4. *Jones–Wenzl idempotents*

The Jones–Wenzl idempotent p_n is the unique element of $\mathcal{TL}_n^n$ such that

- $p_n \neq 0$
- $p_n^2 = p_n$

- $ap_n = 0$ if a is any diagram that includes a strand with both endpoints at the bottom of the rectangle.
- $p_n b = 0$ if b is any diagram that includes a strand with both endpoints at the top of the rectangle.

Because of these last two properties, the Jones–Wenzl idempotents are sometimes referred to as "uncappable." If q is a root of unity, the Jones–Wenzl idempotents do not exist for all n.

A smaller Jones–Wenzl idempotent may be subsumed into a larger one as follows:

$$(p_n \otimes id_k)p_{n+k} = p_{n+k}$$

where id_n is n nonoriented parallel strands, the multiplicative identity in $\mathcal{LT}_n^n$.

3. The Pop-Switch Planar Algebra

3.1. *The pop-switch planar algebra*

Definition 3.1. Let the pop-switch planar algebra $\mathcal{PSPA}$ be the planar algebra generated by oriented strands modulo the following relations.

- The pop-switch relations

- The bubble-bursting relation

$$\bigcirc + \bigcirc = (q + q^{-1})\epsilon,$$

where ϵ denotes the empty diagram.

This contains the TL planar algebra; a non-oriented strand is the sum of each orientation.

We need some tools to move the diagrams around.

Denote n parallel strands oriented in the same direction by a single oriented strand labelled n.

If n is a negative integer, $\uparrow^n = \downarrow_{-n}$

Let ι_n denote n vertical strands oriented up. Let β_n denote n parallel strands that form a bubble oriented counterclockwise. Let α_n denote a β_{-n} inside a β_n.

$$\iota_n = \uparrow^n \quad \beta_n = \bigcirc^n \quad \alpha_n = \left(\bigcirc^{\curvearrowright n}\right)^n.$$

Lemma 3.2. *Suppose $x \in \mathcal{PSPA}_0$ and y is a sequence of $2n$ vertical strands such that n are oriented up and n are oriented down. Then $x \otimes y = y \otimes x$.*

Proof. Use the pop-switch relation repeatedly to create a gap and pass x through. Then use the pop-switch relation repeatedly to restore the original $2n$ vertical strands. □

Lemma 3.3 (The multi-pop-switch relations). *The pop-switch relations hold for multiple strands.*

Proof. Without loss of generality, consider the first equality. Induct on n. The case $n = 1$ is the pop-switch relations. For the case $n = k + 1$, move the innermost β_{-k} across two strands using the previous lemma. Then use the case $n = k$, and finally the case $n = 1$. □

Corollary 3.4. $\iota_k \otimes \alpha_n = \iota_k$ *and* $\iota_{-k} \otimes \alpha_{-n} = \iota_{-k}$ *for* $k \geq n \geq 0$.

Proof. Consider $\iota_k \otimes \alpha_n$. Use the multi-pop-switch relation by popping the innermost β_n of the α_n. Then straighten out the ι_n. The other case is similar. □

Corollary 3.5.

$$\left(\overset{\curvearrowleft}{\bigcirc}\right)^{\curvearrowright n} = \alpha_n \otimes \beta_{n-1} \quad and \quad \left(\bigcirc\right)^{\curvearrowleft n} = \alpha_{-n} \otimes \beta_{-n+1}.$$

Proof. Start with the left side of the first equality. Use a multi-pop-switch relation on the $n - 1$ strands, as shown below:

By Lemma 3.2 we can move the β_{-n+1} into the α_1 to achieve the result.

$$= \left(\bigcirc^{\curvearrowright n}\right)^{\curvearrowright n} \bigcirc^{n-1} = \alpha_n \otimes \beta_{n-1}.$$

The second identity is proved similarly. □

Lemma 3.6. $\iota_n = \beta_{-n} \otimes \iota_n \otimes \beta_n$.

