DUALITIES FOR MODULES OF FINITE PROJECTIVE DIMENSION

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Dedicated to José Antonio de la Peña on the occasion of his sixtieth birthday

Abstract. The purpose of this article is threefold: The first is to provide an overview of old and recent findings which relate finite dimensional tilting modules over a finite dimensional algebra Λ to dualities defined on resolving subcategories of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, the category of Λ-modules of finite projective dimension. The second goal is to supplement the picture available so far by new results, based on the transfer of representation-theoretic information via such partial dualities, and to place the array of contravariant equivalences into a broader context. The third objective is to apply the findings (those on dualities induced by strong tilting modules in particular) to truncated path algebras: A streamlined presentation of existing theory is accompanied by a fully worked example. The pertinent computational methods are developed at the end; they show that the algebras obtained from a truncated path algebra Λ through iterated strong tilting, as well as their $\mathcal{P}^{<\infty}$-categories and $\mathcal{P}^{<\infty}$-approximations, may be constructed from the quiver and Loewy length of Λ.

1. Introduction

During the past four decades, tilting functors have asserted their role as powerful tools for comparing the module categories of finite dimensional algebras. Next to the fact that they often induce "almost-equivalences" on the level of these categories themselves, they give rise to triangle equivalences on the level of the corresponding derived categories. The multi-step trip to this insight started with the reflection functors of Bernstein-Gelfand-Ponomarev which underwent several stages of generalization through work of Auslander-Platzeck-Reiten, Brenner-Butler, Happel-Ringel, Bongartz, Miyashita and others (see [4, 10, 21, 9, 31, 1]). The breakthrough to equivalences linking the derived category of an algebra Λ to those of its tilted companions was ushered in by Happel [19] and Cline-Parshall-Scott [13], then fully established by Rickard [32]; see also Keller's approach by way of DG-algebras [29]. The resulting derived Morita theory has found immediate application to finite dimensional representation theory, e.g., to modular representations of finite groups (see [30] for the connection to Broué's conjecture and [33] for partial confirmations to date).

By contrast, the present article focuses on *contravariant* equivalences of an equally natural format which are induced by tilting modules. These dualities provide a separate set of bridges connecting an algebra Λ to its tilted algebras.

Our notion of a *tilting module* T is Miyashita's generalized version, allowing for arbitrary finite projective dimensions of T, as opposed to restricting to $\operatorname{p\,dim} T \le 1$. (For unexplained italicized terms, we refer to Section 2.) In particular, any tilting (left) module over a finite dimensional algebra Λ belongs to the full subcategory $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ consisting of the finitely generated left Λ-modules with finite projective dimension. Among the tilting modules, distinguished specimens dubbed *strong* were first singled out by Auslander and Reiten in [5]. They are the tilting modules T which are Ext-*injective relative to the objects in* $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, meaning that $\mathrm{Ext}^i_\Lambda(\mathcal{P}^{<\infty}(\Lambda\text{-mod}), T) = 0$ for all $i \geq 1$; in light of Section 2.C, this description of strong tilting modules is equivalent to the original definition. Due to [5], strong tilting modules are unique up to repeats of their indecomposable direct summands; i.e., there is at most one *basic* strong tilting object in Λ-mod, and existence is equivalent to *contravariant finiteness* of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. Subsequently, these special tilting modules were re-encountered from a different angle by Happel and Unger in [22]: Existence provided, the basic strong tilting object in Λ-mod is the unique smallest element in the poset of all basic tilting objects in Λ-mod. The pertinent partial order was introduced by Riedtmann and Schofield [34]; namely, $T_1 \leq T_2$ precisely when the *right perpendicular category* $T_1^\perp$ of T_1 is contained in that of T_2; here $T^\perp$ consists of the objects $Y \in \Lambda$-mod which satisfy $\mathrm{Ext}^i_\Lambda(T, Y) = 0$ for all $i \geq 1$. (Clearly, this poset always has a largest element, namely the basic projective generator in Λ-mod.) Strong tilting modules were further explored for Auslander-Gorenstein algebras by Iyama and Zhang in [28]. In the present context, we will find them to play a particularly prominent role: They are precisely those tilting modules which induce dualities defined on the full category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$.

The underlying theme of the paper is to exhibit the mongrel status of tilting modules which, by definition, combine properties of projective generators with properties of injective cogenerators. As is well-known, the former are the modules whose covariant Hom-functors induce classical Morita equivalences $\Lambda\text{-mod} \longleftrightarrow \Lambda'\text{-mod}$, while the latter are characterized by the fact that their contravariant Hom-functors induce dualities $\Lambda\text{-mod} \longleftrightarrow \text{mod-}\Lambda'$; in each case Λ' is the opposite of the endomorphism ring of the Λ-module inducing the pertinent functors. Suppose that ${}_\Lambda T_{\widetilde{\Lambda}}$ is a tilting bimodule, where $\widetilde{\Lambda} = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$. Equivalences of suitable subcategories of Λ-mod and $\widetilde{\Lambda}$-mod have been amply studied (see, e.g., [30], [1], [35]). As was first discovered by Miyashita in his pivotal 1986 paper (if not advertised in an explicit format), tilting modules also give rise to dualities linking subcategories of Λ-mod to subcategories of mod-$\widetilde{\Lambda}$. These dualities – they reflect the resemblance of tilting modules to injective cogenerators – have received far less attention than the covariant equivalences. We aim at providing the foundations for a systematic exploration.

We begin by reviewing Miyashita's dualities. They are defined on *resolving subcategories* of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and are *strictly exact* in the following sense: A functor $F : \mathfrak{C} \to \mathfrak{C}'$, co- or contravariant, linking two categories $\mathfrak{C}$ and $\mathfrak{C}'$ of modules over algebras Λ and Λ', is strictly exact if F takes any short exact sequence $0 \to C_1 \to C_2 \to C_3 \to 0$ of Λ-modules with $C_i \in \mathfrak{C}$ to a short exact sequence of Λ'-modules. Miyashita's Theorem has a converse: According to [24, Theorem 1], every strictly exact duality defined on a resolving subcategory $\mathfrak{C}$ of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is of the form $\mathrm{Hom}_\Lambda(-, T)$ for a tilting module T, and $\mathfrak{C}$ necessarily has the form $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \cap {}^\perp T$; here ${}^\perp T$ is the *left perpendicular category* of T. Regarding the sizes of the subcategories coupled by tilting-induced equivalences or dualities, the rule of thumb is this: The larger the tilting module T is in the Riedtmann-Schofield order, the closer its traits are to those of a projective generator; the smaller T is in this partial order, the more its behavior resembles that of an injective cogenerator. The case of a projective generator and that of a strong tilting module constitute the extreme positions on this seesaw.

The main theorems on dualities, including comparisons with classical scenarios and links to work of Auslander, Reiten and others, are assembled in Sections 3, 4. In Section 5, we propose several open problems which address connections between, on one hand, the subcategories of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ that carry dualities and naturally associated subcategories of $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ on the other. ($\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ stands for the category of arbitrary left Λ-modules with finite projective dimension.)

In Section 6, we apply the general results on strong tilting to truncated path algebras, i.e., to algebras of the form $\Lambda = \Lambda_L = KQ/\langle\text{all paths of length } L+1\rangle$ for some quiver Q and some positive integer L. We review the findings of [24] without proofs; in parallel we analyze an example that illustrates each step. In a nutshell: If Λ is truncated, $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is always contravariantly finite. This guarantees a basic strong tilting module ${}_\Lambda T$, which induces a duality $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \longleftrightarrow \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}) \cap {}^{\perp}(T_{\widetilde{\Lambda}})$. In general, the righthand category is properly contained in $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$, meaning that T fails to be strong as a tilting module over $\widetilde{\Lambda}$. However, the category $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ always has its own basic strong tilting module. This tilting module of the second generation is in fact strong also as a tilting module over its endomorphism ring $\widetilde{\widetilde{\Lambda}}$, whence the process of iterated strong tilting turns periodic: $\Lambda \rightsquigarrow \widetilde{\Lambda} \rightsquigarrow \widetilde{\widetilde{\Lambda}} \rightsquigarrow \widetilde{\Lambda} \rightsquigarrow \cdots$. In each round, the objects in the $\mathcal{P}^{<\infty}$-categories of Λ, $\widetilde{\Lambda}$, $\widetilde{\widetilde{\Lambda}}$ are characterized in terms of their intrinsic structure. Moreover, the strong tilting objects, as well as the $\mathcal{P}^{<\infty}$-approximations arising in the theory, are algorithmically accessible from Q and L. The constructive aspects are supplemented in Sections 7 and 8.

We point to an undercurrent of Section 6: The homological picture resulting from this section reinforces the idea that many results and tools specific to the theory of hereditary algebras may be carried over to truncated path algebras in a fairly natural manner (note that the hereditary algebras are among the truncated ones). The motivation that drives the study of the more general class of algebras is, in essence, the same as in the hereditary case: Namely, to explore the finite dimensional representation theory of a quiver Q per se (not necessarily acyclic in the general truncated case), unmodified by relations; in the presence of oriented cycles, this is achieved by letting the Loewy lengths $L+1$ of the algebras Λ_L grow. Curiously, the process of strongly tilting a truncated path algebra Λ reveals its kinship to a hereditary algebra more clearly than the original category Λ-mod does (see, e.g., Corollary 31).

In Section 2, we recall some background. However, the article is far from being self-contained in that we include proofs only where we do not have references to cite.

2. Background on tilting, strong tilting, and contravariant finiteness

Throughout the first five sections, Λ *will denote an arbitrary basic finite dimensional algebra over a field* K unless otherwise specified. We fix a full sequence $e_1, \dots, e_n$ of primitive idempotents of Λ. By J we denote the Jacobson radical of Λ and by S_i the simple module $\Lambda e_i / J e_i$.

To fix our terminology, we recall several definitions, some of which have evolved over time.

2.A. $\mathcal{X}$-coresolutions and $\mathcal{X}$-resolutions. Let $\mathcal{X}$ be a subcategory of Λ-Mod. An $\mathcal{X}$-*coresolution* of a left Λ-module M is an exact sequence of the form

$$0 \longrightarrow M \longrightarrow X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

with all $X_i \in \mathcal{X}$; the *length* of the coresolution is the maximal l with $X_l \neq 0$. The concept of an $\mathcal{X}$-*resolution* of M is dual.

By $\mathfrak{co}\text{-}\mathfrak{res}^{<\infty}(\mathcal{X})$ and $\mathfrak{Co}\text{-}\mathfrak{Res}^{<\infty}(\mathcal{X})$ we denote the full subcategories of Λ-mod and Λ-Mod, respectively, which consist of the modules that have finite $\mathcal{X}$-coresolutions. The categories $\mathfrak{res}^{<\infty}(\mathcal{X})$ and $\mathfrak{Res}^{<\infty}(\mathcal{X})$ are defined analogously. In case $\mathcal{X} = \text{add}(X)$ or $\mathcal{X} = \text{Add}(X)$ for a module X, we abbreviate further to $\mathfrak{co}\text{-}\mathfrak{res}^{<\infty}(X)$ and $\mathfrak{res}^{<\infty}(X)$, resp., $\mathfrak{Co}\text{-}\mathfrak{Res}^{<\infty}(X)$ and $\mathfrak{Res}^{<\infty}(X)$.

2.B. (Strong) tilting modules.

Definition. Following Miyashita [31], we call a left Λ-module T a *tilting module* in case (i) T belongs to $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, (ii) $\text{Ext}^i(T,T) = 0$ for all $i \geq 1$, and (iii) the left regular module ${}_\Lambda\Lambda$ belongs to $\mathfrak{co}\text{-}\mathfrak{res}^{<\infty}(T)$.

According to [5], a tilting module ${}_\Lambda T$ is *strong* if $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \subseteq \mathfrak{co}\text{-}\mathfrak{res}^{<\infty}(T)$; due to Proposition 18 in Section 4.B, this definition is equivalent to the description of a strong tilting module given in the introduction (requiring that the functors $\text{Ext}^i_\Lambda(-,T)$ vanish on $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ for all $i \geq 1$).

The module T is *basic* if it has no indecomposable direct summand of multiplicity ≥ 2.

Given an arbitrary tilting module ${}_\Lambda T$ with $\widetilde{\Lambda} = \text{End}_\Lambda(T)^{\text{op}}$, the properties of the bimodule ${}_\Lambda T_{\widetilde{\Lambda}}$ are known to be left-right symmetric in the following sense: $T_{\widetilde{\Lambda}}$ is in turn a tilting module, and Λ is canonically isomorphic to $\text{End}_{\widetilde{\Lambda}}(T)$, i.e., the K-algebra map $\Lambda \to \text{End}_{\widetilde{\Lambda}}(T)$, $\lambda \mapsto (x \mapsto \lambda x)$, is an isomorphism (see [31, Proposition 1.4]); in other words, ${}_\Lambda T_{\widetilde{\Lambda}}$ is a *balanced bimodule.* For the family of covariant equivalences linking suitable subcategories of Λ-mod to subcategories of $\widetilde{\Lambda}$-mod – they originally motivated the exploration of tilting modules – we refer to [31, Theorems 1.14 – 1.16], where the current level of generality is attained. See also [30] and the *Handbook of Tilting Theory*, [1], for numerous additional references.

2.C. Contravariant finiteness of resolving subcategories of Λ-mod**.** The following concepts were introduced by Auslander and Smalø in [7]; for an alternative development of this approximation theory, see [16, 17].

Definition. Let $\mathfrak{C}$ be a full subcategory of Λ-mod. Then $\mathfrak{C}$ is said to be *resolving* if $\mathfrak{C}$ contains the projective modules and is closed under extensions and kernels of surjective homomorphisms in Λ-mod.

Now let $\mathfrak{C} \subseteq \Lambda$-mod be a resolving subcategory. A (right) $\mathfrak{C}$-*approximation* of a module $M \in \Lambda$-mod is a homomorphism $\phi : \mathcal{A} \to M$ such that $\mathcal{A} \in \mathfrak{C}$ and every map in $\text{Hom}_\Lambda(\mathfrak{C}, M)$ factors through ϕ; note that any such approximating map ϕ is surjective, since $\mathfrak{C}$ contains the projectives in Λ-mod. Whenever such an approximation of M exists, there is one of minimal K-dimension, $\mathcal{A}(M) = \mathcal{A}_{\mathfrak{C}}(M)$, which is uniquely determined by M up to isomorphism. In case every $M \in \Lambda$-mod has a $\mathfrak{C}$-approximation, the category $\mathfrak{C}$ is called *contravariantly finite* (in Λ-mod). To explain the terminology, one observes that $\mathfrak{C}$ is contravariantly finite precisely when the restricted contravariant Hom-functor $\mathrm{Hom}(-, M)|_{\mathfrak{C}}$ is finitely generated in the functor category $\mathrm{Fun}(\mathfrak{C}, \mathbf{Ab})$.

Evidently, $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite whenever Λ has either finite global dimension or vanishing left finitistic dimension. But the condition has traction only in cases lying strictly between these extremes. The following sufficient condition for contravariant finiteness of a resolving subcategory $\mathfrak{C}$ was established in [5, Proposition 3.7]: If each of the simple left Λ-modules $S_1, \dots, S_n$ has a $\mathfrak{C}$-approximation, then $\mathfrak{C}$ is contravariantly finite. Moreover, by [5, Proposition 3.8], for contravariantly finite $\mathfrak{C}$, the modules in $\mathfrak{C}$ are precisely those in $\mathrm{add}\big(\mathrm{filt}\big(\mathcal{A}_{\mathfrak{C}}(S_1), \dots, \mathcal{A}_{\mathfrak{C}}(S_n)\big)\big)$, where $\mathcal{A}_{\mathfrak{C}}(S_i)$ is the minimal $\mathfrak{C}$-approximation of S_i. (Notation: For any choice of $M_1, \dots, M_u \in \Lambda$-mod, the category $\mathrm{filt}(M_1, \dots, M_u)$ consists of the modules permitting finite filtrations with consecutive factors among the M_i.) Clearly, this implies $\sup\{\mathrm{p\,dim}\, C \mid C \in \mathfrak{C}\} = \sup\{\mathrm{p\,dim}\, \mathcal{A}_{\mathfrak{C}}(S_i) \mid 1 \le i \le n\}$.

Of special interest to us will be resolving subcategories of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, in particular the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ itself. By dint of the upcoming theory, the following homological assets entailed by contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ will become relevant (we summarily point to [7, 5, 27, 25, 2] for proofs). Suppose for the moment that $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite. Then this category has relative Auslander-Reiten sequences, a bonus that singles it out for an internal representation-theoretic analysis. Moreover, in light of the preceding paragraph, the basic building blocks of the modules in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ are the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations $\mathcal{A}(S_i)$ of the simples $S_i \in \Lambda$-mod. The same holds for the objects of $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$, as they turn out to be the direct limits of directed systems in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. In particular, the little and big left finitistic dimensions, $\mathrm{fin\,dim}\,\Lambda$ and $\mathrm{Fin\,dim}\,\Lambda$, coincide and are equal to the maximum of the projective dimensions of the $\mathcal{A}(S_i)$. It is the key role played by the $\mathcal{A}(S_i)$ that motivates us to pin them down whenever contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is confirmed.

While finite dimensional algebras Λ with the property that $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite abound, it may be difficult to decide the status of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ for a given algebra Λ. The condition appears to slice through most major classes of algebras which have been delineated by other shared properties (see [11, 26, 14] for exceptional classes, and [20] for test criteria). One of the classes enjoying this property consists of the truncated path algebras; in Section 6, we will use it to illustrate the theory laid out in Sections 2-5.

2.D. Relative splitting injectivity versus relative Ext-injectivity, and duals. Suppose $\mathfrak{C} \subseteq \Lambda$-mod is a full subcategory which is closed under isomorphic objects, direct summands and finite direct sums.

