On the Burau Representation of B_4 modulo p

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Abstract

The problem of faithfulness of the (reduced) Burau representation for $n = 4$ is known to be equivalent to the problem of whether certain two matrices A and B generate a free group of rank two. It is known that A^3 and B^3 generate a free group of rank two [Mor], [Wit-Zar], [Ber-Tra$_2$]. We prove that they also generate a free group when considered as matrices over the $\mathbb{Z}_p[t, t^{-1}]$ for any integer $p > 1$.

1 Introduction

The faithfulness problem of the Burau representation is open in just one case, that of $n = 4$ [Big], [Lon-Pat], [Mood]. On the other hand, for $n = 4$ the representation is faithful if and only if certain two matrices A and B generate a free group $\langle A, B\rangle$ of rank two [Bir], [Ber-Tra$_2$].

Let $\rho_4 : B_4 \to GL\left(3, \mathbb{Z}\left[t, t^{-1}\right]\right)$ be the reduced Burau representation of the braid group B_4. Consider the matrices A and B as follows:

$$A = \rho_4\left(a^{-1}\right) = \begin{bmatrix} 0 & 0 & -t^{-1} \\ 0 & -t & -t^{-1}+t \\ -1 & 0 & -t^{-1}+1 \end{bmatrix}, \tag{1}$$

$$B = \rho_4\left(b\right) = \begin{bmatrix} -t^{-1} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -t \end{bmatrix}, \tag{2}$$

where $a = \sigma_1\sigma_2\sigma_1^{-1}\sigma_3\sigma_2^{-1}\sigma_1^{-1}$ and $b = \sigma_3\sigma_1^{-1}$ ($\sigma_i,\ i = 1, 2, 3$ are standard generators of B_4). It is known that the group $\langle a, b\rangle$ generated by a and

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b is a free group which contains the kernel of the Burau map $\rho_4 : B_4 \to GL\left(3, Z[t,t^{-1}]\right)$ [Bok-Ves], [Ber-Tra$_2$].

In the paper it is shown that there exists an order four matrix T, which satisfies the following equality:

$$A = TBT^{-1}, \quad A^{-1} = T^{-1}BT, \quad B^{-1} = T^2BT^2. \tag{3}$$

Consequently, by a suitable substitution in a word

$$\omega = A^{m_k}B^{n_k}\dots A^{n_2}B^{n_2}A^{m_1}B^{n_1}, \quad n_i, m_i \in \mathbb{Z} \tag{4}$$

we obtain a word in T, T^{-1} and B^n, $n \in \mathbb{N}$. Therefore, the faithfulness problem of the Burau representation for $n = 4$ reduces to showing that any word in which B appears only with positive exponents, and T and T^{-1} appear one at a time, does not give the identity matrix. We will give a simple proof of this, which works equally well if the coefficients are in the ring $\mathbb{Z}_p[t,t^{-1}]$ for any integer $p > 1$.

2 Matrices conjugating B^{-1}, A and A^{-1} to B

Lemma 2.1. *There exists a matrix T which satisfies the following equality:*

$$A = TBT^{-1}, \quad A^{-1} = T^{-1}BT, \quad B^{-1} = T^2BT^2. \tag{5}$$

This matrix T is of order four as an element of $GL\left(3, \mathbb{Z}\left[t,t^{-1}\right]\right)$.

Proof. It is easily checked that the matrices A, B, A^{-1} and B^{-1} have the same eigenvalues $-t^{-1}, -t$ and 1. Therefore, they are conjugate to the same diagonal matrix:

$$\Delta = \begin{bmatrix} -t^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -t \end{bmatrix}. \tag{6}$$

Let us consider transformation matrices T_A and T_B

$$T_A = \begin{bmatrix} -1 & t^{-1} & 0 \\ -1 & t^{-1}-1 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \quad T_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & t^{-1}+1 & 0 \\ 0 & t^{-1} & 1 \end{bmatrix}. \tag{7}$$

so that the following equalities hold.

$$A = T_A \Delta T_A^{-1}, \quad B = T_B \Delta T_B^{-1}. \tag{8}$$

We obtain that

$$A = T_A T_B^{-1} B T_B T_A^{-1}. \tag{9}$$

If we calculate the matrix $T = T_A T_B^{-1}$, then we obtain that:

$$T = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, T^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}. \tag{10}$$

The direct calculation shows that the matrix T is an element of order four in $GL\left(3, \mathbb{Z}\left[t, t^{-1}\right]\right)$ and $B^{-1} = T^2 B T^2$. □

As we mentioned above the problem of faithfulness of the Burau representation for $n = 4$ is equivalent to the problem of whether A and B generate a free group of rank 2 in $GL\left(3, \mathbb{Z}\left[t, t^{-1}\right]\right)$ [Bir], [Ber-Tra$_2$]. This means that we need to prove that for every non–empty non–reducible word in letters A, B, A^{-1} and B^{-1} the corresponding product of matrices A, B, A^{-1} and B^{-1} is not equal to the unit matrix. If a word like this does exist we may as well consider one with suffix $B^{-i}, i \geq 1$ replacing the given word with a conjugate if necessary. However, for such words we have the following.