Proof. This follows from the multi-pop switch relations.

$$\iota_n = \uparrow^n = \overset{n}{\uparrow} = \mho^n \uparrow^n \mho^n = \beta_{-n} \otimes \iota_n \otimes \beta_n. \qquad \square$$

Now we give some relations involving the Jones–Wenzl idempotents p_n. First, we need some notation for them. We will use a rectangle to represent p_n. It should always be assumed that $p_n \in P_n^n$ even if the strands are not drawn.

Notation 3.7.

$$p_n = \boxed{\begin{array}{c} \cdots \\ p_n \\ \cdots \end{array}} = \boxed{}\, |n = \boxed{} = \boxed{}\, |n \; .$$

We can make use of the fact that they are uncappable.

Lemma 3.8. $\quad \overline{}\!\!\bigcup_n \;=\; (-1)^{n+1}\, \overline{}\!\!\bigcup_n \mho^n$.
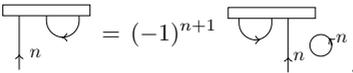
This relation remains true if all arrows are reversed.

Proof. For the case $n = 0$, use the fact that an unoriented cap gives zero. For the case $n = 1$, use the case $n = 0$ and the pop-switch relation.

For the general case, use induction on n. Start by using the case $n = k$ as follows:

$$\overline{}\!\!\bigcup_{k+1} \;=\; \overline{}\!\!\bigcup_k \;=\; (-1)^{k+1}\, \overline{}\!\!\bigcup_k \mho^k .$$

Next use the case $n = 1$, followed by Lemma 3.2, to achieve the result.

$$= (-1)^{k+2}\, \overline{}\!\!\bigcup_k \mho^k \;=\; (-1)^{k+2}\, \overline{}\!\!\bigcup_k \mho^k$$

$$= (-1)^{k+2}\, \overline{}\!\!\bigcup_{k+1} \mho^{k+1} . \qquad \square$$

4. Proof of the Main Theorem

The aim of this section is to prove the following.

Theorem 4.1. *The nth Jones–Wenzl idempotent is isomorphic to a direct sum of $n + 1$ diagrams*:

$$p_n \simeq \bigoplus_{i=0}^{n} \iota_{-i} \otimes \iota_n \otimes \iota_{-i}$$

$$= \uparrow^n \quad \oplus \quad \left| \uparrow^n \right| \downarrow \quad \oplus \cdots \oplus \quad \downarrow_n \left| \uparrow^n \right| \downarrow_n.$$

Proof. Since p_n is an idempotent, $p_n = p_n^2 = p_n \mathrm{id}_n p_n$, where id_n is n nonoriented parallel strands. Now write id_n as a sum of 2^n different ways of orienting n vertical strands. Break this sum into $n+1$ sums depending on how many strands are oriented up.

Definition 4.2. Let p_{n-k}^k denote the sum of $\binom{n}{k}$ diagrams obtained from $p_n \mathrm{id}_n p_n$ by orienting k strands up and $n-k$ strands down in the id_n.

Then $p_n = p_n^0 + p_{n-1}^1 + \cdots + p_0^n$. If $k_1 \neq k_2$, then $p_{n-k_1}^{k_1} p_{n-k_2}^{k_2} = 0$. Thus, by Lemma 2.4,

$$p_n \simeq p_n^0 \oplus p_{n-1}^1 \oplus \cdots \oplus p_0^n.$$

It remains only to show

$$p_l^k \simeq \iota_{-l} \otimes \iota_{k+l} \otimes \iota_{-l}.$$

This is done in Lemma 4.9. □

To prove Lemma 4.9, we first define X_l^k, which we will show is equal a scalar times p_l^k in Lemma 4.8.

Definition 4.3.

$$X_l^k = \quad$$
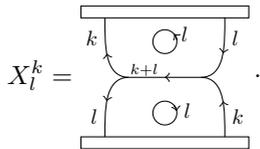
Lemmas 4.4 and 4.5 are similar and begin the inductive step of the proof of Lemma 4.8.