Definition. We call an object Y of $\mathfrak{C}$ *relatively splitting injective in* $\mathfrak{C}$ if every injective morphism in $\mathrm{Hom}_\Lambda(Y, C)$ with $C \in \mathfrak{C}$ splits (compare with [7, Section 2]). Moreover, call Y *relatively* Ext-*injective in* $\mathfrak{C}$ if $\mathrm{Ext}^i_\Lambda(\mathfrak{C}, Y) = 0$ for all $i \ge 1$. The concepts of *relative splitting projectivity* and *relative* Ext-*projectivity* are dual.

Remarks 1. Suppose that $\mathfrak{C}$ is a resolving subcategory of Λ-mod and hence closed under syzygies. Then every relatively splitting injective object of $\mathfrak{C}$ is relatively Ext-injective in $\mathfrak{C}$. However, the converse fails in general; see Example 5 below. As for the dual attributes: The relative splitting projective objects of $\mathfrak{C}$ coincide with the projectives in Λ-mod, since projective covers of objects in $\mathfrak{C}$ belong to $\mathfrak{C}$. The relative Ext-projective objects of $\mathfrak{C}$ are projective as well. To see this, let $Y \in \mathfrak{C}$ be relatively Ext-projective, and suppose that $f : C \to Y$ with $C \in \mathfrak{C}$ is a surjective homomorphism. Since $\mathrm{Ker}(f) \in \mathfrak{C}$, we have $\mathrm{Ext}^1_\Lambda(Y, \mathrm{Ker}(f)) = 0$, which implies splitness of f.

The following invariant measures the deviation of an object X of $\mathfrak{C}$ from being relatively Ext-injective. It will become relevant in Section 4.

Definition. Let $X \in \mathfrak{C}$. The Ext-*injective dimension* of X *relative to* $\mathfrak{C}$ is

$$\mathrm{Ext.idim}_{\mathfrak{C}} X := \inf\{r \in \mathbb{N}_0 \mid \mathrm{Ext}^j_\Lambda(-, X)|_{\mathfrak{C}} = 0 \text{ for all } j \geq r + 1\},$$

with the understanding that $\mathrm{Ext.idim}_{\mathfrak{C}} X = \infty$ if the specified set is empty. By definition, $\mathrm{Ext.idim}_{\mathfrak{C}} X = 0$ precisely when X is a relatively Ext-injective object of $\mathfrak{C}$.

The Ext-*projective dimension* of X *relative to* $\mathfrak{C}$ is defined analogously.

2.E. Well-known connections between tilting and contravariant finiteness. A priori, the notion of a contravariantly finite subcategory of Λ-mod does not indicate the tight connection with tilting theory which will surface in consecutive steps in the sequel.

Let $\mathfrak{C}$ be as in Section 2.D. Call Λ-module Y a *cogenerator in* $\mathfrak{C}$, if $Y \in \mathfrak{C}$ and every object of $\mathfrak{C}$ embeds into some power Y^k in Λ-mod.

Lemma 2. *Suppose that* $\mathfrak{C}$ *is a contravariantly finite resolving subcategory of* Λ-mod*. Then* $\mathfrak{C}$ *has a relatively splitting injective cogenerator* (*obviously unique up to multiplicities of indecomposable direct summands*)*, namely* $\mathcal{A}_{\mathfrak{C}}(E)$*, the minimal* $\mathfrak{C}$*-approximation of the basic injective cogenerator* E *in* Λ-mod*.*

Proof. For splitness of any injective homomorphism $\mathcal{A}_{\mathfrak{C}}(E) \hookrightarrow C$ with $C \in \mathfrak{C}$, see e.g. [7], or [14, Lemma 5.1] for a proof in the current terminology. The cogenerator property of $\mathcal{A}_{\mathfrak{C}}(E)$ for $\mathfrak{C}$ is immediate from the cogenerator property of E in Λ-mod. $\square$

The first link between tilting and contravariant finiteness is provided by the dual of Auslander and Reiten's [5, Theorem 5.5(b)]:

Theorem 3. Correspondence between basic tilting modules and contravariantly finite subcategories of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. *There is a one-to-one correspondence between the isomorphism classes of basic tilting objects in* Λ-mod *on one hand and the contravariantly finite subcategories of* $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ *on the other. It assigns to any basic tilting module* $T \in \Lambda$-mod *the subcategory* $\mathfrak{co\text{-}res}^{<\infty}({}_\Lambda T)$ *of* Λ-mod (cf. 2.A)*.*

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Theorem 4. Existence and uniqueness of strong tilting modules. ([5, Proposition 6.3]; use Theorem 5.5, loc. cit., to correct the statement of 6.3.) *The category* Λ-mod *has a strong tilting module if and only if* $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ *is contravariantly finite in* Λ-mod*. In the positive case, the basic strong tilting module is the direct sum of the distinct* (*i.e., pairwise nonisomorphic*) *indecomposable relatively* Ext-*injective objects of* $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$*.*

In other words, if $\mathfrak{C} = \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite and T is the corresponding basic strong tilting module, the modules $X \in \mathfrak{C}$ with $\operatorname{Ext.idim}_{\mathfrak{C}} X = 0$ are precisely the objects of $\operatorname{add}(T)$.

Example 5. This example illustrates the fact that relative splitting injectivity in a resolving subcategory $\mathfrak{C} \subseteq \Lambda\text{-mod}$ is stronger than relative Ext-injectivity in general. Let Λ be the Kronecker algebra. Label the quiver of Λ so that the two equi-directed arrows start in e_1 and end in e_2. Clearly, the subcategory $\mathfrak{C}$ of Λ-mod which consists of all projective left Λ-modules is resolving and contravariantly finite. The basic strong tilting module associated to this category is the left regular module ${}_\Lambda\Lambda$, evidently relatively Ext-injective in $\mathfrak{C}$. On the other hand, the simple projective module Λe_2 fails to be relatively splitting injective in $\mathfrak{C}$. For later use, observe that the minimal $\mathfrak{C}$-approximation of the basic injective cogenerator in Λ-mod is $(\Lambda e_1)^3$.

However, in case $\mathfrak{C} = \mathcal{P}^{<\infty}(\Lambda\text{-mod})$, relative splitting injectivity of an object in $\mathfrak{C}$ amounts to the same as relative Ext-injectivity, since $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is closed under cokernels of injective homomorphisms in that case. In fact, we obtain

Proposition 6. Comparison of strong tilting modules with approximations of injective cogenerators. [24, Supplement II] *Suppose that* $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ *is contravariantly finite,* $T \in \Lambda\text{-mod}$ *a strong tilting module, and* $\mathcal{A}(E)$ *the minimal* $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$*-approximation of the basic injective cogenerator* E *in* Λ-mod. *Then*

$$\operatorname{add}(T) = \operatorname{add}(\mathcal{A}(E)).$$

In Section 4, this proposition will be supplemented as follows: If $T \in \Lambda\text{-mod}$ is an arbitrary basic tilting module and $\mathfrak{C}$ is the contravariantly finite subcategory $\mathfrak{co\text{-}res}^{<\infty}(T)$ which is associated to T (see Theorem 3), then $\operatorname{add}\big(\mathcal{A}_{\mathfrak{C}}(E)\big) \subseteq \operatorname{add}(T)$. On the other hand, Example 5 shows that the reverse inclusion fails in general.

3. Dualities of subcategories of $\mathcal{P}^{<\infty}$-categories. Consequences

Let Λ' be another finite dimensional K-algebra. For a Λ-Λ'-bimodule ${}_\Lambda Y_{\Lambda'}$, we denote by ${}^\perp({}_\Lambda Y)$ the left perpendicular category of ${}_\Lambda Y$ in Λ-mod, namely the full subcategory of Λ-mod consisting of the Λ-modules X with $\operatorname{Ext}^i_\Lambda(X, Y) = 0$ for all $i \geq 1$. Clearly, ${}^\perp({}_\Lambda Y)$ is a resolving subcategory of Λ-mod, whence so is the intersection $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \cap {}^\perp({}_\Lambda Y)$. Analogously, one defines the left perpendicular category ${}^\perp(Y_{\Lambda'}) \subseteq \text{mod-}\Lambda'$.

3.A. The main results on dualities. The upcoming fact is readily deduced from Miyashita's Theorem 3.5 in [31].

Theorem and Notation 7. Miyashita's duality. *Given a tilting bimodule* ${}_\Lambda T_{\widetilde{\Lambda}}$*, i.e.,* $\widetilde{\Lambda} = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$*, set*

$$\mathfrak{C} = \mathfrak{C}({}_\Lambda T) := \mathcal{P}^{<\infty}(\Lambda\text{-mod}) \,\cap\, {}^\perp({}_\Lambda T) \quad \textit{and} \quad \widetilde{\mathfrak{C}} = \widetilde{\mathfrak{C}}(T_{\widetilde{\Lambda}}) := \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}) \,\cap\, {}^\perp(T_{\widetilde{\Lambda}}).$$

Then the restricted Hom-functors $\operatorname{Hom}_\Lambda(-, T)|_{\mathfrak{C}}$ *and* $\operatorname{Hom}_{\widetilde{\Lambda}}(-, T)|_{\widetilde{\mathfrak{C}}}$ *are inverse dualities* $\mathfrak{C} \longleftrightarrow \widetilde{\mathfrak{C}}$.

By the definition of ${}^{\perp}({}_{\Lambda}T)$ and ${}^{\perp}(T_{\widetilde{\Lambda}})$, the inverse dualities secured by this theorem are *strictly exact*, meaning that they take those sequences $0 \to X_1 \to X_2 \to X_3 \to 0$ (in the specified subcategories of Λ-mod, resp., mod-$\widetilde{\Lambda}$) which are exact in the ambient full module categories to sequences of the same ilk.

Clearly, the inclusion "$\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \subseteq {}^{\perp}({}_{\Lambda}T)$" is tantamount to relative Ext-injectivity of the tilting module ${}_{\Lambda}T$ in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. Theorem 7 thus shows that every tilting bimodule ${}_{\Lambda}T_{\widetilde{\Lambda}}$ which is strong on both sides yields a duality $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \longleftrightarrow \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$. The converse provided by Corollary 9 below places additional emphasis on the distinguished role of strong tilting modules relative to categories of modules of finite projective dimension.

We supplement Miyashita's theorem by showing that strictly exact dualities between resolving subcategories of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\text{mod-}\Lambda')$ are always afforded by tilting bimodules according to the blueprint of Theorem 7.

The analogy of the upcoming result with classical Morita duality will be be further stressed by Proposition 18 and Observation 20 below. (For artinian rings Λ and Λ', a "classical Morita duality" between Λ and Λ' is a contravariant equivalence $\Lambda\text{-mod} \longleftrightarrow \text{mod-}\Lambda'$, well-known to always be induced by a contravariant Hom-functor which is determined by an injective cogenerator in Λ-mod.) The connection with dualities linking the $\mathcal{P}^{<\infty}$-categories of Λ-mod and mod-Λ' is evident in case Λ is a Gorenstein algebra (i.e., the left and right regular modules have finite injective dimension). In that case, the Λ-modules of finite injective dimension coincide with those of finite projective dimension, whence the basic injective cogenerator $E \in \Lambda$-mod is a tilting module, obviously strong. Observe that ${}_{\Lambda}E$ then gives rise to the standard Morita duality $\text{Hom}_{\Lambda}(-, E) \cong \text{Hom}_K(-, K) : \Lambda\text{-mod} \longleftrightarrow \text{mod-}\Lambda$, which restricts to a duality $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \longleftrightarrow \mathcal{P}^{<\infty}(\text{mod-}\Lambda)$. However, beyond this scenario, the dualities encountered in Theorems 7 and 8 are not induced by dualities of classical Morita type in that they do not result from restrictions of dualities defined on Λ-mod.

Theorem 8. Correspondence between tilting modules and dualities defined on resolving subcategories of $\mathcal{P}^{<\infty}$-categories. [24, Theorem 1] *Let Λ and Λ' be finite dimensional algebras, and let $\mathfrak{C} \subseteq \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathfrak{C}' \subseteq \mathcal{P}^{<\infty}(\text{mod-}\Lambda')$ be resolving subcategories of* Λ-mod *and* mod-Λ'*, respectively.*

Suppose $\mathfrak{C}$ is dual to $\mathfrak{C}'$ by way of strictly exact contravariant additive functors

$$F : \mathfrak{C} \longrightarrow \mathfrak{C}' \quad \text{and} \quad F' : \mathfrak{C}' \longrightarrow \mathfrak{C}$$

such that $F' \circ F$ and $F \circ F'$ are isomorphic to the pertinent identity functors.

Then there exists a tilting bimodule ${}_{\Lambda}T_{\Lambda'}$ such that:

- **(a)** $F \cong \text{Hom}_{\Lambda}(-, T)\,|_{\mathfrak{C}}$ *and* $F' \cong \text{Hom}_{\Lambda'}(-, T)\,|_{\mathfrak{C}'}$*;*
- **(b)** $\mathfrak{C} = \mathcal{P}^{<\infty}(\Lambda\text{-mod}) \cap {}^{\perp}({}_{\Lambda}T)$, *and* $\mathfrak{C}' = \mathcal{P}^{<\infty}(\text{mod-}\Lambda') \cap {}^{\perp}(T_{\Lambda'})$*; in other words,* $\mathfrak{C} = \mathfrak{C}(T)$ *and* $\mathfrak{C}' = \widetilde{\mathfrak{C}}(T)$*;*
- **(c)** $\mathfrak{C} = \mathfrak{co\text{-}res}^{<\infty}({}_{\Lambda}T)$ *and* $\mathfrak{C}' = \mathfrak{co\text{-}res}^{<\infty}(T_{\Lambda'})$*, i.e., the modules in $\mathfrak{C}$ are precisely those left Λ-modules which have finite coresolutions by objects in* $\text{add}({}_{\Lambda}T)$*, and the modules in $\mathfrak{C}'$ are those right Λ'-modules which have finite coresolutions by objects in* $\text{add}(T_{\Lambda'})$*.*

In particular, the functors F and F' are "minimal dualities", in the sense that they do not induce any duality $\mathfrak{C}_0 \longleftrightarrow \mathfrak{C}'_0$ between proper resolving subcategories $\mathfrak{C}_0$ of $\mathfrak{C}$ and $\mathfrak{C}'_0$ of $\mathfrak{C}'$.

Addendum: *Provided that $\mathfrak{C}$ and $\mathfrak{C}'$ are closed also under cokernels of injective morphisms in the ambient module categories, arbitrary additive dualities $F : \mathfrak{C} \longrightarrow \mathfrak{C}'$ and $F' : \mathfrak{C}' \longrightarrow \mathfrak{C}$ are strictly exact.*

We follow with several consequences. For the first, note that the categories $\mathfrak{C} = \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathfrak{C}' = \mathcal{P}^{<\infty}(\text{mod-}\Lambda')$ satisfy the conditions spelled out in the addendum to Theorem 8.

Corollary 9. *Let Λ and Λ' be finite dimensional algebras. Then any pair of inverse dualities (F, F')*

$$\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \longleftrightarrow \mathcal{P}^{<\infty}(\text{mod-}\Lambda')$$

is isomorphic to a pair of functors $\big(\mathrm{Hom}_\Lambda(-,T),\, \mathrm{Hom}_{\Lambda'}(-,T)\big)$ for some tilting bimodule ${}_\Lambda T_{\Lambda'}$ which is strong on both sides.

Corollary 10. *Again, let Λ, Λ' be finite dimensional algebras. If there are resolving subcategories of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\text{mod-}\Lambda')$, respectively, which are dual via strictly exact functors, then Λ and Λ' are derived equivalent.*

On combining Theorems 7 and 8 with Theorem 3, we can further sharpen the picture, since these results imply $\mathfrak{C}({}_\Lambda T) = \mathfrak{co}\text{-}\mathfrak{res}^{<\infty}({}_\Lambda T)$ for any tilting module ${}_\Lambda T$.

Corollary 11. *The resolving subcategories $\mathfrak{C}$ of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ which are strictly dual to resolving subcategories $\mathfrak{C}'$ of $\mathcal{P}^{<\infty}(\text{mod-}\Lambda')$ (for a suitable finite dimensional algebra Λ' depending on $\mathfrak{C}$) are precisely the contravariantly finite ones, i.e., those of the form $\mathfrak{co}\text{-}\mathfrak{res}^{<\infty}({}_\Lambda T)$ for some tilting module ${}_\Lambda T$.*

We stress the special case of finite global dimension.

Corollary 12. *Suppose Λ has finite global dimension. Then there is a one-to-one correspondence between the isomorphism classes of basic tilting modules in Λ-mod and the equivalence classes of strictly exact dualities $\mathfrak{C} \longleftrightarrow \mathfrak{C}'$ which are defined on resolving subcategories $\mathfrak{C}$ of Λ-mod; in each case, $\mathfrak{C}'$ is a resolving subcategory of mod-Λ' for some algebra Λ' of finite global dimension that depends on $\mathfrak{C}$.*

Comment. We do not have an example of inverse dualites F, F' between resolving subcategories $\mathfrak{C} \subseteq \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathfrak{C}' \subseteq \mathcal{P}^{<\infty}(\text{mod-}\Lambda')$ for which F or F' fails to be strictly exact. In general, relative epimorphisms in a resolving subcategory $\mathfrak{C}$ of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ need not be surjections though. For instance, take Λ to be $KQ/\langle\beta\alpha\rangle$, where Q is the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, and let $\mathfrak{C}$ be the full subcategory of $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) = \Lambda\text{-mod}$ consisting of the projective modules. Then the map $f \in \mathrm{Hom}_\Lambda(\Lambda e_2, \Lambda e_1)$ with $f(e_2) = \alpha$ is a relative epimorphism in $\mathfrak{C}$. Nonetheless, every duality between $\mathfrak{C}$ and some resolving $\mathfrak{C}' \subseteq \text{mod-}\Lambda'$ is easily seen to be strictly exact in this case.