Corollary 2.2. *Let w be a formally irreducible non–empty word in letters A, B, A^{-1} and B^{-1} with suffix $B^{-i}, i \geq 1$. Then the corresponding product of matrices A, B, A^{-1} and B^{-1} may be written in the form*

$$T^m B^{n_k} \cdots T^{m_2} B^{n_2} T^{m_1} B^{n_1} T^2, \; n_i \in \mathbb{N}, \; m_i = \pm 1, \; m \in \{-1, 0, 1, 2\}. \tag{11}$$

Proof. We replace A, A^{-1} and B^{-1} with $T^{-1}BT, TBT^{-1}$ and T^2BT^2 respectively. In the process we may obtain $TT^2, T^{-1}T^2, T^2T$ and T^2T^{-1} between two consecutive powers of B but these are equal to T^{-1}, T, T^{-1} and T respectively (because $T^4 = 1$). The need to allow $m = 2$ or 0 arises from the possibility that the last multiplication might be by B^{-1} or by B. □

3 The group $\langle A^3, B^3 \rangle$ modulo p

Let $q_p : GL\left(3, \mathbb{Z}\left[t, t^{-1}\right]\right) \to GL\left(3, \mathbb{Z}_p\left[t, t^{-1}\right]\right)$ be the map given by reducing coefficient modulo p. Let the map

$$\rho_4^p : B_4 \to GL\left(3, \mathbb{Z}_p\left[t, t^{-1}\right]\right) \tag{12}$$

be given by $\rho_4^p = q_p \circ \rho_4$. Note that we use same notation of image matrices under the maps ρ_4 and ρ_4^p. If it is necessary, we just say that a given matrix is over the $\mathbb{Z}_p\left[t,t^{-1}\right]$. Therefore, our aim is to show that any combination of T, T^{-1} (appearing one at a time) and positive powers of B^n does not give the identity matrix over the $\mathbb{Z}_p\left[t,t^{-1}\right]$.

Note that from now on T, T^{-1} and B are considered as matrices over $\mathbb{Z}_p\left[t,t^{-1}\right]$, which means that coefficients of polynomials are considered modulo p, where p is an integer greater than 1.

Theorem 3.1. *Let ω be a word of the letters B, T and T^{-1} of the following form*

$$\omega = T^m B^{n_k}\cdots T^{m_2}B^{n_2}T^{m_1}B^{n_1}T^2,\ n_i\in\mathbb{N},\ m_i=\pm1,\ m\in\{-1,0,1,2\}. \tag{13}$$

If for every i we have $n_i\geq 2$, whenever $m_{i-1}=1$ and $n_i\geq 3$, whenever $m_{i-1}=-1$ then the product matrix is not identity.

Proof. To prove the statement, we use the method of the ping–pong lemma. Let X_1 be the set of such vectors that lowest degree of the first coordinate is smaller by 2 or more than lowest degree of second and third coordinates. Similarly, X_2 is the set of such vectors that lowest degree of all coordinates are equal and X_3 is the set of such vectors that lowest degree of second coordinate is two less than lowest degree of first and third coordinates. Note that if a coordinate is zero (trivial polynomial) that it is considered that the lowest degree is $+\infty$. Let B, T and T^{-1} acts on X_1, X_2 and X_3 by left multiplication. Our aim is to prove that:

1. $T^2v_0\in X_1$, for some $v_0\notin X_1\bigcup X_2\bigcup X_3$;
2. $TX_1\subset X_2$;
3. $T^{-1}X_1\subset X_3$;
4. $B^nX_2\subset X_1,\ \forall\, n\geq 2$;
5. $B^nX_3\subset X_1,\ \forall\, n\geq 3$.

In this case, for ω as in the previous theorem, $\omega v_0\neq v_0$ and so it is not the identity matrix. We now prove each of these five items.