Lemma 4.4. $p_{k+l}(X_l^{k-1} \otimes \iota_1) p_{k+l} = (-1)^l X_l^k \otimes \beta_{-l}.$

Proof.

$$p_{k+l}(X_l^{k-1} \otimes \iota_1) p_{k+l} = \quad$$

By a pop-switch relation we have the following:

$$= \quad \text{(diagram)}$$

Then by Lemma 3.8 we can move the arc across the $l - 1$ strands creating a β_{l-1} on the right. Next we use Lemma 3.6 to replace the arc with $\beta_{-1} \otimes \iota_1 \otimes \beta_1$.

$$= (-1)^l \quad \text{(diagram)}$$

Move the innermost β_1 from the β_l to the far right across $l - 1$ strands in both directions by Lemma 3.2.

$$= (-1)^l \quad \text{(diagram)}$$

Then remove the two bubbles on the bottom right of the diagram by Lemma 3.6.

$$= (-1)^l \quad \text{(diagram)}$$

Lastly, by Lemma 3.2 move the β_{l-1} into the β_1 and the β_{-1} into the $\beta_{-(l-1)}$.

$$= (-1)^l \quad \text{(diagram)}$$

$$= (-1)^l X_l^k \otimes \beta_{-l}.$$ $\qquad \square$

Lemma 4.5. $p_{k+l}(X_{l-1}^k \otimes \iota_{-1})p_{k+l} = (-1)^k X_l^k \otimes \beta_k.$

Proof.

$$p_{k+l}(X_{l-1}^k \otimes \iota_{-1})p_{k+l} = \quad \text{(diagram)}.$$

By a pop-switch relation we have the following:

Then by Lemma 3.8 we can move the arc across the $k-1$ strands creating a $\beta_{-(k-1)}$ on the right. Next we use Lemma 3.2 to move the β_1 across the $l-1$ strands in both directions into the β_{l-1}.
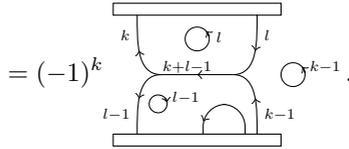
Replace the arc with $\beta_{-1} \otimes \iota_1 \otimes \beta_1$.
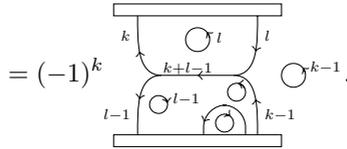
Lastly, by Lemma 3.2 move the β_1 across the $k-1$ strands in both directions into the β_{k-1}. By the same lemma, move the $\beta_{-(l-1)}$ into the β_{-1}.
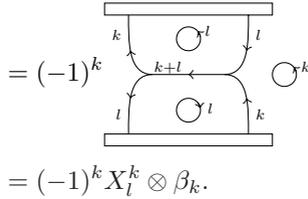
$$= (-1)^k X_l^k \otimes \beta_k. \qquad \square$$

Lemma 4.6 is the key to proving Lemma 4.7, which is required to complete the proof of Lemma 4.8. It is worth noting that in Lemma 4.7 the X_l^k merely acts as a catalyst to provide enough strands to use 4.6. All that is necessary is the presence of ι_{-l+1} and ι_{k-1} on the left as specified in Lemma 4.6 for the purpose of implementing Corollary 3.4.

Lemma 4.6. *For $k \geq n-1$,*

$$\iota_k \otimes \beta_n = [n]\iota_k \otimes \beta_1 - [n-1]\iota_k$$

and

$$\iota_{-k} \otimes \beta_{-n} = [n]\iota_{-k} \otimes \beta_{-1} - [n-1]\iota_{-k}.$$

Proof. We prove the first identity, since the second is similar. Consider the case $n = 2$ with $k \geq 1$. Use the bubble-bursting relation on the innermost loop of β_2. Corollary 3.4 then gives the result.

$$\iota_k \otimes \beta_2 = [2]\iota_k \otimes \beta_1 - \iota_k \otimes \alpha_1 = [2]\iota_k \otimes \beta_1 - \iota_k.$$