3.B. Cotilting modules and dualities of $\mathcal{I}^{<\infty}$-categories. We briefly address the cotilting situation by spelling out the duals of Theorems 7 and 8. (The corollaries to these theorems have obvious duals as well.) We now focus on a *cotilting* module $C \in \text{mod-}\Lambda$, the K-dual of a tilting object in Λ-mod. The concept of a *strong* cotilting module is in turn dual to that of a strong tilting module (see [5, definition preceding Proposition 6.3]), and the attributes "*coresolving*" / "*covariantly finite*" for subcategories of mod-Λ are the obvious duals of "resolving" / "contravariantly finite" (see [7] and [5]). In explicit terms: A cotilting module C_Λ is strong in case every module in $\mathcal{I}^{<\infty}(\text{mod-}\Lambda)$ (= the subcategory of mod-Λ consisting of the modules of finite injective dimension) has a finite add(C)-resolution; that is, every right Λ-module M of finite injective dimension admits a resolution $0 \to C^{(m)} \to \cdots \to C^{(0)} \to M \to 0$ with $C^{(i)} \in \text{add}(C)$. Parallel to Theorem 4, a strong cotilting module C is a relatively Ext-projective object of $\mathcal{I}^{<\infty}(\text{mod-}\Lambda)$.

By applying the main results of Section 3.A to the tilting module $D(C)$, where $D = \text{Hom}_K(-, K)$, we obtain

Theorem 13. Correspondence between cotilting modules and dualities defined on coresolving subcategories of $\mathcal{I}^{<\infty}(\text{mod-}\Lambda)$.

(A) Twin of Miyashita's theorem*: Suppose* $C \in \text{mod-}\Lambda$ *is a cotilting module with endomorphism ring* $\widehat{\Lambda}$*. Then the functors* $\text{Hom}_\Lambda(-, C)$ *and* $\text{Hom}_{\widehat{\Lambda}}(-, C)$ *restrict to mutually inverse dualities*

$$\mathcal{I}^{<\infty}(\text{mod-}\Lambda) \cap (C_\Lambda)^\perp \quad \longleftrightarrow \quad \mathcal{I}^{<\infty}(\widehat{\Lambda}\text{-mod}) \cap ({}_{\widehat{\Lambda}}\text{C})^\perp.$$

Equivalently: The covariant Hom-functors $\text{Hom}_\Lambda(C, -)$ *and* $\text{Hom}_{\widehat{\Lambda}}(C, -)$ *induce inverse equivalences between* $\mathcal{I}^{<\infty}(\text{mod-}\Lambda) \cap (C_\Lambda)^\perp$ *and* $\mathcal{P}^{<\infty}(\widehat{\Lambda}\text{-mod}) \cap {}^\perp(\text{D(C)}_{\widehat{\Lambda}})$.

(B) Twin of Theorem 8*: Let* $\widehat{\Lambda}$ *be another finite dimensional algebra, and suppose that* $\mathfrak{C} \subseteq \mathcal{I}^{<\infty}(\text{mod-}\Lambda)$ *and* $\widehat{\mathfrak{C}} \subseteq \mathcal{I}^{<\infty}(\widehat{\Lambda}\text{-mod})$ *are coresolving subcategories of* mod-Λ *and* $\widehat{\Lambda}$-mod*, respectively. Moreover, suppose that* $\mathfrak{C}$ *is dual to* $\widehat{\mathfrak{C}}$ *by way of mutually inverse strictly exact contravariant additive functors*

$$F : \mathfrak{C} \longrightarrow \widehat{\mathfrak{C}} \quad \textit{and} \quad \widehat{F} : \widehat{\mathfrak{C}} \longrightarrow \mathfrak{C}.$$

Then there exists a cotilting bimodule ${}_{\widehat{\Lambda}}C_\Lambda$ *such that:*

- **(a)** $F \cong \text{Hom}_\Lambda(-, C)\,|_{\mathfrak{C}}$ *and* $\widehat{F} \cong \text{Hom}_{\widehat{\Lambda}}(-, C)\,|_{\widehat{\mathfrak{C}}}$*;*
- **(b)** $\mathfrak{C} = \mathcal{I}^{<\infty}(\text{mod-}\Lambda) \cap (C_\Lambda)^\perp = \mathfrak{res}^{<\infty}(C_\Lambda)$ *and* $\widehat{\mathfrak{C}} = \mathcal{I}^{<\infty}(\widehat{\Lambda}\text{-mod}) \cap ({}_{\widehat{\Lambda}}\text{C})^\perp = \mathfrak{res}^{<\infty}({}_{\widehat{\Lambda}}\text{C})$. (*See* 2.A for notation.)

Addendum: *Provided that* $\mathfrak{C}$ *and* $\widehat{\mathfrak{C}}$ *are closed also under kernels of surjective morphisms in the ambient module categories, arbitrary additive dualities* $F : \mathfrak{C} \longrightarrow \widehat{\mathfrak{C}}$ *and* $\widehat{F} : \widehat{\mathfrak{C}} \longrightarrow \mathfrak{C}$ *are strictly exact.*

Remark 14. As a special case, we find that any strong cotilting module C_Λ induces inverse (covariant) equivalences $\mathcal{I}^{<\infty}(\text{mod-}\Lambda) \longleftrightarrow \mathcal{P}^{<\infty}(\text{mod-}\widehat{\Lambda}) \cap {}^\perp\big(D(C)_{\widehat{\Lambda}}\big)$. Specializing further, we retrieve Proposition 6.6 of [5]: If ${}_{\widehat{\Lambda}}C_\Lambda$ is a cotilting bimodule which is strong on both sides, the functor $\text{Hom}_\Lambda(C, -)$ induces an equivalence $\mathcal{I}^{<\infty}(\text{mod-}\Lambda) \to \mathcal{P}^{<\infty}(\text{mod-}\widehat{\Lambda})$. Indeed, by definition, bilateral strongness makes C relatively Ext-projective in both $\mathcal{I}^{<\infty}(\text{mod-}\Lambda)$ and $\mathcal{I}^{<\infty}(\widehat{\Lambda}\text{-mod})$, whence $D(C)_{\widehat{\Lambda}}$ is relatively Ext-injective in the category $\mathcal{P}^{<\infty}(\text{mod-}\widehat{\Lambda})$. This amounts to the equalities $\mathcal{I}^{<\infty}(\text{mod-}\Lambda) \cap (C_\Lambda)^\perp = \mathcal{I}^{<\infty}(\text{mod-}\Lambda)$ and $\mathcal{P}^{<\infty}(\text{mod-}\widehat{\Lambda}) \cap {}^\perp(D(C)_{\widehat{\Lambda}}) = \mathcal{P}^{<\infty}(\text{mod-}\widehat{\Lambda})$. In light of Theorem 13, the converse of this implication holds as well.

Note moreover that, for any cotilting module C_Λ, the categories $\mathcal{I}^{<\infty}(\text{mod-}\Lambda) \cap (C_\Lambda)^\perp$ and $\mathcal{I}^{<\infty}(\widehat{\Lambda}\text{-mod}) \cap ({}_{\widehat{\Lambda}}C)^\perp$ are covariantly finite in mod-Λ and $\widehat{\Lambda}$-mod, respectively (combine the dual of Theorem 3 with Theorem 13.B).

Proof of Theorem 13 from Theorems 7, 8. Set $T = D(C)$. For part **(A)**, apply Theorem 7 to the tilting module ${}_\Lambda T$ with $\operatorname{End}_\Lambda(T)^{\mathrm{op}} = \widehat{\Lambda}$, so as to obtain inverse dualites $\operatorname{Hom}_\Lambda(-,T)$ and $\operatorname{Hom}_{\widehat{\Lambda}}(-,T)$

$$\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \cap {}^\perp({}_\Lambda T) \longleftrightarrow \mathcal{P}^{<\infty}(\text{mod-}\widehat{\Lambda}) \cap {}^\perp(T_{\widehat{\Lambda}}).$$

Since D takes $(C_\Lambda)^\perp$ to ${}^\perp({}_\Lambda T)$, and ${}^\perp(T_{\widehat{\Lambda}})$ to $({}_{\widehat{\Lambda}}C)^\perp$, this yields the following commutative diagram.

$$\begin{tikzcd}
\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \cap {}^\perp({}_\Lambda T) \arrow[r, "\operatorname{Hom}_\Lambda(-,T)"] & \mathcal{P}^{<\infty}(\text{mod-}\widehat{\Lambda}) \cap {}^\perp(T_{\widehat{\Lambda}}) \arrow[d, "D"] \\
\mathcal{I}^{<\infty}(\text{mod-}\Lambda) \cap (C_\Lambda)^\perp \arrow[u, "D"] \arrow[ur, "\operatorname{Hom}_\Lambda(C,-)"] \arrow[r, "\operatorname{Hom}_\Lambda(-,C)"] & \mathcal{I}^{<\infty}(\widehat{\Lambda}\text{-mod}) \cap ({}_{\widehat{\Lambda}}C)^\perp
\end{tikzcd}$$

A straightforward computation thus yields the first claim. Part **(B)** is deduced from Theorem 8 by a similar argument. □

3.C. Connection with the Riedtmann-Schofield partial order. As mentioned in the introduction, Riedtmann and Schofield [34] introduced a partial order on the set of all basic tilting objects in Λ-mod: $T_1 \le T_2$ if and only if $T_1^\perp \subseteq T_2^\perp$. The orientation of this partial order is in synchrony with the sizes of the domains of the partial equivalences induced by the T_i; in [31, Theorem 1.16], this was first spelled out for the current notion of a tilting module. To put it in rough terms: The larger T is in this partial order, the more comprehensive the equivalences induced by T. In this vein, the unique largest tilting object in Λ-mod, namely the basic projective generator for Λ-mod, induces an actual Morita equivalence on Λ-Mod.

We next observe that, as the domains of the covariant equivalences induced by tilting modules grow, the domains of Miyashita's contravariant equivalences shrink in tandem. More precisely:

Observation 15. *For tilting modules $T_1, T_2 \in \Lambda$-mod,*

$$T_1 \le T_2 \quad \textit{if and only if} \quad \mathfrak{C}(T_1) \supseteq \mathfrak{C}(T_2).$$

Proof. To see this, recall that $\mathfrak{C}(T) = \mathcal{P}^{<\infty}(\Lambda\text{-mod}) \cap {}^\perp T$. We first remark that the proof of [22, Theorem 2.1] establishes the equivalence "$T_1 \le T_2 \iff T_1 \in T_2^\perp$". We supplement it by showing that the condition $T_1 \in T_2^\perp$, i.e., $T_2 \in {}^\perp T_1$, implies $\mathfrak{C}(T_2) \subseteq \mathfrak{C}(T_1)$. So let

$X \in \mathfrak{C}(T_2)$. In view of $\mathfrak{C}(T_2) = \mathfrak{co\text{-}res}^{<\infty}(T_2)$, this means that X has a finite $\operatorname{add}(T_2)$-coresolution $0 \to X \to U^{(0)} \to \cdots \to U^{(m)} \to 0$ with $U^{(j)} \in \operatorname{add}(T_2)$. Since T_2 is left perpendicular to T_1 by hypothesis, we obtain $\operatorname{Ext}^i_\Lambda(U^{(j)}, T_1) = 0$ for $i \geq 1$. The given coresolution of X therefore yields $\operatorname{Ext}^i_\Lambda(X, T_1) = 0$ for $i \geq 1$. □

This viewpoint, namely that small tilting modules in the Riedtmann-Schofield partial order are closer to injective cogenerators than large ones, is reinforced in Section 4.

4. Tiilting modules viewed as relatively Ext-injective cogenerators

4.A. Projective resolutions versus $\operatorname{add}(T)$-coresolutions. Let $\mathfrak{C}$ be a resolving contravariantly finite subcategory of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. As we glean from Theorems 3, 8 and Corollary 11, there exists a basic tilting module ${}_\Lambda T$ such that $\mathfrak{C} = \mathfrak{C}({}_\Lambda T) = \mathfrak{co\text{-}res}^{<\infty}({}_\Lambda T)$. By definition, the objects of $\mathfrak{C}({}_\Lambda T)$ have finite $\operatorname{add}({}_\Lambda T)$-coresolutions, whence they have nonnegative $\operatorname{add}(T)$-codimension in the following sense: For $X \in \mathfrak{C}$, we denote by $\operatorname{codim}_{\operatorname{add}(T)}(X)$ the minimal length of an $\operatorname{add}(T)$-coresolution $0 \to X \to T^{(0)} \to \cdots \to T^{(m)} \to 0$. Again, we set $\widetilde{\Lambda} = \operatorname{End}_\Lambda(T)^{\text{op}}$ and $\widetilde{\mathfrak{C}} = \mathfrak{C}(T_{\widetilde{\Lambda}})$.

Lemma and Terminology 16. *Keep the above notation.*

(a) *The functor* $F = \operatorname{Hom}_\Lambda(-, T)|_{\mathfrak{C}} : \mathfrak{C} \to \widetilde{\mathfrak{C}}$ *takes the exact sequences* $0 \to X \to T^{(0)} \to \cdots \to T^{(m)} \to 0$ *and* $0 \to P^{(r)} \to \cdots \to P^{(0)} \to X \to 0$, *with* $X \in \mathfrak{C}$, $T^{(i)} \in \operatorname{add}({}_\Lambda T)$ *and* $P^{(j)} \in \operatorname{add}({}_\Lambda \Lambda)$, *to projective resolutions, resp.,* $\operatorname{add}(T_{\widetilde{\Lambda}})$*-coresolutions, of* $F(X)$ *in* $\widetilde{\mathfrak{C}}$. *These resolutions have the same lengths as the corresponding ones in* $\mathfrak{C}$. *Thus*

$$\operatorname{p\,dim}_\Lambda X = \operatorname{codim}_{\operatorname{add}(T_{\widetilde{\Lambda}})} \operatorname{Hom}_\Lambda(X, T) \quad \textit{and} \quad \operatorname{codim}_{\operatorname{add}({}_\Lambda T)} X = \operatorname{p\,dim}_{\widetilde{\Lambda}} \operatorname{Hom}_\Lambda(X, T).$$

Analogous statements hold for the functor $\operatorname{Hom}_{\widetilde{\Lambda}}(-, T)|_{\widetilde{\mathfrak{C}}} : \widetilde{\mathfrak{C}} \to \mathfrak{C}$.

(b) *The values of* $\operatorname{codim}_{\operatorname{add}({}_\Lambda T)} X$ *are uniformly bounded on* $\mathfrak{C}$, *and those of* $\operatorname{codim}_{\operatorname{add}(T_{\widetilde{\Lambda}})}(Y)$ *are uniformly bounded on* $\widetilde{\mathfrak{C}}$.

Proof. To derive part (a) from Miyashita's duality, it suffices to observe that all consecutive kernels arising in the displayed exact sequences, read from right to left, again belong to $\mathfrak{C}$. As for the uniform boundedness claim in (b): $\mathfrak{C}$ and $\widetilde{\mathfrak{C}}$ are contravariantly finite in Λ-mod, resp., mod-$\widetilde{\Lambda}$ by Theorem 3. Therefore the Λ-, resp., $\widetilde{\Lambda}$-projective dimensions of the objects in $\mathfrak{C}$, resp., $\widetilde{\mathfrak{C}}$, are bounded from above by the maximum of the projective dimensions of the minimal $\mathfrak{C}$-, resp., minimal $\widetilde{\mathfrak{C}}$-approximations, of the simples in Λ-mod, resp., mod-$\widetilde{\Lambda}$; see Section 2. □

To stress the analogy of the dualities induced by tilting modules with the scenarios of more classical dualities of module categories, we translate the $\operatorname{add}(T)$-codimension of an object X of $\mathfrak{C} = \mathfrak{C}({}_\Lambda T)$ into the Ext-injective dimension of X in $\mathfrak{C}$, as introduced in 2.D.

Corollary 17. *Retain the above hypotheses and notation.*

Then $\operatorname{codim}_{\operatorname{add}(T)} X = \operatorname{Ext.idim}_{\mathfrak{C}} X$ *for all* $X \in \mathfrak{C}$.

Moreover, the following invariants associated to $\mathfrak{C}$ *coincide* (*and hence are all finite*): $\sup\{\operatorname{p\,dim}_\Lambda X \mid X \in \mathfrak{C}\} = \sup\{\operatorname{Ext.idim}_{\mathfrak{C}} X \mid X \in \mathfrak{C}\} = \sup\{\operatorname{p\,dim}_{\widetilde{\Lambda}} Y \mid Y \in \widetilde{\mathfrak{C}}\} = \sup\{\operatorname{Ext.idim}_{\widetilde{\mathfrak{C}}} Y \mid Y \in \widetilde{\mathfrak{C}}\}$.

Proof. In light of $\mathrm{Ext.idim}_{\mathfrak{C}}\, T = 0$, the initial claim follows from a straightforward induction on m, applied to an $\mathrm{add}({}_\Lambda T)$-coresolution $0 \to X \to T^{(0)} \to \cdots \to T^{(m)} \to 0$ of minimal length.