1. Let $v_0=\begin{bmatrix}0\\0\\1\end{bmatrix}$, then $T^2v_0=\begin{bmatrix}1\\0\\0\end{bmatrix}\in X_1$.

2. Let $v\in X_1$, then it has the following form:

$$v=\begin{bmatrix} a_0t^n+a_1t^{n+1}+a_2t^{n+2}+a_3t^{n+3}+\dots\\ b_2t^{n+2}+b_3t^{n+3}+\dots\\ c_2t^{n+2}+c_3t^{n+3}+\dots\end{bmatrix} \tag{14}$$

where $a_0 \neq 0 \in \mathbb{Z}_p$. Then we have:

$$Tv = T \begin{bmatrix} a_0 t^n + a_1 t^{n+1} + a_2 t^{n+2} + a_3 t^{n+3} + \dots \\ b_2 t^{n+2} + b_3 t^{n+3} + \dots \\ c_2 t^{n+2} + c_3 t^{n+3} + \dots \end{bmatrix} =$$

$$= \begin{bmatrix} (-a_0)t^n + (-a_1)t^{n+1} + (-a_2 + b_2)t^{n+2} + (-a_3 + b_3)t^{n+3} + \dots \\ (-a_0)t^n + (-a_1)t^{n+1} + (-a_2 + c_2)t^{n+2} + (-a_3 + c_3)t^{n+3} + \dots \\ (-a_0)t^n + (-a_1)t^{n+1} + (-a_2)t^{n+2} + (-a_3)t^{n+3} + \dots \end{bmatrix}. \tag{15}$$

Therefore, $Tv \in X_2$;

3. Let apply on the vector v given by (14) with T^{-1}:

$$T^{-1}v = T^{-1} \begin{bmatrix} a_0 t^n + a_1 t^{n+1} + a_2 t^{n+2} + a_3 t^{n+3} + \dots \\ b_2 t^{n+2} + b_3 t^{n+3} + \dots \\ c_2 t^{n+2} + c_3 t^{n+3} + \dots \end{bmatrix} =$$

$$= \begin{bmatrix} (-c_2)t^{n+2} + (-c_3)t^{n+3} + \dots \\ a_0 t^n + a_1 t^{n+1} + (a_2 - c_2)t^{n+2} + (a_3 - c_3)t^{n+3} + \dots \\ (b_2 - c_2)t^{n+2} + (b_3 - c_3)t^{n+3} + \dots \end{bmatrix}. \tag{16}$$

Hence, $T^{-1}v \in X_2$ for all $v \in X_1$;

4. Let $v \in X_2$, then we have:

$$v = \begin{bmatrix} a_0 t^n + a_1 t^{n+1} + a_2 t^{n+2} + \dots \\ b_0 t^n + b_1 t^{n+1} + b_2 t^{n+2} + \dots \\ c_0 t^n + c_1 t^{n+1} + c_2 t^{n+2} + \dots \end{bmatrix}, \tag{17}$$

where $a_0, b_0, c_0 \neq 0 \in \mathbb{Z}_p$. Therefore,

$$B^n v = B^n \begin{bmatrix} a_0 t^n + a_1 t^{n+1} + a_2 t^{n+2} + \dots \\ b_0 t^n + b_1 t^{n+1} + b_2 t^{n+2} + \dots \\ c_0 t^n + c_1 t^{n+1} + c_2 t^{n+2} + \dots \end{bmatrix} =$$

$$= B^{n-1} \begin{bmatrix} (-a_0)t^{n-1} + (-a_1 + b_0)t^n + (-a_2 + b_1)t^{n+1} + \dots \\ b_0 t^n + b_1 t^{n+1} + \dots \\ b_0 t^n + (b_1 - c_0)t^{n+1} + \dots \end{bmatrix}. \tag{18}$$

As we see, we have at least one more B to multiply the obtained matrix by and therefore, the result belongs to X_1;

5. Let $v \in X_3$, then

$$v = \begin{bmatrix} a_2 t^{n+2} + a_3 t^{n+3} + \dots \\ b_0 t^n + b_1 t^{n+1} + b_2 t^{n+2} + b_3 t^{n+3} + \dots \\ c_2 t^{n+2} + c_3 t^{n+3} + \dots \end{bmatrix}. \tag{19}$$

Consequently, we have

$$B^n v = B^n \begin{bmatrix} a_2 t^{n+2} + a_3 t^{n+3} + \dots \\ b_0 t^n + b_1 t^{n+1} + b_2 t^{n+2} + b_3 t^{n+3} + \dots \\ c_2 t^{n+2} + c_3 t^{n+3} + \dots \end{bmatrix} =$$

$$= B^{n-1} \begin{bmatrix} b_0 t^n + (-a_2 + b_1) t^{n+1} + (-a_3 + b_2) t^{n+2} + \dots \\ b_0 t^n + b_1 t^{n+1} + b_2 t^{n+2} + b_3 t^{n+3} + \dots \\ b_0 t^n + b_1 t^{n+1} + b_2 t^{n+2} + (b_3 - c_3) t^{n+3} + \dots \end{bmatrix}. \qquad (20)$$

Therefore, Bv belongs to X_2 and by the previous case, we obtain that $B^n v \in X_1$ for all $n \geq 3$. □

Corollary 3.2. *The matrices A^3 and B^3 generate a non–abelian free group of rank 2 over $\mathbb{Z}_p[t, t^{-1}]$ for any integer $p > 1$.*

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