Now assume $k \geq n - 1$. Use the bubble-bursting relation on the innermost loop of β_n. Corollaries 3.5, 3.4, and induction give

$$
\begin{aligned}
\iota_k \otimes \beta_n &= [2]\iota_k \otimes \beta_{n-1} - \iota_k \otimes \beta_{n-2} \\
&= [2]([n-1]\iota_k \otimes \beta_1 - [n-2]\iota_k) - ([n-2]\iota_k \otimes \beta_1 - [n-3]\iota_k) \\
&= ([2][n-1] - [n-2])\iota_k \otimes \beta_1 - ([2][n-] - [n-3])\iota_k \\
&= [n]\iota_k \otimes \beta_1 - [n-1]\iota_k.
\end{aligned}
$$ $\square$

Lemma 4.7. *If $k + l = n$ then*

$$\begin{bmatrix} n-1 \\ l \end{bmatrix} X_l^k \otimes \beta_{-l} + \begin{bmatrix} n-1 \\ k \end{bmatrix} X_l^k \otimes \beta_k = \begin{bmatrix} n \\ k \end{bmatrix} X_l^k.$$

Proof. Note that every term in the equation contains X_l^k. However, the result will hold so long as there are both a ι_{-l+1} and ι_{k-1} on the left of each diagram in order to use Lemma 4.6. Thus it suffices to prove

$$\begin{bmatrix} n-1 \\ l \end{bmatrix} ([l]\beta_{-1} - [l-1]) + \begin{bmatrix} n-1 \\ k \end{bmatrix} ([k]\beta_1 - [k-1]) = \begin{bmatrix} n \\ k \end{bmatrix}.$$

Use the identity

$$\begin{bmatrix} n-1 \\ l \end{bmatrix} [l] = \begin{bmatrix} n-1 \\ k \end{bmatrix} [k],$$

and the bubble bursting relation $\beta_{-1} + \beta_1 = [2]$ to eliminate β_{-1} and β_1 from the left side. Then simplify further using the identity $[2][l] - [l-1] = [l+1]$. We obtain

$$\begin{bmatrix} n-1 \\ l \end{bmatrix} [l+1] - \begin{bmatrix} n-1 \\ k \end{bmatrix} [k-1].$$

By Corollary 2.3, this is equal to $\begin{bmatrix} n \\ k \end{bmatrix}$, as desired. $\square$

Lemma 4.8. $p_l^k = (-1)^{kl} \begin{bmatrix} k+l \\ k \end{bmatrix} X_l^k.$

Proof. Induct on $n = k + l$. Notice $p_0^1 = \iota_1 = X_0^1$ and $p_1^0 = \iota_{-1} = X_1^0$. Assume $k > 0$ and $l > 0$. Then

$$p_l^k = p_{k+l}(p_l^{k-1} \otimes \iota_1)p_{k+l} + p_{k+l}(p_{l-1}^k \otimes \iota_{-1})p_{k+l}$$

By Lemmas 4.4 and 4.5,

$$= (-1)^{kl} \begin{bmatrix} k+l-1 \\ l \end{bmatrix} X_l^k \otimes \beta_{-l} + (-1)^{kl} \begin{bmatrix} k+l-1 \\ k \end{bmatrix} X_l^k \otimes \beta_k.$$

By Lemma 4.7,

$$= (-1)^{kl} \begin{bmatrix} k+l \\ k \end{bmatrix} X_l^k. \qquad \square$$

Lemma 4.9. $p_l^k \simeq \iota_{-l} \otimes \iota_{k+l} \otimes \iota_{-l}.$

Proof. The explicit isomorphisms are:

$$f = (-1)^{kl} \begin{bmatrix} k+l \\ k \end{bmatrix} \quad \text{} \quad , \qquad g = \quad \text{} \quad .$$

Then $f \circ g = (-1)^{kl} \begin{bmatrix} k+l \\ k \end{bmatrix} X_l^k = p_l^k$ by Lemma 4.8. Thus $f \circ g$ is the identity morphism from p_l^k to p_l^k.