As for the string of additional equalities: Let c be the maximum of the finite set $\{\mathrm{p\,dim}_\Lambda X \mid X \in \mathfrak{C}\}$. Since the category $\mathfrak{C}$ is resolving, the Λ-projective dimension of X coincides with the relative Ext-projective dimension of X in $\mathfrak{C}$ (see Remark 1). Therefore, Lemma 16 yields $c = \inf\{j \ge 0 \mid \mathrm{Ext}^{j+1}(X, -)|_{\mathfrak{C}} = 0 \text{ for all } X \in \mathfrak{C}\} = \inf\{j \ge 0 \mid \mathrm{Ext}^{j+1}(-,-)|_{\mathfrak{C}\times\mathfrak{C}} = 0\} = \inf\{j \ge 0 \mid \mathrm{Ext}^{j+1}(-, X)|_{\mathfrak{C}} = 0 \mid \text{ for all } X \in \mathfrak{C}\} = \sup\{\mathrm{Ext.idim}_{\mathfrak{C}}\, X \mid X \in \mathfrak{C}\}$. The second equality is immediate from Lemma 16(a), the final one is symmetric to the first. ☐

4.B. The case of a strong tilting module. The results of this section are closely intertwined with the theory developed by Auslander and Reiten in [5]. Recall that an object $T \in \mathfrak{C}$ is a *cogenerator in* $\mathfrak{C}$ if every object in $\mathfrak{C}$ embeds into a power of T. The following characterization of strong tilting modules buttresses the viewpoint that dualties $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \longleftrightarrow \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ should be viewed as variants of the standard duality $\Lambda\text{-mod} \longleftrightarrow \text{mod-}\Lambda$ in the special case when Λ is Gorenstein. When viewed from this angle, condition (2) below reveals contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ to be a weakened Gorenstein condition for Λ.

Proposition 18. (See [24, Proposition 4].) *Let Λ be any finite dimensional algebra. For $T \in \Lambda$-mod, the following conditions are equivalent:*

(1) *T is a strong tilting module in the sense of Auslander-Reiten, i.e., T is a tilting module and $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) = \mathfrak{co\text{-}res}^{<\infty}({}_\Lambda T)$.*

(2) *T is a relatively* Ext*-injective cogenerator in the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, and all objects in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ have finite relative* Ext*-injective dimension.*

(3) *T is a tilting module which is relatively* Ext*-injective in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$.*

In case these conditions are satisfied, T is even relatively splitting injective in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ (cf. Proposition 6)*. Moreover,*

$$\sup\{\mathrm{Ext.idim}_{\mathcal{P}^{<\infty}(\Lambda\text{-mod})} X \mid X \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})\} = \mathrm{fin\,dim}\,\Lambda = \mathrm{Fin\,dim}\,\Lambda,$$

where $\mathrm{fin\,dim}\,\Lambda$ *and* $\mathrm{Fin\,dim}\,\Lambda$ *denote the* left *little and big finitistic dimensions of Λ. This number is smaller than or equal to the right finitistic dimension of $\widetilde{\Lambda}$.*

Corollary 19. (See [24, Corollary 5] and [14, Section 8].) *Suppose that ${}_\Lambda T_{\widetilde{\Lambda}}$ is a tilting bimodule which is strong on both sides. Then the coinciding left finitistic dimensions of Λ are equal to the coinciding right finitistic dimensions of $\widetilde{\Lambda}$.*

4.C. Arbitrary tilting modules. In order to emphasize the conceptual connection between "tilting dualities" and standard dualities, we extend Proposition 18 so as to fit arbitrary tilting modules into an analogous pattern. The description is immediate from Theorems 7, 8 and the results of Section 4.A.

Observation 20. *Suppose $T \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is a basic tilting module. Moreover, let $\mathfrak{C} = \mathfrak{C}({}_\Lambda T)$ and $\widetilde{\mathfrak{C}} = \mathfrak{C}(T_{\widetilde{\Lambda}})$, where $\widetilde{\Lambda} = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$. Then T is a relatively Ext-injective* *cogenerator in* $\mathfrak{C}$*, and* $\operatorname{Ext.idim}_{\mathfrak{C}} X < \infty$ *for all* $X \in \mathfrak{C}$*. Moreover, the following coinciding invariants are finite:*

$$\sup\{\operatorname{p\,dim}_\Lambda X \mid X \in \mathfrak{C}\} \;=\; \sup\{\operatorname{Ext.idim}_{\mathfrak{C}} X \mid X \in \mathfrak{C}\} \;=$$
$$\sup\{\operatorname{p\,dim}_{\widetilde{\Lambda}} Y \mid Y \in \widetilde{\mathfrak{C}}\} \;=\; \sup\{\operatorname{Ext.idim}_{\widetilde{\mathfrak{C}}} Y \mid Y \in \widetilde{\mathfrak{C}}\}.$$

As for relative splitting injectivity of T in $\mathfrak{C}$, the upcoming observation will show that the number of those indecomposable direct summands of T which are splitting injective in $\mathfrak{C}$ equals the number of non-isomorphic direct summands of the minimal approximation $\mathcal{A}_{\mathfrak{C}}(E)$ of any injective cogenerator $E \in \Lambda\text{-mod}$. Indeed, on combining Observation 20 with Lemma 2, we encounter the announced extension of Proposition 6:

Corollary 21. *Let* $T \in \Lambda\text{-mod}$ *be a basic tilting module, and let* $\mathfrak{C}$ *be the contravariantly finite category* $\mathfrak{C}({}_\Lambda T)$*. Then the minimal* $\mathfrak{C}$*-approximation* $\mathcal{A}_{\mathfrak{C}}(E)$ *of the basic injective cogenerator* E *in* $\Lambda\text{-mod}$ *belongs to* $\operatorname{add}(T)$*; in other words, every indecomposable direct summand of* $\mathcal{A}_{\mathfrak{C}}(E)$ *is also a direct summand of* T*. Thus* T^k *is a* $\mathfrak{C}$*-approximation of* E *for some* $k \geq 1$ *depending on the multiplicities of the indecomposable direct summands of* $\mathcal{A}_{\mathfrak{C}}(E)$*.*

In case $\mathfrak{C}$ *is closed under arbitrary cokernels of monomorphisms, equality holds:* $\operatorname{add}(T) = \operatorname{add}\big(\mathcal{A}_{\mathfrak{C}}(E)\big)$*.*

Proof. Observation 20 guarantees an embedding $\mathcal{A}_{\mathfrak{C}}(E) \subseteq T^k$ for some k. Since, by Lemma 2, $\mathcal{A}_{\mathfrak{C}}(E)$ is relatively splitting injective in $\mathfrak{C}$, any such inclusion splits. □

Corollary 21 is another indication of the intermediate position between a projective generator and an injective cogenerator which is held by a tilting module: Any tilting module T comes paired with a homomorphism $\phi(T) : T \to E$ such that $\phi(T)^k : T^k \to E$ is a $\mathfrak{C}({}_\Lambda T)$-approximation of E.

We conclude the section with some open problems triggered by Observation 20 and Corollary 21.

Problem 1. Compare $\operatorname{fin\,dim} \Lambda$ with

$$\sup\{\operatorname{fin\,dim} \mathfrak{C}({}_\Lambda T) \mid T \in \Lambda\text{-mod} \;\text{ is a tilting module}\}.$$

Problem 2. Keep the notation of Corollary 21. For which basic tilting modules ${}_\Lambda T$ is the category $\mathfrak{C}({}_\Lambda T)$ closed under cokernels of injective homomorphisms? This problem leads to the following more probing one: Let $\alpha(T)$ be the number of isomorphism classes of indecomposable objects in $\operatorname{add}\big(\mathcal{A}_{\mathfrak{C}(T)}(E)\big)$; we know that this is the number of indecomposable direct summands of T which are relatively splitting injective in $\mathfrak{C}({}_\Lambda T)$ (as opposed to only being relatively Ext-injective). Clearly, $\alpha(T) \leq \operatorname{rank} K_0(\Lambda)$, with equality holding when T is strong. The invariant $\alpha(T)$ gauges the "degree of relative injectivity" of T or, alternatively, the completeness of $\mathfrak{C}(T)$ with respect to cokernels of embeddings $T \hookrightarrow C$, for $C \in \mathfrak{C}$. Obviously, this measure is far coarser than the Riedtmann-Schofield partial order. We include a simple instance in which $\mathfrak{C}({}_\Lambda T_1) \subsetneqq \mathfrak{C}({}_\Lambda T_2)$ does not imply $\alpha(T_1) < \alpha(T_2)$. Pick up the notation of Example 5 for the Kronecker algebra Λ, and denote by R_2 the indecomposable preprojective module with top dimension 2. For the tilting module $T_1 = {}_\Lambda\Lambda$, we obtained $\alpha(T) = 1$. For the alternate choice $T_2 = \Lambda e_1 \oplus R_2$, we find

$\mathfrak{C}(T_2) = \mathcal{P}^{<\infty}(\Lambda\text{-mod}) \cap {}^{\perp}T_2 \supsetneqq \mathfrak{C}(T_1) = \mathrm{add}({}_\Lambda\Lambda)$; on the other hand, the minimal $\mathfrak{C}(T_2)$-approximation of the basic injective cogenerator equals $(R_2)^2$, whence $\alpha(T_1) = \alpha(T_2) = 1$.

Given a finite dimensional algebra Λ, determine the values among $\{1, \dots, \mathrm{rank}\, K_0(\Lambda)\}$ which are attained by the function $\alpha(T)$, as T traces the tilting objects in Λ-mod. May "gaps" of size ≥ 2 occur? Comment: If the left finitistic dimension of Λ is zero, the only value attained by $\alpha(T)$ is $\alpha({}_\Lambda\Lambda) = \mathrm{rank}\, K_0(\Lambda)$.

5. The big companion categories of $\mathfrak{co\text{-}res}^{<\infty} T$. Further open problems

Throughout, $T \in \Lambda$-mod stands for a basic tilting module. Again, $\mathfrak{C} = \mathfrak{C}({}_\Lambda T) = \mathfrak{co\text{-}res}^{<\infty}({}_\Lambda T)$ and $\widetilde{\mathfrak{C}} = \mathfrak{C}(T_{\widetilde{\Lambda}}) = \mathfrak{co\text{-}res}^{<\infty}(T_{\widetilde{\Lambda}})$, where $\widetilde{\Lambda} = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$ and $\mathfrak{C}({}_\Lambda T) = \mathcal{P}^{<\infty}(\Lambda\text{-mod}) \cap {}^{\perp}({}_\Lambda T)$. Recall from Section 2.A that $\mathfrak{Co\text{-}Res}^{<\infty}({}_\Lambda T)$ denotes the full subcategory of Λ-Mod which consists of the modules with finite coresolutions by objects in $\mathrm{Add}(T)$. Evidently, $\mathfrak{Co\text{-}Res}^{<\infty}({}_\Lambda T)$ contains $\mathfrak{C}({}_\Lambda T)$, $\mathrm{Add}({}_\Lambda\Lambda)$ and $\mathrm{Add}({}_\Lambda T)$.

Problem 3. Compare the categories $\mathfrak{Co\text{-}Res}^{<\infty}({}_\Lambda T)$ and $\mathcal{P}^{<\infty}(\Lambda\text{-Mod}) \cap {}^{\perp}({}_\Lambda T)$.

- It is immediate from the definition that $\mathfrak{Co\text{-}Res}^{<\infty}({}_\Lambda T) \subseteq \mathcal{P}^{<\infty}(\Lambda\text{-Mod}) \cap {}^{\perp}({}_\Lambda T)$.
- In case ${}_\Lambda T$ is a strong tilting module, equality holds by [25]. Indeed, in this case $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ consists of the direct limits of objects in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$; all of these belong to ${}^{\perp}({}_\Lambda T)$, since pure injectivity of T guarantees that the functors $\mathrm{Ext}^i_\Lambda(-, T)$ take direct limits to inverse limits (see [3]).

Our interest in the category $\mathfrak{Co\text{-}Res}^{<\infty}({}_\Lambda T)$ is buttressed by the following observation:

Observation 22. *The functors $F = \mathrm{Hom}_\Lambda(-, T)$ and $\widetilde{F} = \mathrm{Hom}_{\widetilde{\Lambda}}(-, T)$ induce functors*

$$\mathcal{P}^{<\infty}(\Lambda\text{-Mod}) \cap {}^{\perp}({}_\Lambda T) \longrightarrow \mathfrak{Co\text{-}Res}^{<\infty}(T_{\widetilde{\Lambda}}), \quad \textit{and}$$
$$\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda}) \cap {}^{\perp}(\mathrm{T}_{\widetilde{\Lambda}}) \longrightarrow \mathfrak{Co\text{-}Res}^{<\infty}({}_\Lambda \mathrm{T}),$$

respectively. In particular, F and $\widetilde{F}$ induce functors

$$\mathfrak{Co\text{-}Res}^{<\infty}({}_\Lambda T) \rightleftarrows \mathfrak{Co\text{-}Res}^{<\infty}(T_{\widetilde{\Lambda}}).$$

Obviously these are not dualities in the infinite dimensional setting, but they induce Miyashita's dualities on finite dimensional modules. Moreover, the displayed restrictions of F and $\widetilde{F}$ are strictly exact on the indicated subcategories of Λ-Mod *and* Mod-$\widetilde{\Lambda}$.

Proof. We only show the first claim, the second being symmetric and the following one a consequence. For $M \in \mathcal{P}^{<\infty}(\Lambda\text{-Mod}) \cap {}^{\perp}({}_\Lambda T)$, consider a finite projective resolution

$$(\dagger) \qquad 0 \to P_r \to \cdots \to P_0 \to M \to 0.$$

Write $P_j \cong \bigoplus_{1 \leq i \leq n} (\Lambda e_i)^{(A_{ji})}$. Since the syzygies of the resolution $(\dagger)$ again belong to ${}^{\perp}({}_\Lambda T)$, application of the functor F yields an exact sequence $(\ddagger)$ $0 \to F(M) \to \prod_{1 \leq i \leq n} T_i^{A_{0i}} \to \cdots \to \prod_{1 \leq i \leq n} T_i^{A_{ri}} \to 0$, where the $T_i = \mathrm{Hom}_\Lambda(\Lambda e_i, T)$ are objects of $\mathrm{Add}(T_{\widetilde{\Lambda}})$. Since $T_{\widetilde{\Lambda}}$ is endofinite, direct products of objects in $\mathrm{Add}(T_{\widetilde{\Lambda}})$ again belong to this category, which shows $(\ddagger)$ to be a finite $\mathrm{Add}(T_{\widetilde{\Lambda}})$-coresolution of $F(M)$. The final comment is obvious in light of the inclusion $\mathfrak{Co\text{-}Res}^{<\infty}({}_\Lambda T) \subseteq {}^{\perp}({}_\Lambda T)$. $\square$

Problem 4. Recall that $\mathfrak{C}({}_\Lambda T) = \mathfrak{co\text{-}res}^{<\infty}({}_\Lambda T)$, and denote by $\varinjlim \mathfrak{C}({}_\Lambda T)$ the closure of $\mathfrak{C}({}_\Lambda T)$ under direct limits. Compare the categories

$$\varinjlim \mathfrak{C}({}_\Lambda T) \quad \text{and} \quad \mathfrak{Co\text{-}Res}^{<\infty}({}_\Lambda T).$$

We again add some comments:

- Always, $\varinjlim \mathfrak{C}({}_\Lambda T) \subseteq \mathfrak{Co\text{-}Res}^{<\infty}({}_\Lambda T)$ by the upcoming proposition.
- When ${}_\Lambda T$ is strong, the two categories being compared are equal, since $\varinjlim \mathfrak{C}({}_\Lambda T) = \mathcal{P}^{<\infty}(\Lambda\text{-Mod}) = \mathcal{P}^{<\infty}(\Lambda\text{-Mod}) \cap {}^\perp({}_\Lambda T)$ in that case.
- Whenever $\varinjlim \mathfrak{C}({}_\Lambda T) = \mathfrak{Co\text{-}Res}^{<\infty}({}_\Lambda T)$, the finitistic dimension $\operatorname{fin\,dim} \mathfrak{Co\text{-}Res}^{<\infty}({}_\Lambda T)$ is finite; here $\operatorname{fin\,dim} \mathcal{X}$ denotes the supremum of the finite projective dimensions attained on a category $\mathcal{X}$ of modules.

Proposition 23. $\varinjlim \mathfrak{C}({}_\Lambda T) \subseteq \mathfrak{Co\text{-}Res}^{<\infty}({}_\Lambda T)$.

Proof. Let $\mathcal{X} = \big((C_i)_{i \in \mathcal{I}}, (f_{ij})_{i \le j}\big)$ be a directed system of objects in $\mathfrak{C}({}_\Lambda T)$. To show that the direct limit of $\mathcal{X}$ belongs to $\mathfrak{Co\text{-}Res}^{<\infty}({}_\Lambda T)$, we first shift the system to mod-$\widetilde{\Lambda}$ via $\operatorname{Hom}_\Lambda(-, T)$, to arrive at an inverse system $\mathcal{Y} = \big((\widetilde{C}_i)_{i \in \mathcal{I}}, (\widetilde{f}_{ji})_{i \le j}\big)$ in $\mathfrak{Co\text{-}Res}^{<\infty}(T_{\widetilde{\Lambda}})$ with $\widetilde{f}_{ji} = \operatorname{Hom}_\Lambda(f_{ij}, T)$. Keeping in mind that the projective dimensions of the modules in $\mathfrak{C}(T_{\widetilde{\Lambda}})$ are uniformly bounded (see Observation 20), by an integer c say, we will construct an inverse system of projective resolutions

$$(\dagger) \qquad 0 \to P_{ic} \xrightarrow{p_{ic}} P_{i,c-1} \xrightarrow{p_{i,c-1}} \cdots \xrightarrow{p_{i1}} P_{i0} \xrightarrow{p_{i0}} \widetilde{C}_i \to 0$$

of the $\widetilde{C}_i$ in the big category Mod-$\widetilde{\Lambda}$. More precisely, by induction on $0 \le k \le c-1$, one constructs the modules P_{ik} and maps p_{ik} as shown, next to maps $\widetilde{g}_{ji}^{(k)} \in \operatorname{Hom}_\Lambda(P_{jk}, P_{ik})$ for $i \le j$, so as to obtain inverse systems $\mathcal{P}_k = \big((P_{ik})_{i \in \mathcal{I}}, (\widetilde{g}_{ji}^{(k)})_{i \le j}\big)$, together with morphisms $(p_{ik})_{i \in \mathcal{I}}$ of inverse systems, resulting in an exact sequence $\mathcal{P}_{c-1} \to \cdots \mathcal{P}_0 \to \mathcal{Y} \to 0$ of inverse systems. As for the left-most term in $(\dagger)$: Due to the fact that $\operatorname{p\,dim}_{\widetilde{\Lambda}} \widetilde{C}_i \le c$ for all i, the kernels of the morphisms $p_{i,c-1}$ then constitute an inverse system $\mathcal{P}_c$ of projective $\widetilde{\Lambda}$-modules as well.

The construction for $k \le c-1$ is dual to that of [12, Lemma 9.5*]. For $k = 0$, we let P_{i0} be the free right $\widetilde{\Lambda}$-module with the set $\widetilde{C}_i$ as basis, and define p_{i0} to be the epimorphism $P_{i0} \to \widetilde{C}_i$ which sends any element in the distinguished basis of P_{i0} to its namesake in $\widetilde{C}_i$. Furthermore, we define the required $\widetilde{g}_{ji}^{(0)}$ for $i \le j$ in terms of the fixed bases, by sending any basis element b of P_{j0} to the distinguished basis element $\widetilde{f}_{ji}(b)$ of P_{i0}. Clearly, this ensures that $p_{i0} \circ \widetilde{g}_{ji}^{(0)} = \widetilde{f}_{ji} \circ p_{j0}$ whenever $i \le j$. To proceed to $k = 1$, one observes that the kernel of the morphism $(p_{i0})_{i \in \mathcal{I}}$ from the inverse system $\mathcal{P}_0 = \big((P_{i0})_{i \in \mathcal{I}}, (\widetilde{g}_{ji}^{(0)})_{i \le j}\big)$ to $\mathcal{Y}$ is an inverse system of kernels $\mathcal{K}_0 = \big(\operatorname{Ker}(p_{i0})\big)_{i \in \mathcal{I}}$ with the induced maps. We now repeat the initial step for $\mathcal{K}_0$ in place of $\mathcal{Y}$.

Next we flip the constructed inverse systems in $\mathfrak{Co\text{-}Res}^{<\infty}(T_{\widetilde{\Lambda}})$ back to Λ-Mod via the functor $\widetilde{F} = \operatorname{Hom}_{\widetilde{\Lambda}}(-, T)$. Since this functor is strictly exact on the subcategory

$\mathfrak{Co}\text{-}\mathfrak{Res}^{<\infty}(T_{\widetilde{\Lambda}})$ of Mod-$\widetilde{\Lambda}$ and the C_i are reflexive relative to T, this produces direct systems

$$\mathcal{T}_k = \big((\mathrm{Hom}_{\widetilde{\Lambda}}(P_{ik}, T))_{i \in \mathcal{I}}, (\mathrm{Hom}_{\widetilde{\Lambda}}(\widetilde{g}_{ji}, T)_{i \le j}\big)$$

in $\mathfrak{Co}\text{-}\mathfrak{Res}^{<\infty}({}_\Lambda T)$ for $0 \le k \le c$, giving rise to an exact sequence

$$0 \to C_i \to \mathcal{T}_0 \to \cdots \to \mathcal{T}_c \to 0$$

of directed systems. Observe that the left Λ-modules $\mathrm{Hom}_{\widetilde{\Lambda}}(P_{ik}, T)$ are direct products of objects in $\mathrm{Add}({}_\Lambda T)$ and thus belong to $\mathrm{Add}({}_\Lambda T)$ by endofiniteness of ${}_\Lambda T$. Applying the direct limit functor, we finally obtain an exact sequence

$$(\ddagger) \qquad 0 \to \varinjlim C_i \to \varinjlim \mathcal{T}_0 \to \cdots \to \varinjlim \mathcal{T}_c \to 0.$$

By [25, Observation 3.1], the category $\mathrm{Add}({}_\Lambda T)$ is also closed under direct limits, showing $(\ddagger)$ to be a finite $\mathrm{Add}({}_\Lambda T)$-coresolution of $\varinjlim C_i$ as required. $\square$

6. Applications to truncated path algebras

In the sequel, Λ *will denote a truncated path algebra*, say

$$\Lambda = KQ/\langle \text{all paths of length } L+1 \rangle$$

for some quiver Q and positive integer L. We identify the vertices of Q with a full sequence of primitive idempotents $e_1, \dots, e_n$ of Λ. A vertex e of Q (alias a primitive idempotent of Λ) is *precyclic* in case there is an oriented path in Q which starts in e and ends on an oriented cycle; the vertex e is *postcyclic* if the dual holds, and *critical* if it is both pre- and postcyclic. In particular, every vertex that lies on an oriented cycle is critical. Moreover, we say that a simple module S_i is precyclic (postcyclic, critical) if the vertex e_i has the corresponding property.

Due to the key homological difference between precyclic and nonprecyclic vertices of Q, we will fix a convenient indexing: $e_1, \dots, e_m$ are the distinct precyclic vertices, $e_{m+1}, \dots, e_n$ the nonprecyclic ones. Moreover, we will adopt the following notation:

$$\varepsilon = \sum_{\text{nonprecyclic}} e_i = \sum_{m+1 \le i \le n} e_i.$$

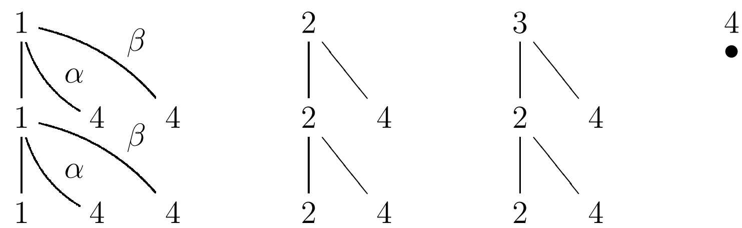
We summarily refer to [15], [14] and [24] for proofs of results that are not explicitly referenced. The theory will be illustrated with a fully worked example, which we set up at the outset. Alongside the development of the theory, we will display (without proofs) the results of our computations for this example. The methods for obtaining them will be developed in Section 7, the actual computations carried out in Section 8.

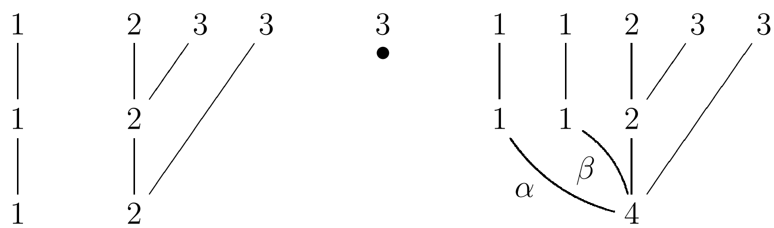
6.A. Reference example. For our graphing conventions, we point to [18, Section 2].

Let $\Lambda = KQ/I$, where Q is the following quiver, and I is the ideal of the path algebra KQ which is generated by all paths of length 3.

$$\begin{tikzcd} 1 \arrow[loop left, "u_1"] \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 4 & 3 \arrow[l, "\gamma"'] \arrow[r, "\delta"] & 2 \arrow[loop right, "u_2"] \arrow[ll, bend left, "\tau"] \end{tikzcd}$$

The precyclic vertices are e_1, e_2, e_3 in this case, with e_1, e_2 critical and e_3 a precyclic source. For immediate use in the upcoming section we display the indecomposable projectives and the indecomposable injectives in Λ-mod:





6.B. Characterization of the objects in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. Observe that $(1-\varepsilon)\Lambda\varepsilon = 0$. Hence, for any $M \in \Lambda$-Mod, the subspace εM is a Λ-submodule of M. The pivotal feature of the left Λ-modules M of finite projective dimension is the structure of $M/\varepsilon M$.

Theorem and Notation 24. [14, Theorem 3.1] *Let $M \in \Lambda$-Mod. Then $\operatorname{p\,dim}_\Lambda M < \infty$ if and only if $M/\varepsilon M$ is projective as a module over the algebra $\Lambda/\Lambda\varepsilon\Lambda = \Lambda/\varepsilon\Lambda$.*

In other words, $\operatorname{p\,dim}_\Lambda M < \infty$ precisely when $M/\varepsilon M$ is a direct sum of copies of the modules $\mathcal{A}_i = \Lambda e_i/\varepsilon\Lambda e_i = \Lambda e_i/\varepsilon J e_i$ for the precyclic vertices e_i $(1 \le i \le m)$.

In our **reference example**, the indecomposable projective left $(\Lambda/\Lambda\varepsilon\Lambda)$-modules are $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ determined by the graphs

$$\begin{array}{ccccc} 1 & \quad & 2 & \quad & 3 \\ | & & | & & | \\ 1 & & 2 & & 2 \\ | & & | & & | \\ 1 & & 2 & & 2 \end{array}$$

Remark. The category $(\varepsilon\Lambda\varepsilon)$-mod is always a subcategory of Λ-mod because $\varepsilon\Lambda = \Lambda\varepsilon\Lambda$. It is, in fact, a localizing subcategory, which is contained in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ since the quiver of $\varepsilon\Lambda\varepsilon$ is acyclic. By Theorem 24, the quotient category $\big(\mathcal{P}^{<\infty}(\Lambda\text{-mod})/\big((\varepsilon\Lambda\varepsilon)\text{-mod}\big)$ has finite representation type. However, assuming that Q has at least one postcyclic vertex which fails to be critical, we find: No matter what the representation type of $(\varepsilon\Lambda\varepsilon)$-mod, the extensions $\operatorname{Ext}^1_\Lambda(M/\varepsilon M, \varepsilon M)$ for $M \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ always grow sufficiently complex for increasing L' to entail infinite representation type of $\mathcal{P}^{<\infty}\big(\Lambda_{L'}\text{-mod}\big)$ for $L' \gg 0$; here $\Lambda_{L'} = KQ/\langle\text{the paths of length } L'+1\rangle$. The following minimal quiver Q illustrating this was brought to the authors' attention by R. Kinser: Take $\Lambda_{L'} = \mathbb{C}Q/\langle$the paths of length $L'+1\rangle$ based on the quiver

$$\begin{tikzcd} 1 \arrow[loop left] \arrow[r] & 2 \end{tikzcd} .$$

The authors checked that $\mathcal{P}^{<\infty}(\Lambda_{L'}\text{-mod})$ has infinite representation type for $L' \geq 8$ by considering the generic number of parameters of the variety $\mathfrak{Grass}_{\mathbf{d}}^{S_1^2}$ for $\mathbf{d} = \big(2(L'{+}1), L'\big)$. Note that all points of this variety encode modules of finite projective dimension by Theorem 24; for our notation and for techniques to obtain generic numbers of parameters, we refer to [18]. On the side, we mention that $\Lambda_{L'}$ has wild representation type for $L' \geq 4$; see [23].

6.C. Contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and the basic strong tilting module.

Theorem 25. Contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. [14, Theorem 4.2] *The category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite, and the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation $\mathcal{A}(M)$ of any $M \in \Lambda\text{-mod}$ is $P/\varepsilon\operatorname{Ker}(f)$, where $f: P \to M$ is a projective cover of M. In particular, $\mathcal{A}(S_i) = \mathcal{A}_i = \Lambda e_i/\varepsilon J e_i$ for $i \leq n$, and the nonprecyclic simple modules $S_{m+1}, \dots, S_n$ coincide with their minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations.*

Therefore Λ-mod has strong tilting modules in light of Theorem 4. Let T be the basic one. We again write $\widetilde{\Lambda} = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. The following explicit description of the module ${}_\Lambda T$ permits its construction from quiver and Loewy length of Λ; this description is readily derived from Proposition 6 and Theorem 25.

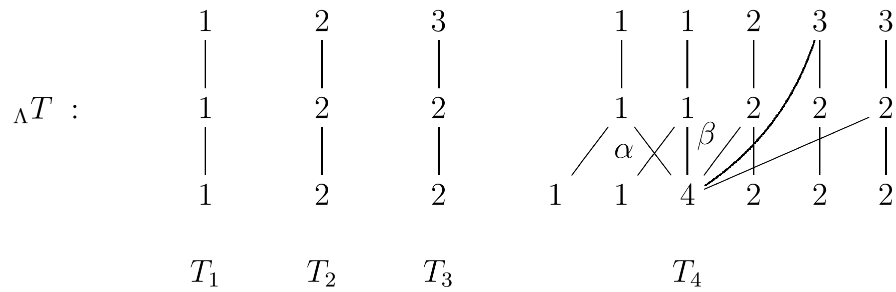
Theorem 26. [14, Theorem 5.3] *Again, let $\mathcal{A}_i = \mathcal{A}(S_i)$ be the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of S_i, and let $E_i = E(S_i)$ be the injective envelope of S_i. Then the basic strong tilting module in Λ-mod is $T = \bigoplus_{1 \leq i \leq n} T_i$, where $T_i = \mathcal{A}_i$ for $1 \leq i \leq m$, and $T_i = \mathcal{A}(E_i)$ for $m+1 \leq i \leq n$.*

Supplement to Theorem 26. [14, Theorem 5.3] We provide further structural detail on the T_i in the two cases "e_i precyclic" and "e_i nonprecyclic" (referring to T_i and S_i as "precyclic" or "nonprecyclic" in case e_i has the pertinent property):

• Any precyclic T_i is a local module with a tree graph, all of whose simple composition factors are in turn precyclic.

• Now suppose that T_i is nonprecyclic. Then the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation $T_i = \mathcal{A}(E_i) \twoheadrightarrow E_i$ shows E_i to also be a factor module of T_i. The socle of T_i contains precisely one nonprecyclic simple direct summand, namely $S_i = \operatorname{soc}(E_i)$, and every submodule $U \subseteq T_i$ with $U \not\subseteq JT_i$ contains S_i in its socle. Moreover, the factor module $T_i/\varepsilon T_i$ is isomorphic to a direct sum of copies of precyclic T_j's, the multiplicity of any critical T_j as a direct summand of $T_i/\varepsilon T_i$ being the number of paths of length L from e_j to e_i. Finally, the graph of T_i is again a tree (as an undirected graph) and thus determines T_i up to isomorphism.

In our **reference example**, the basic strong tilting module in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is $T = \bigoplus_{i=1}^4 T_i$, where the T_i are pinned down, up to isomorphism, by the following graphs:



Here the summands $T_i = \mathcal{A}_i$ for $1 \le i \le 3$ correspond to the precyclic vertices of Q; the only nonprecyclic indecomposable direct summand of T is $T_4 = \mathcal{A}(E_4)$ with $T_4/\varepsilon T_4 \cong T_1^2 \oplus T_2 \oplus T_3^2$.

As was already observed by Auslander and Reiten in [5, Proposition 6.5], one-sided strongness of a tilting bimodule does not necessarily carry over to strongness on the opposite side. This asymmetry still occurs over truncated truncated path algebras, our reference example being an instance; note that e_3 is a precyclic source in this example, and apply the upcoming criterion.

Again, we set $\widetilde{\Lambda} = \operatorname{End}_\Lambda(T)^{\rm op}$. There is a handy criterion for twosided strongness of ${}_\Lambda T_{\widetilde{\Lambda}}$ in the present, truncated, setting:

Criterion 27. Left-right symmetry of strongness. [14, Corollary 7.2] *Let ${}_\Lambda T$ be a strong tilting module, and $\widetilde{\Lambda} = \operatorname{End}_\Lambda(T)^{\rm op}$. Then $T_{\widetilde{\Lambda}}$ is strong also in* mod-$\widetilde{\Lambda}$ *if and only if all precyclic vertices of Q are critical* (*equivalently, if Q does not have a precyclic source*).

6.D. The pivotal endofunctors of the strongly tilted category Mod-$\widetilde{\Lambda}$**.** In the following, $T = \bigoplus_{1 \le i \le n} T_i$ will always stand for the basic strong tilting module in Λ-mod, and $\widetilde{\Lambda}$ for the opposite of its endomorphism ring. Moreover, $\widetilde{J}$ will denote the Jacobson radical of $\widetilde{\Lambda}$. The strongly tilted algebra $\widetilde{\Lambda}$ is in turn a path algebra modulo relations (see [24, Observation 10]), but hardly ever truncated unless Q is acyclic. Write $\widetilde{\Lambda} = K\widetilde{Q}/\widetilde{I}$.

Theorem 7 guarantees that the contravariant Hom-functors $\operatorname{Hom}_\Lambda(-, T)$ and $\operatorname{Hom}_{\widetilde{\Lambda}}(-, T)$ induce inverse dualities

$$\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \quad \longleftrightarrow \quad \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}) \cap {}^{\perp}(T_{\widetilde{\Lambda}}).$$

Since the righthand category is properly contained in $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ in general, this duality does not a priori permit us to transfer information from $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ to $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, or vice versa. The upcoming analysis of $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ through the lens of Q and L additionally hinges on the following **refined partition of the vertices** e_i **of** Q, and on a synchronized partition of the vertices of $\widetilde{Q}$.

Further Conventions. We assume the n vertices $e_1, \dots, e_n$ to be ordered as follows:

- $e_1, \dots, e_r$ are the *critical* vertices of Q, i.e., those that are both pre- and postcyclic.
- $e_{r+1}, \dots, e_m$ are the *precyclic vertices which fail to be postcyclic.*
- $e_{m+1}, \dots, e_s$ are the vertices which are *postcyclic, but not precyclic*; and
- $e_{s+1}, \dots, e_n$ are the vertices which are *neither pre- nor postcyclic.*

Theorem 26 provided us with a natural bijection $\{e_1, \dots, e_n\} \leftrightarrow \{T_1, \dots, T_n\}$. We supplement it by a bijection $\{T_1, \dots, T_n\} \leftrightarrow \{\text{vertices of } \widetilde{\Lambda}\}$ as follows:

The preferred primitive idempotents $\widetilde{e}_i$ of $\widetilde{\Lambda}$, alias the vertices of $\widetilde{Q}$:

$$\widetilde{e}_i = \iota_i \circ \pi_i : \ T \twoheadrightarrow \ T_i \ \hookrightarrow T,$$

where the ι_i and π_i are the canonical injections and projections, respectively.