On the other hand, $g \circ f = \iota_{-l} \otimes \iota_{k+l} \otimes \iota_{-l}$, the identity morphism from $\iota_{-l} \otimes \iota_{k+l} \otimes \iota_{-l}$ to $\iota_{-l} \otimes \iota_{k+l} \otimes \iota_{-l}$.

$$g \circ f = (-1)^{kl} \begin{bmatrix} k+l \\ k \end{bmatrix} \quad \text{}$$

$$= (-1)^{kl} \begin{bmatrix} k+l \\ k \end{bmatrix} \quad \text{}$$

$$= \iota_{-l} \otimes \iota_{k+l} \otimes \iota_{-l}.$$

The second equality holds by performing two multi-pop-switch relations: one on the β_{-l} at the top with the l strands to the left and the l strands on the bottom left, and the other on the β_l and the l strands on the right and top right. Now expand the Jones–Wenzl idempotent. The only non-zero term come from one of the following TL diagrams:

Thus the result of $g \circ f$ must be a scalar times $\iota_{-l} \otimes \iota_{k+l} \otimes \iota_{-l}$. Since $f \circ g$ is the identity and $g \circ f$ is a scalar times the identity, that scalar must be 1. $\qquad \square$

5. Graph Planar Algebra and the Temperley–Lieb Planar Algebra

This section is motivation for the definition of the PSPA. We start with a summary of the definition of the graph planar algebra, first defined in [5].

Throughout this section, fix a simple graph Γ. For Jones, all planar algebras are shaded, and Γ is required to be bipartite. We will ignore this issue.

Let μ be a function from the vertices of Γ to $\mathbf{R}_{>0}$. We will define the graph planar algebra $\mathcal{P}$ corresponding to (Γ, μ).

For each $k > 0$, let $\mathcal{P}_{2k}$ be the vector space of complex valued functions on the set of loops of length $2k$ on Γ.

Suppose T is a tangle. For each input disk of T, let v_b be a corresponding input vector. We must define a corresponding output vector v. Thus we must define $v(\gamma)$ for every loop γ in Γ that has length equal to the number of endpoints on the outer boundary of T.

A *state* σ of T is a function from the set of regions of T to the set of vertices of Γ such that adjacent regions are sent to adjacent vertices.

Suppose r is a region of T. This is a planar surface with boundary that may include some right-angled corners. The *Euler measure* $e(r)$ is defined in a similar way to the Euler characteristic, using the usual formula $V - E + F$ for a triangulation of r. The difference is, every corner must be a vertex and only counts as $\frac{1}{4}$, any other vertex on a boundary only counts as $\frac{1}{2}$, and every edge on a boundary only counts as $\frac{1}{2}$.

We are finally ready to define the image vector v of the vectors v_b under the action of the tangle T.

$$v(\gamma) = \sum_{\sigma} \left(\prod_r \mu(\sigma(r))^{e(r)} \right) \left(\prod_b v_b(\sigma|_{\partial b}) \right).$$

The sum is over all states σ that are compatible with γ. The first product is over all regions r of T. The second product is over all input disks b of T.

The TL planar algebra is a subfactor planar algebra of type A_∞. It can be found inside the graph planar algebra associated to $\Gamma = A_\infty$, which is the ray with vertices indexed by positive integers. The function μ assigns the quantum integer $[n]$ to the nth vertex. (Note we are still assuming q is not a root of unity. If q is a primitive $(n+1)$th root of unity then we should use the graph A_n.)

Suppose T is an oriented tangle. Define a state of T to be a function from the set of regions of T to the set of vertices of A_∞ such that, for any strand of T, if the region to its right is sent to vertex n then the region to its left is sent to vertex $n + 1$. Thus, a state is determined by the vertex associated to a single region. In a sense, the orientation on the strands removes the ambiguity in the state of a TL diagram.

Now suppose T and T' differ by a pop-switch relation. There is an obvious correspondence between states of T and states off T'. Furthermore, the total Euler measure of the region associated to any given vertex is the same. We therefore have a well-defined embedding of the PSPA in the graph planar algebra of the graph A_∞.

One can think of the PSPA as a diagrammatic way to keep track of computations inside the graph planar algebra of A_∞.

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