Criticality in $\widetilde{\Lambda}$. In our analysis of the homology of $\widetilde{\Lambda}$, the role played by the precyclic vertices $e_1, \dots, e_m$ will be taken over by the critical vertices $\widetilde{e}_1, \dots, \widetilde{e}_r$ of $\widetilde{\Lambda}$, namely those which correspond to the critical idempotents $e_1, \dots, e_r$ of Λ. More precisely: An idempotent $\widetilde{e}_i$ of $\widetilde{\Lambda}$ will be referred to as (*tilted*) *critical*, (*tilted*) *pre-* or *postcyclic* precisely when the corresponding idempotent e_i of Λ has the specified property. In fact, in the sequel we will drop the qualifier "tilted" in reference to criticality (etc.) of the $\widetilde{e}_i$, since we will not incur any danger of ambiguity. (Observe, however, that criticality in this sense is not in line with the position of the vertex $\widetilde{e}_i$ relative to oriented cycles of the quiver $\widetilde{Q}$ of $\widetilde{\Lambda}$; in fact, even for a nonprecyclic vertex e_i of Q, the vertex $\widetilde{e}_i$ of $\widetilde{Q}$ will typically lie on multiple oriented cycles of $\widetilde{Q}$; cf. the reference example below.) We moreover extend the use of the attributes "pre(post)cyclic" and "critical" from the e_i and $\widetilde{e}_i$ to the indecomposable projective modules $\Lambda e_i \in \Lambda\text{-mod}$ and $\widetilde{e}_i\widetilde{\Lambda} \in \text{mod-}\widetilde{\Lambda}$, as well as to the simple modules $S_i = \Lambda e_i / J e_i \in \Lambda\text{-mod}$ and $\widetilde{S}_i = \widetilde{e}_i\widetilde{\Lambda}/\widetilde{e}_i\widetilde{J} \in \text{mod-}\widetilde{\Lambda}$.

Finally, we set $\widetilde{\mu} := \sum_{1 \le i \le r} \widetilde{e}_i = \sum_{e_i \text{ critical}} \widetilde{e}_i$.

Definition. The endofunctors ∇ and Δ, and the critical core. We introduce three endofunctors, ∇, Δ and $\mathcal{C}$, of Mod-$\widetilde{\Lambda}$ as follows. Let $\widetilde{M} \in \text{Mod-}\widetilde{\Lambda}$.

- $\nabla(\widetilde{M}) := \widetilde{M}\,\widetilde{\mu}\,\widetilde{\Lambda}$. Thus $\nabla(\widetilde{M})$ is the unique smallest $\widetilde{\Lambda}$-submodule of $\widetilde{M}$ with the property that $\big(\widetilde{M}/\nabla(\widetilde{M})\big)\,\widetilde{\mu} = 0$. Note that the top of $\nabla(\widetilde{M})$ is a direct sum of critical simples.
- $\Delta(\widetilde{M}) := \{x \in \widetilde{M} \mid x\widetilde{\Lambda}\widetilde{\mu} = 0\} = \text{ann}_{\widetilde{M}}(\widetilde{\Lambda}\widetilde{\mu})$ is the unique largest $\widetilde{\Lambda}$-submodule of $\widetilde{M}$ with $\Delta(\widetilde{M})\,\widetilde{\mu} = 0$. In particular, the socle of $\widetilde{M}/\Delta(\widetilde{M})$ is a sum of critical simples.
- $\widetilde{\Delta} := \Delta(\widetilde{\Lambda}_{\widetilde{\Lambda}})$ is a two-sided ideal of $\widetilde{\Lambda}$, referred to as the *noncritical ideal*. Clearly, $\widetilde{M} \cdot \widetilde{\Delta} \subseteq \Delta(\widetilde{M})$ with equality holding in case $\widetilde{M}$ is projective.
- $\mathcal{C}(\widetilde{M}) := \nabla(\widetilde{M})/\Delta(\nabla(\widetilde{M}))$ is called the *critical core* of $\widetilde{M}$.

By definition, both the top and the socle of $\mathcal{C}(\widetilde{M})$ are direct sums of critical simples in mod-$\widetilde{\Lambda}$. On the other hand, typically $\mathcal{C}(\widetilde{M})\widetilde{\mu} \subsetneqq \mathcal{C}(\widetilde{M})$, i.e., $\mathcal{C}(\widetilde{M})$ has also noncritical composition factors in general (see, e.g., $\widetilde{M} = \widetilde{e}_1\widetilde{\Lambda}$ in the reference example displayed below). The critical cores $\mathcal{C}(\widetilde{e}_i\widetilde{\Lambda})$ will play a crucial role in the sequel.

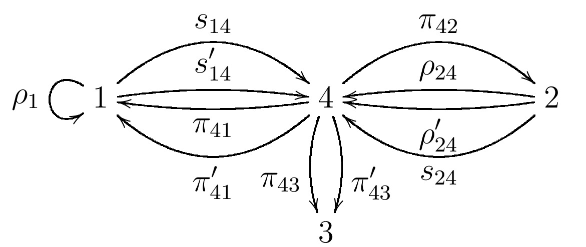
Proposition 28. The critical cores of the $\widetilde{e}_i\widetilde{\Lambda}$. [24, Proposition 6]

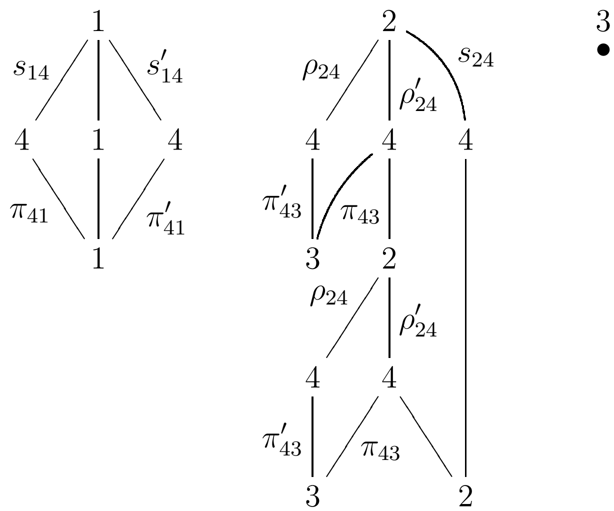
(a) The critical $\widetilde{e}_i$ $(i \le r)$: $\quad \mathcal{C}_i = \mathcal{C}(\widetilde{e}_i\widetilde{\Lambda}) = \widetilde{e}_i\widetilde{\Lambda}/\widetilde{e}_i\widetilde{\Delta}$.

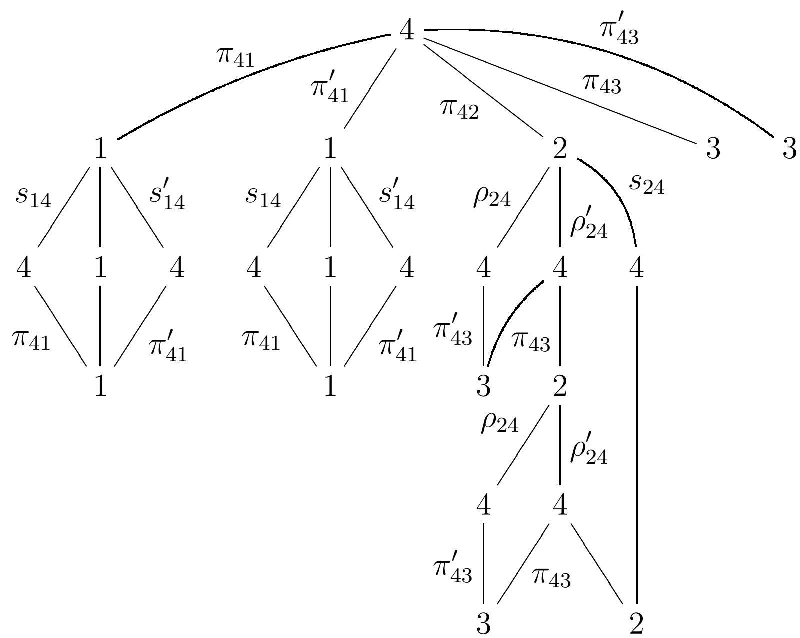
(b) The post- but not precyclic $\widetilde{e}_i$ $(m+1 \le i \le s)$: $\mathcal{C}(\widetilde{e}_i\widetilde{\Lambda}) = \bigoplus_{k=1}^{r} \mathcal{C}_k^{\mathfrak{m}_{ik}}$, where $\mathfrak{m}_{ik}$ is the number of paths of length L from the vertex e_k of Q to the vertex e_i.

(c) The remaining $\widetilde{e}_i$ $(i \in \{r+1, \dots, m\} \cup \{s+1, \dots, n\})$: $\quad \mathcal{C}(\widetilde{e}_i\widetilde{\Lambda}) = 0$.

Return to the reference example. As in the general setting, write $\widetilde{\Lambda} = K\widetilde{Q}/\widetilde{I}$. The quiver $\widetilde{Q}^{\mathrm{op}}$ and the indecomposable projective right $\widetilde{\Lambda}$-modules are as follows; we omit labels on edges connecting vertices a and b when there is only a single arrow from a to b.







We combine these graphs with the information that $\widetilde{I}$ is generated by monomial relations and binomial relations of the form $\widetilde{p} - \widetilde{q}$, where $\widetilde{p}$ and $\widetilde{q}$ are paths in $K\widetilde{Q}$. Taking this subsidiary information into account, we find the indecomposable projective modules $\widetilde{e}_i\widetilde{\Lambda}$ to be pinned down up to isomorphism by their graphs.

The critical cores of the $\widetilde{e}_i\widetilde{\Lambda}$ are as follows. Namely, $\mathcal{C}(\widetilde{e}_1\widetilde{\Lambda}) = \widetilde{e}_1\widetilde{\Lambda}$. Further, $\mathcal{C}(\widetilde{e}_2\widetilde{\Lambda})$ is the quotient $\widetilde{e}_2\widetilde{\Lambda}/U$, where U is the direct sum of the two copies of the uniserial module with composition factors $(\widetilde{S}_4, \widetilde{S}_3)$ plus the copy of $\widetilde{S}_4$ in the socle of $\widetilde{e}_2\widetilde{\Lambda}$; since there is only a single arrow from $\widetilde{e}_4$ to $\widetilde{e}_2$, the socle of $\widetilde{e}_2\widetilde{\Lambda}$ indeed contains a copy of $\widetilde{S}_4$. Thus $\mathcal{C}(\widetilde{e}_2\widetilde{\Lambda})$ has graph

$$\begin{tikzcd} 2 \arrow[d, dash, "\rho'_{24}"'] \arrow[ddd, dash, bend left, "s_{24}"] \\ 4 \arrow[d, dash] \\ 2 \arrow[d, dash, "\rho'_{24}"'] \\ 4 \arrow[d, dash] \\ 2 \end{tikzcd}$$

Moreover, $\mathcal{C}(\widetilde{e}_3\widetilde{\Lambda}) = 0$ and $\mathcal{C}(\widetilde{e}_4\widetilde{\Lambda}) \cong \mathcal{C}(\widetilde{e}_1\widetilde{\Lambda})^2 \oplus \mathcal{C}(\widetilde{e}_2\widetilde{\Lambda})$. Observe that the critical core of $\widetilde{e}_4\widetilde{\Lambda}$ is neither a sub- nor a factor module of $\widetilde{e}_4\widetilde{\Lambda}$, but is properly sandwiched between $\nabla(\widetilde{e}_4\widetilde{\Lambda})$ and $\Delta(\nabla(\widetilde{e}_4\widetilde{\Lambda}))$.

By Criterion 27, the tilting bimodule ${}_\Lambda T_{\widetilde{\Lambda}}$ fails to be strong in mod-$\widetilde{\Lambda}$, since Q has a precyclic source. We will see however that $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ is in turn contravariantly finite and consequently has its own strong tilting module $\widetilde{T} \in \text{mod-}\widetilde{\Lambda}$.

6.E. The structure of the objects in $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$.

Theorem 29. [24, Theorem 13] *As before, $\mathcal{C}_j = \mathcal{C}(\widetilde{e}_j\widetilde{\Lambda})$ is the critical core of $\widetilde{e}_j\widetilde{\Lambda}$.*

A module $\widetilde{M} \in \text{Mod-}\widetilde{\Lambda}$ has finite projective dimension if and only if its critical core $\mathcal{C}(\widetilde{M})$ is a direct sum of copies of $\mathcal{C}_1, \dots, \mathcal{C}_r$; i.e.,

$$\operatorname{p\,dim}_{\widetilde{\Lambda}} \widetilde{M} < \infty \quad \iff \quad \mathcal{C}(\widetilde{M}) \in \operatorname{Add}(\mathcal{C}_1, \dots, \mathcal{C}_r).$$

Supplement for use in Section 7: $\operatorname{p\,dim}_{\widetilde{\Lambda}} \widetilde{M} < \infty$ *if and only if $\widetilde{M}\widetilde{\mu}$ is projective as a right $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$-module, if and only if $\widetilde{M}/\Delta(\widetilde{M})$ is projective as a right module over the algebra $\widetilde{\Lambda}/\widetilde{\Delta}$.*

For the notation of Theorem 29, we refer back to 6.D. Regarding the proof of the supplement: The second of the equivalences added on is not explicitly stated in [24, Theorem 13], but is addressed in the argument given there.

To derive a first consequence: This criterion permits us to recognize the simple right $\widetilde{\Lambda}$-modules of finite projective dimension. In particular, we find that, outside the case where Q has no precyclic source, mod-$\widetilde{\Lambda}$ has more simples of finite projective dimension than Λ-mod does. More precisely:

Corollary 30. [24, Corollary 15] *For $j \in \{1, \dots, n\}$, the simple right $\widetilde{\Lambda}$-module $\widetilde{S}_j$ has finite projective dimension if and only if $\widetilde{S}_j$ is noncritical* (*i.e., $j \in \{r+1, \dots, n\}$*).

Next we address a latent weak heredity property of Λ, which surfaces only on the level of the tilted algebra $\widetilde{\Lambda}$. In our example, the appearance of two copies of $\widetilde{e}_1\widetilde{\Lambda}$ in the radical of $\widetilde{e}_4\widetilde{\Lambda}$, next to a copy of $\widetilde{e}_2\widetilde{\Lambda}$, is not coincidental. Indeed, if $\widetilde{P} = \widetilde{e}_4\widetilde{\Lambda}$, then $\widetilde{P}\widetilde{\mu}\widetilde{\Lambda}$ has top $\widetilde{S}_1^2 \oplus \widetilde{S}_2$ (recall that $\widetilde{\mu}$ is the sum of the critical idempotents $\widetilde{e}_i$, $i \le r$). Hence the following consequence of Theorem 29 predicts this outcome.

Corollary 31. Weak heredity property of $\widetilde{\Lambda}_{\widetilde{\Lambda}}$. [24, Corollary 16] *For any projective right $\widetilde{\Lambda}$-module $\widetilde{P}$, the submodule $\nabla(\widetilde{P}) = \widetilde{P}\widetilde{\mu}\widetilde{\Lambda}$ is again projective.*

The corollary allows us to obtain a large portion of the structure of $\widetilde{e}_k\widetilde{\Lambda}$ for a post- but not precyclic vertex $\widetilde{e}_k$ from the structure of the critical projectives $\widetilde{e}_i\widetilde{\Lambda}$, via mere inspection of the left Λ-module $T_k \subseteq T$. Namely: The number of arrows in $\widetilde{Q}^{\mathrm{op}}$ from $\widetilde{e}_k$ to a critical $\widetilde{e}_i$ equals the dimension of $e_i(T_k/JT_k)$. Each such arrow, α say, generates a copy $\widetilde{P}(\alpha)$ of $\widetilde{e}_i\widetilde{\Lambda}$ in $\widetilde{e}_k\widetilde{J}$ such that $\widetilde{e}_k\widetilde{J} = \bigoplus_\alpha \widetilde{P}(\alpha) \ + \ X$, where α ranges over the arrows starting in $\widetilde{e}_k$ and ending in some critical vertex $\widetilde{e}_i$, and X has only noncritical composition factors. We leave the justification to the reader.

Finally, we compare the "diagnostic homological filtrations" of the modules in Λ-Mod and Mod-$\widetilde{\Lambda}$.

• Each $M \in \Lambda$-Mod has a unique submodule U with the property that all simple composition factors of U have finite projective dimension, while those of M/U have infinite projective dimension (namely $U = \varepsilon M$). This submodule U gives rise to the criterion: $\operatorname{p\,dim}_\Lambda M < \infty$ if and only if M/U is a direct sum of copies of $\mathcal{A}_1, \dots, \mathcal{A}_m$.

• Every module $\widetilde{M} \in$ Mod-$\widetilde{\Lambda}$ contains a unique submodule chain $\widetilde{U} \subseteq \widetilde{V} \subseteq \widetilde{M}$ $\big($namely, $\widetilde{V} = \nabla(\widetilde{M}) \supseteq \widetilde{U} = \Delta(\nabla(\widetilde{M}))\big)$ such that $\widetilde{U}$ and $\widetilde{M}/\widetilde{V}$ have only composition factors of finite projective dimension, while socle and top of $\widetilde{V}/\widetilde{U}$ consist of simples with infinite projective dimension. This chain gives rise to the criterion: $\operatorname{p\,dim}_{\widetilde{\Lambda}} \widetilde{M} < \infty$ if and only if $\widetilde{V}/\widetilde{U}$ is a direct sum of copies of $\mathcal{C}_1, \dots, \mathcal{C}_r$.

Schematically, these filtrations of the objects in Λ-Mod and Mod-$\widetilde{\Lambda}$ look as follows.

	$M \in \Lambda$-Mod
M/U	comp. factors of infinite $\operatorname{p\,dim}$
U	comp. factors of finite $\operatorname{p\,dim}$

$\widetilde{M} \in$ Mod-$\widetilde{\Lambda}$	
comp. factors of finite $\operatorname{p\,dim}$	$\widetilde{M}/\widetilde{V}$
critical core	$\widetilde{V}/\widetilde{U}$
comp. factors of finite $\operatorname{p\,dim}$	$\widetilde{U}$

6.F. Contravariant finiteness of $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$.

Theorem 32. [24, Theorem 19] $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ *is contravariantly finite in* mod-$\widetilde{\Lambda}$. *Moreover, the minimal* $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$*-approximations of the simple right* $\widetilde{\Lambda}$*-modules* $\widetilde{S}_i$ *can be characterized and constructed from quiver and Loewy length of* Λ (on the basis of Supplement 34 below). *Hence* mod-$\widetilde{\Lambda}$ *has a strong tilting module* $\widetilde{T}_{\widetilde{\Lambda}}$.

By Corollary 30, the noncritical simples $\widetilde{S}_{r+1}, \dots, \widetilde{S}_n$ belong to $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$, meaning that they coincide with their minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations. We describe the minimal approximations of the critical simples $\widetilde{S}_1, \dots, \widetilde{S}_r$. For $k \in \{1, \dots, r\}$, let $\mathcal{C}_k$ again be the critical core of $\widetilde{e}_k\widetilde{\Lambda}$. Fix an injective envelope $\widetilde{E}(\mathcal{C}_k)$ of $\mathcal{C}_k$.

Lemma 33. [24, Lemma 18] *Still* $k \in \{1, \dots, r\}$. *There is a unique submodule* $\widetilde{\mathcal{A}}_k \subseteq \widetilde{E}(\mathcal{C}_k) = \widetilde{E}(\widetilde{e}_k\widetilde{\Lambda}/\widetilde{e}_k\widetilde{\Delta})$ *which is maximal relative to the following two properties:*

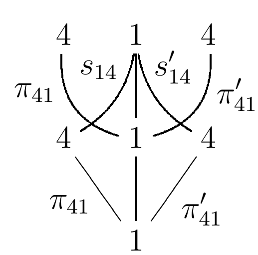
(i) *The generator* $a_k := \widetilde{e}_k + \widetilde{e}_k\widetilde{\Delta}$ *of* $\mathcal{C}_k$ *belongs to* $\widetilde{\mathcal{A}}_k \setminus \widetilde{\mathcal{A}}_k\widetilde{J}$;

(ii) $\widetilde{\mathcal{A}}_k$ *is an object of* $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$.

Observe that existence of $\widetilde{\mathcal{A}}_k$ follows from the fact that $\mathcal{C}_k$ satisfies (i) and (ii).

Supplement 34. (to Theorem 32) [24, Theorem 19] *The minimal* $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$*-approximation of any critical simple* $\widetilde{S}_k \in \text{mod-}\widetilde{\Lambda}$ *is* $\widetilde{\mathcal{A}}_k$.

Return to the reference example. The two critical simple right $\widetilde{\Lambda}$-modules are $\widetilde{S}_i$ for $i = 1, 2$. For $i = 1$, the minimal $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation $\widetilde{\mathcal{A}}_1$ of $\widetilde{S}_1$ is as shown below. It is a proper submodule of the injective envelope $\widetilde{E}(\widetilde{S}_1) = \widetilde{T}_1$, with $\widetilde{T}_1$ as displayed at the end of Section 6.G. On assuming this equality, we visually identify the graph of $\widetilde{\mathcal{A}}_1$ from that of $\widetilde{T}_1$ by using the description in Lemma 33. As for $i = 2$: The minimal $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation $\widetilde{\mathcal{A}}_2$ of $\widetilde{S}_2$ equals the critical core $\mathcal{C}(\widetilde{e}_2\widetilde{\Lambda})$, which is shown in Section 6.D above. To back this, we preempt the equality $\widetilde{E}(\widetilde{S}_2) = \widetilde{T}_2$, with $\widetilde{T}_2$ again given in Section 6.G. From the graph of $\widetilde{T}_2$, it is then obvious that that $\widetilde{\mathcal{A}}_2 = \widetilde{T}_2\widetilde{J} = \mathcal{C}(\widetilde{e}_2\widetilde{\Lambda})$. (Justifications of the claimed equalities $\widetilde{E}(\widetilde{S}_i) = \widetilde{T}_i$ are to follow in Section 8.)



6.G. Iterated strong tilting. For any module $\widetilde{M} \in \text{mod-}\widetilde{\Lambda}$, we fix a $\widetilde{\Lambda}$-injective envelope $\widetilde{E}(\widetilde{M})$ and set $\widetilde{E}_i = \widetilde{E}(\widetilde{S}_i)$. Then $\widetilde{E} := \bigoplus_{1 \le i \le n} \widetilde{E}_i$ is the basic injective cogenerator in mod-$\widetilde{\Lambda}$. The minimal $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation of $\widetilde{M}$ is denoted by $\widetilde{\mathcal{A}}(\widetilde{M})$.

In light of Theorem 32, $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ has a strong tilting module $\widetilde{T}$, and from Proposition 6 we deduce

$$\operatorname{add}(\widetilde{T}_{\widetilde{\Lambda}}) = \operatorname{add}\left(\bigoplus_{i=1}^{n} \widetilde{\mathcal{A}}(\widetilde{E}_i)\right).$$

Now assume $\widetilde{T}$ to be basic. Then $\widetilde{T}_{\widetilde{\Lambda}}$ is the direct sum of one copy of each of the n indecomposable objects in the righthand category.

Question. If we define $\widetilde{\widetilde{\Lambda}}$ to be the endomorphism ring $\operatorname{End}_{\widetilde{\Lambda}}(\widetilde{T})$, is the tilting bimodule ${}_{\widetilde{\widetilde{\Lambda}}}\widetilde{T}_{\widetilde{\Lambda}}$ strong also over $\widetilde{\widetilde{\Lambda}}$?

Towards the positive answer given in the final theorem of this section, one uses the following auxiliary fact, which holds some independent interest. Clearly, the condition that the injective $\widetilde{\Lambda}$-module $\widetilde{E}_i$ embeds into its approximation $\widetilde{\mathcal{A}}(\widetilde{E}_i)$ is equivalent to $\widetilde{E}_i \in \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$. This fails in general. On the other hand, the following weaker statement holds.

Lemma 35. [24, Lemma 20] *The simple right $\widetilde{\Lambda}$-module $\widetilde{S}_i$ is contained in the socle of $\widetilde{\mathcal{A}}(\widetilde{E}_i)$ for each $i \le n$.*

Due to [5, Proposition 6.5], strongness of ${}_{\widetilde{\widetilde{\Lambda}}}\widetilde{T}$ is equivalent to the requirement that all simple right $\widetilde{\Lambda}$-modules embed into $\widetilde{T}_{\widetilde{\Lambda}}$. Hence the lemma immediately leads to

Theorem 36. [24, Theorem 21] *Let $\widetilde{T}_{\widetilde{\Lambda}}$ be the basic strong tilting module in $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ and $\widetilde{\widetilde{\Lambda}} = \operatorname{End}_{\widetilde{\Lambda}}(\widetilde{T})$. Then the $\widetilde{\widetilde{\Lambda}}$-$\widetilde{\Lambda}$ tilting bimodule $\widetilde{T}$ is strong also as a left $\widetilde{\widetilde{\Lambda}}$-module, and the strongly tilted algebra $\operatorname{End}_{\widetilde{\widetilde{\Lambda}}}(\widetilde{T})^{\text{op}}$ is isomorphic to $\widetilde{\Lambda}$.*

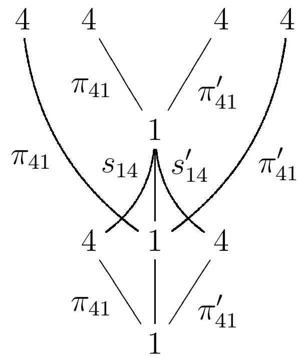
In particular, this shows that the process of iterated strong tilting of Λ-mod becomes stationary. Indeed, combined with Theorem 7, Theorem 36 yields dualities

$$\operatorname{Hom}_{\widetilde{\Lambda}}(-,\widetilde{T}) : \ \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}) \quad \longleftrightarrow \quad \mathcal{P}^{<\infty}(\widetilde{\widetilde{\Lambda}}\text{-mod}) \ : \operatorname{Hom}_{\widetilde{\widetilde{\Lambda}}}(-,\widetilde{T}).$$

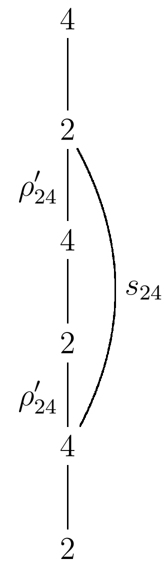
Clearly, these inverse dualities permit to pivot structural information garnered for the objects of $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ to a full complement of information on $\mathcal{P}^{<\infty}(\widetilde{\widetilde{\Lambda}}\text{-mod})$ and vice versa.

Return to the reference example. Decompose $\widetilde{T}$ into its indecomposable direct summands, $\widetilde{T} = \bigoplus_{1\le i\le 4} \widetilde{T}_i$. We display the $\widetilde{T}_i$; for their construction, we refer to the final section.

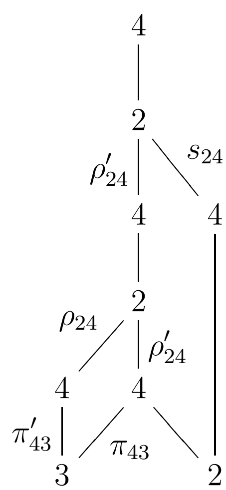
$\widetilde{T}_1 = \widetilde{E}_1$:



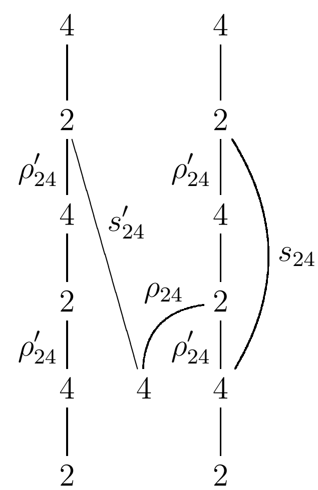
$\widetilde{T}_2 = \widetilde{E}_2$:



$\widetilde{T}_3$:



$\widetilde{T}_4$:



6.H. Representation type of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ and $\mathcal{P}^{<\infty}(\widetilde{\widetilde{\Lambda}}\text{-mod})$. The following observation picks up on the comment at the end of Section 6.B. It is an immediate consequence of Theorems 7, 32 and 36.

Observation 37. *There is an injection from the set of (isomorphism classes of) indecomposable left Λ-modules of finite projective dimension to the set of indecomposable right $\widetilde{\Lambda}$-modules of finite projective dimension, and the latter are in one-to-one correspondence with the indecomposable left $\widetilde{\widetilde{\Lambda}}$-modules of finite projective dimension.*

In light of the characterization of the objects in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ (Theorem 24), it is often manageable to determine the representation type of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ from that of $\mathcal{P}^{<\infty}(\varepsilon\Lambda\varepsilon\text{-mod})$ and the extensions in $\operatorname{Ext}^1\big(\Lambda e_i/\varepsilon J e_i, \mathcal{P}^{<\infty}(\varepsilon\Lambda\varepsilon\text{-mod})\big)$ for $i \le m$. In such situations, one obtains an effective handle on the representation types of $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$

and $\mathcal{P}^{<\infty}(\widetilde{\widetilde{\Lambda}}\text{-mod})$ which allows bypassing the cumbersome computation of quiver and relations of $\widetilde{\Lambda}$ and of the critical cores of the $\widetilde{e}_i\widetilde{\Lambda} \in \text{mod-}\widetilde{\Lambda}$. To illustrate this, we modify the small example given at the end of Section 6.B. If Q is the quiver

$$\omega \circlearrowright 1 \longrightarrow 2\ ,$$

and Λ is the corresponding truncated path algebra of Loewy length 3, then it is not difficult to deduce from Theorem 24 that the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ has finite representation type (note that the factor algebra $\Lambda/\Lambda\omega^2$ is a string algebra of finite type) containing precisely 5 isomorphism classes of indecomposable objects; all of them are local. Given that Q does not have a precyclic source, ${}_\Lambda T_{\widetilde{\Lambda}}$ is strong on both sides by Criterion 27. Hence, by applying $\text{Hom}_\Lambda(-, T)$, one finds that also $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ has precisely 5 isomorphism classes of indecomposables, each having a simple socle. It is, in fact, routine to pin them down explicitly. Here $\widetilde{\widetilde{\Lambda}} \cong \Lambda$ because the tilting module T is strong on both sides.

7. Constructing the minimal $\mathcal{P}^{<\infty}$-approximations in mod-$\widetilde{\Lambda}$

7.A. Results targeting general truncated path algebras. In this section, we assume Λ to be an arbitrary truncated path algebra and $\widetilde{\Lambda} = \text{End}({}_\Lambda T)^{\text{op}}$, where $T \in \Lambda\text{-mod}$ is the basic strong tilting module. We keep all conventions of Section 6. In particular, we will freely use the endofunctors Δ, ∇ and $\mathcal{C}(-)$ of Mod-$\widetilde{\Lambda}$ which were introduced in 6.D. As before, $\widetilde{\Delta}$ denotes the twosided ideal $\Delta(\widetilde{\Lambda}_{\widetilde{\Lambda}})$ of $\widetilde{\Lambda}$. Motivated by Theorem 29, we will switch back and forth among modules over the algebras $\widetilde{\Lambda}$, $\widetilde{\Lambda}/\widetilde{\Delta}$ and $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$.

Proposition 38. *Suppose $\widetilde{M} \in \text{mod-}\widetilde{\Lambda}$ satisfies $\Delta(\widetilde{M}) = 0$. Then the minimal $\mathcal{P}^{<\infty}$-approximations of $\widetilde{M}$ in* Mod-$\widetilde{\Lambda}$ *and* Mod-$(\widetilde{\Lambda}/\widetilde{\Delta})$ *coincide.*

Proof. Let $p : \widetilde{X} \longrightarrow \widetilde{M}$ be the minimal $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation of $\widetilde{M}$. Recall that, for any $\widetilde{Y} \in \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$, all simple composition factors of the submodule $\Delta(\widetilde{Y})$ are noncritical, whence also $\widetilde{Y}/\Delta(\widetilde{Y})$ belongs to $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ by Corollary 30. By hypothesis, every homomorphism $f \in \text{Hom}_{\widetilde{\Lambda}}(\widetilde{Y}, \widetilde{M})$ induces a homomorphism $\bar{f} : \widetilde{Y}/\Delta(\widetilde{Y}) \to \widetilde{M}$. Thus the approximation property of p shows that all homomorphisms from a module $\widetilde{Y}$ in $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ to $\widetilde{M}$ factor through $\bar{p} : \widetilde{X}/\Delta(\widetilde{X}) \to \widetilde{M}$. Minimality of p consequently yields $\Delta(\widetilde{X}) = 0$, which makes $\widetilde{X}$ a module over $\widetilde{\Lambda}/\widetilde{\Delta}$; indeed, $\dim \widetilde{X}$ is minimal among the K-dimensions of the $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximations of $\widetilde{M}$. It is readily checked that, in this situation, finiteness of $\operatorname{p\,dim}_{\widetilde{\Lambda}} \widetilde{X}$ amounts to the same as finiteness of $\operatorname{p\,dim}_{\widetilde{\Lambda}/\widetilde{\Delta}} \widetilde{X}$, which shows p to also be a minimal $\mathcal{P}^{<\infty}(\text{mod-}(\widetilde{\Lambda}/\widetilde{\Delta})$-approximation of $\widetilde{M}$. $\square$

The following result says that $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximations of arbitrary $\widetilde{\Lambda}$-modules can be constructed from approximations of $\widetilde{\Lambda}$-modules $\widetilde{N}$ with $\Delta(\widetilde{N}) = 0$.

Proposition 39. *Let $\widetilde{M} \in \text{mod-}\widetilde{\Lambda}$, and let $\pi : \widetilde{M} \longrightarrow \widetilde{M}/\Delta(\widetilde{M})$ be the canonical projection. If $p : \widetilde{Y} \longrightarrow \widetilde{M}/\Delta(\widetilde{M})$ is the minimal $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation of $\widetilde{M}/\Delta(\widetilde{M})$ then the map $q : \widetilde{X} \longrightarrow \widetilde{M}$ arising in the following* (bi)*cartesian square is the minimal*

$\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$*-approximation of* $\widetilde{M}$*:*

$$\begin{tikzcd}
0 \arrow[r] & \Delta(\widetilde{M}) \arrow[r] \arrow[d, equal] & \widetilde{X} \arrow[r, "\widetilde{\pi}"] \arrow[d, "q"'] \arrow[dr, phantom, "\blacksquare", pos=0.2] \arrow[dr, phantom, "\bullet", pos=0.8] & \widetilde{Y} \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & \Delta(\widetilde{M}) \arrow[r] & \widetilde{M} \arrow[r, "\pi"] & \widetilde{M}/\Delta(\widetilde{M}) \arrow[r] & 0
\end{tikzcd}$$

Proof. Suppose $\widetilde{Z} \in \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ and $f \in \operatorname{Hom}_{\widetilde{\Lambda}}(\widetilde{Z}, \widetilde{M})$. Then $\pi \circ f$ factors through p, i.e., there is a morphism $g : \widetilde{Z} \longrightarrow \widetilde{Y}$ such that $p \circ g = \pi \circ f$. The universal property of pullbacks then yields a unique morphism $h : \widetilde{Z} \longrightarrow \widetilde{X}$ such that $\widetilde{\pi} \circ h = g$ and $q \circ h = f$. In particular, f factors through q, and hence q is a $\mathcal{P}^{<\infty}$-approximation of $\widetilde{M}$.

For the minimality of q, note that, by Proposition 38, we know that $\Delta(\widetilde{Y}) = 0$. Looking at the upper row in the commutative diagram of the statement, we conclude that $\Delta(\widetilde{X}) \cong \Delta(\widetilde{M})$ and $\widetilde{Y} \cong \widetilde{X}/\Delta(\widetilde{X})$. It is obvious that, after these identifications, we may view $\widetilde{\pi}$ as the canonical projection $\widetilde{X} \twoheadrightarrow \widetilde{X}/\Delta(\widetilde{X})$ and p as the map $\widetilde{q} : \widetilde{X}/\Delta(\widetilde{X}) \longrightarrow \widetilde{M}/\Delta(\widetilde{M})$ induced by q.

Now suppose $u : \widetilde{X} \longrightarrow \widetilde{X}$ is an endomorphism such that $q \circ u = q$. Then clearly $\widetilde{q} \circ \widetilde{u} = \widetilde{q}$, where $\widetilde{u} : \widetilde{X}/\Delta(\widetilde{X}) \longrightarrow \widetilde{X}/\Delta(\widetilde{X})$ is the endomorphism induced by u. By the right minimality of $\widetilde{q} \cong p$, the latter equality implies that $\widetilde{u}$ is an isomorphism. We deduce that u is an isomorphism: Since $q_{|\Delta(\widetilde{X})} : \Delta(\widetilde{X}) \xrightarrow{\cong} \Delta(\widetilde{M})$ is an isomorphism, we find $u_{|\Delta(\widetilde{X})} : \Delta(\widetilde{X}) \longrightarrow \Delta(\widetilde{X})$ to be an isomorphism. When combined with the fact that $\widetilde{u}$ is an isomorphism, this shows that, indeed, u is an isomorphism. $\square$

The task of calculating minimal $\mathcal{P}^{<\infty}$-approximations of $\widetilde{\Lambda}$-modules is now reduced to calculating them for those $\widetilde{\Lambda}$-modules $\widetilde{F}$ which satisfy $\Delta(\widetilde{F}) = 0$. To tackle it, we pick up on Supplement 34, where the minimal approximations of the critical simples in mod-$\widetilde{\Lambda}$ are described.

Definition. Let $\widetilde{F}$ be a finitely generated right $\widetilde{\Lambda}$-module such that $\Delta(\widetilde{F}) = 0$, and let $q : \widetilde{P} \longrightarrow \nabla(\widetilde{F})$ be the projective cover of $\nabla(\widetilde{F})$ in Mod-$(\widetilde{\Lambda}/\widetilde{\Delta})$. A Δ*-extension of* q will be a pair $(\widetilde{Y}, \varphi)$ satisfying the following conditions:

(1) $\widetilde{Y}$ is a submodule of a fixed injective envelope $\widetilde{E}(\widetilde{P})$ of $\widetilde{P}$ in mod-$\widetilde{\Lambda}$ such that $\widetilde{Y}$ contains $\widetilde{P}$ and $\widetilde{Y}/\widetilde{P}$ has only noncritical composition factors.

(2) $\varphi : \widetilde{Y} \longrightarrow \widetilde{F}$ is a morphism in mod-$\widetilde{\Lambda}$ such that the restriction $\varphi_{|\widetilde{P}}$ is the composition $\widetilde{P} \xrightarrow{q} \nabla(\widetilde{F}) \overset{j}{\hookrightarrow} \widetilde{F}$, where $j = j_{\widetilde{F}}$ is the inclusion.

We equip the set of Δ-extensions of $q : \widetilde{P} \to \nabla(\widetilde{F})$ (in a fixed injective envelope $\widetilde{E}(\widetilde{P})$) with the partial order of pairs induced by inclusion of the first components and prove:

Proposition 40. *Let* $\widetilde{F} \in \text{mod-}\widetilde{\Lambda}$ *with* $\Delta(\widetilde{F}) = 0$*, and let* $q : \widetilde{P} \longrightarrow \nabla(\widetilde{F})$ *be the projective cover of* $\nabla(\widetilde{F})$ *in* mod-$(\widetilde{\Lambda}/\widetilde{\Delta})$*. The poset of* Δ*-extensions of* q *in* $\widetilde{E}(\widetilde{P})$ *is upward directed and has a maximum. If* $(\widetilde{X}, p)$ *is the largest element, then* $p : \widetilde{X} \longrightarrow \widetilde{F}$ *is the minimal* $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$*-approximation of* $\widetilde{F}$*.*

Proof. Let us denote by $\mathcal{S}$ the set of Δ-extensions of q. Existence of a maximum in $\mathcal{S}$ will follow once we have proved that this set is directed: Indeed, finiteness of $\dim \widetilde{E}(\widetilde{P})$ will then yield a unique maximal element in $\mathcal{S}$.

So let $(\widetilde{Y}_1, \varphi_1)$ and $(\widetilde{Y}_2, \varphi_2)$ be elements of $\mathcal{S}$. We check that the restrictions of φ_1 and φ_2 to $\widetilde{Y}_1 \cap \widetilde{Y}_2$ are equal. Indeed, by condition (1) of the above definition, all composition factors of $(\widetilde{Y}_1 \cap \widetilde{Y}_2)/\widetilde{P}$ are noncritical. Since $(\varphi_1 - \varphi_2)(\widetilde{P}) = 0$ by condition (2), this shows $(\varphi_1 - \varphi_2)(\widetilde{Y}_1 \cap \widetilde{Y}_2) \subseteq \Delta(\widetilde{F}) = 0$. Consequently, we obtain a well-defined homomorphism $\varphi : \widetilde{Y}_1 + \widetilde{Y}_2 \longrightarrow \widetilde{F}$ which restricts to the φ_i on the summands $\widetilde{Y}_i$. In other words, the pair $(\widetilde{Y}_1 + \widetilde{Y}_2, \varphi)$ is an element of $\mathcal{S}$ which majorizes both of the original pairs.

Note moreover that, for any pair $(\widetilde{Y}, \varphi) \in \mathcal{S}$, we have $\nabla(\widetilde{Y}) = \widetilde{P}$; this is due to condition (1) and the fact that the top of $\widetilde{P}$ is a sum of critical simples.

Now let $p : \widetilde{X} \twoheadrightarrow \widetilde{F}$ be the minimal $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation of $\widetilde{F}$. We will show that $(\widetilde{X}, p)$ is isomorphic to a Δ-extension of q, i.e., we will prove existence of an element $(\widetilde{X}', p') \in \mathcal{S}$, together with an isomorphism $\xi : \widetilde{X} \xrightarrow{\cong} \widetilde{X}'$, such that $p' \circ \xi = p$.

To verify this claim, we will in a first step reduce the problem to finding an element $(\widetilde{Y}, \varphi) \in \mathcal{S}$, together with a morphism $u : \widetilde{X} \longrightarrow \widetilde{Y}$, such that $\varphi \circ u = p$. Indeed, suppose we are in possession of $(\widetilde{Y}, \varphi) \in \mathcal{S}$ and u as specified. Minimality of p then shows u to be a split monomorphism giving rise to a decomposition $\widetilde{Y} = \mathrm{Im}(u) \oplus \widetilde{Y}'$ with $\varphi_{|\widetilde{Y}'} = 0$. This clearly entails that $(\widetilde{X}, p)$ is isomorphic to $(\mathrm{Im}(u), \varphi_{|\mathrm{Im}(u)})$ via the isomorphism $u : \widetilde{X} \xrightarrow{\cong} \mathrm{Im}(u)$. To ascertain that the latter pair belongs to $\mathcal{S}$, we observe that $\widetilde{P} = \nabla(\widetilde{Y}) = \nabla(\mathrm{Im}(u)) \oplus \nabla(\widetilde{Y}')$. Since $\varphi_{|\widetilde{P}}$ induces the projective cover $q : \widetilde{P} \longrightarrow \nabla(\widetilde{F})$ in mod-$(\widetilde{\Lambda}/\widetilde{\Delta})$, we obtain a matrix decomposition $q \cong \begin{pmatrix} \varphi_{|\nabla(\mathrm{Im}(u))} & \varphi_{|\widetilde{Y}'} \end{pmatrix} : \widetilde{P} = \nabla(\widetilde{Y}) = \nabla(\mathrm{Im}(u)) \oplus \nabla(\widetilde{Y}') \longrightarrow \nabla(\widetilde{F})$. In light of $\varphi_{|\widetilde{Y}'} = 0$, minimality of q thus implies $\nabla(\widetilde{Y}') = 0$, i.e., $\widetilde{Y}'$ is a submodule of $\widetilde{E}(\widetilde{P})$ which has only noncritical composition factors. From $\Delta(\widetilde{P}) = 0$, we therefore deduce that $\widetilde{P} \cap \widetilde{Y}' = 0$, which yields $\widetilde{Y}' = 0$. Hence $(\mathrm{Im}(u), \varphi_{|\mathrm{Im}(u)})$ indeed belongs to $\mathcal{S}$, and the first step is complete.

Let us prove the existence of $(\widetilde{Y}, \varphi)$ and u. By Proposition 38, we have $\Delta(\widetilde{X}) = 0$, i.e., $\nabla(\widetilde{X})$ is the critical core of $\widetilde{X}$ and thus projective as a right $\widetilde{\Lambda}/\widetilde{\Delta}$-module by Theorem 29. Hence the induced map $p_{|\nabla(\widetilde{X})} : \nabla(\widetilde{X}) \longrightarrow \nabla(\widetilde{F})$ factors through q, which yields a morphism $g : \nabla(\widetilde{X}) \longrightarrow \widetilde{P}$ with $q \circ g = p_{|\nabla(\widetilde{X})}$. We thus obtain a commutative diagram as shown below, where the upper square is the pushout of g and the inclusion $j_{\widetilde{X}} : \nabla(\widetilde{X}) \hookrightarrow \widetilde{X}$:

$$\begin{tikzcd}
\nabla(\widetilde{X}) \arrow[r, hook, "j_{\widetilde{X}}"] \arrow[d, "g"] \arrow[dd, bend right=50, "p|_{\nabla(\widetilde{X})}"'] & \widetilde{X} \arrow[d, "\widetilde{g}"'] \arrow[dd, bend left=50, "p"] \\
\widetilde{P} \arrow[r, hook, "\iota"] \arrow[d, "q"] & \widetilde{Z} \arrow[d, dashed, "h"'] \\
\nabla(\widetilde{F}) \arrow[r, hook, "j_{\widetilde{F}}"] & \widetilde{F}
\end{tikzcd}$$

A homomorphism h as indicated by the dotted arrow exists by the pushout property. In view of $\Delta(\widetilde{F}) = 0$, we moreover have a factorization $h : \widetilde{Z} \xrightarrow{\rho} \widetilde{Z}/\Delta(\widetilde{Z}) \xrightarrow{\bar{h}} \widetilde{F}$, where ρ is the canonical map. We set $\widetilde{Y} := \widetilde{Z}/\Delta(\widetilde{Z})$. It remains to be checked that the composition $\widetilde{P} \overset{\iota}{\hookrightarrow} \widetilde{Z} \xrightarrow{\rho} \widetilde{Y}$ is an essential monomorphism whose cokernel has only noncritical composition factors. Once this is in place, we know that, up to isomorphism, the monomorphism $\rho \circ \iota$ may be wiewed as an embedding of $\widetilde{Y}$ into $\widetilde{E}(\widetilde{P})$ and that the choice $\varphi = \bar{h}$ establishes the status of $(\widetilde{X}, p)$ as a copy of an element of $\mathcal{S}$.

To verify the final auxiliary claim, note that $\mathrm{Ker}(\rho \circ \iota) = \widetilde{P} \cap \Delta(\widetilde{Z}) \subseteq \Delta(\widetilde{P}) = 0$. As for essentiality of the image of $\rho \circ \iota$, the above diagram yields epimorphisms $\widetilde{X}/\nabla(\widetilde{X}) \cong \mathrm{Coker}(j_{\widetilde{X}}) \twoheadrightarrow \mathrm{Coker}(\iota) \twoheadrightarrow \mathrm{Coker}(\rho \circ \iota)$, and hence all composition factors of $\mathrm{Coker}(\rho \circ \iota)$ are noncritical. Given any submodule $\widetilde{Y}'$ of $\widetilde{Y}$ with $\mathrm{Im}(\rho \circ \iota) \cap \widetilde{Y}' = 0$, we infer that $\widetilde{Y}'$ is free of critical composition factors as well, i.e., $\widetilde{Y}' \subseteq \Delta(\widetilde{Y}) = 0$. Thus $\rho \circ \iota$ is indeed an essential monomorphism.

We have shown that $(\widetilde{X}, p) \cong (\widetilde{X}', p')$ with $(\widetilde{X}', p') \in \mathcal{S}$. It is now straightforward to verify that the latter pair is, in fact, maximal in $\mathcal{S}$. We leave the detail to the reader. □

7.B. The critical corner $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$ of a truncated path algebra. The following general observation can be found in [24, Section 6, 1st paragraph]. Let $\mu = \sum_{1 \le i \le r} e_i$ be the sum of the critical idempotents of Λ, and (as before) let $\widetilde{\mu} = \sum_{1 \le i \le r} \widetilde{e}_i$ be the corresponding sum in $\widetilde{\Lambda}$.

Lemma 41. *The algebras $\mu\Lambda\mu$ and $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$ are isomorphic. Any representative of the isomorphism class of these algebras will be called the* critical corner *of Λ or $\widetilde{\Lambda}$.*

We are interested in functorial connections between the category Mod-$\widetilde{\Lambda}$ and the module category of the critical corner $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$ of $\widetilde{\Lambda}$. Recall that the functor $H : \text{Mod-}\widetilde{\Lambda} \longrightarrow \text{Mod-}\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$ which takes $\widetilde{M}$ to $\to \widetilde{M}\widetilde{\mu}$ is naturally isomorphic to the functor $\mathrm{Hom}_{\widetilde{\Lambda}}(\widetilde{\mu}\widetilde{\Lambda}, -) : \text{Mod-}\widetilde{\Lambda} \longrightarrow \text{Mod-}\mathrm{End}_{\widetilde{\Lambda}}(\widetilde{\mu}\widetilde{\Lambda})$. As such it is exact and has a left adjoint, namely $G = - \otimes_{\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}} \widetilde{\mu}\widetilde{\Lambda} : \text{Mod-}\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu} \longrightarrow \text{Mod-}\widetilde{\Lambda}$. The unit of this adjunction $\eta : 1_{\text{Mod-}\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}} \longrightarrow H \circ G$ is readily seen to be a natural equivalence. These comments lead to the following conclusion:

Proposition 42. *Suppose that e_i is a critical idempotent of Λ (equivalently, $\widetilde{e}_i$ is critical in $\widetilde{\Lambda}$). If $\widetilde{E}_i = \widetilde{E}(\widetilde{S}_i)$ is the corresponding indecomposable injective right $\widetilde{\Lambda}$-module, then $\widetilde{E}_i\widetilde{\mu}$ is an indecomposable injective right $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$-module.*

In particular: If the critical corner $\mu\Lambda\mu$ is a self-injective algebra, then the projective dimension of $\widetilde{E}_i$ is finite.

Proof. Let $j : \widetilde{V} \to \widetilde{W}$ be a monomorphism in mod-$\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$, and denote by $\widetilde{U}$ the kernel of $G(j)$. In a first step, we show that $\mathrm{Hom}_{\widetilde{\Lambda}}(\widetilde{U}, \widetilde{E}_i) = 0$. Indeed, by applying H to the exact sequence

$$(\dagger) \qquad 0 \longrightarrow \widetilde{U} \longrightarrow G(\widetilde{V}) \xrightarrow{G(j)} G(\widetilde{W}),$$

and keeping in mind that $(H \circ G)(j) \cong j$ is a monomorphism, we obtain $\widetilde{U}\widetilde{\mu} = H(\widetilde{U}) = 0$; in other words, $\widetilde{U}$ is devoid of critical composition factors. Since $\widetilde{E}_i$ has a critical socle, this confirms the equality $\mathrm{Hom}_{\widetilde{\Lambda}}(\widetilde{U}, \widetilde{E}_i) = 0$.

To infer that $j^* := \mathrm{Hom}_{\widetilde{\Lambda}}(j, \widetilde{E}_i\widetilde{\mu})$ is surjective, we use the adjointness of G and H, to obtain the commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{\widetilde{\Lambda}}(G(\widetilde{W}),\, \widetilde{E}_i) \arrow[rr, "\mathrm{Hom}(G(j),\widetilde{E}_i)"] \arrow[d, leftrightarrow, "\cong"'] & & \mathrm{Hom}_{\widetilde{\Lambda}}(G(\widetilde{V}),\, \widetilde{E}_i) \arrow[d, leftrightarrow, "\cong"] \\
\mathrm{Hom}_{\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}}(\widetilde{W}, \widetilde{E}_i\widetilde{\mu}) \arrow[rr, "j^*"] & & \mathrm{Hom}_{\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}}(\widetilde{V}, \widetilde{E}_i\widetilde{\mu})
\end{tikzcd}$$

In view of the preceding paragraph, injectivity of $\widetilde{E}_i$ as a right $\widetilde{\Lambda}$-module ensures that $\mathrm{Hom}_{\widetilde{\Lambda}}(G(j), \widetilde{E}_i)$ is surjective. Hence so is j^*. This proves the first of our two assertions.

We deduce the second from the first: In case the critical corner of $\widetilde{\Lambda}$ is self-injective, $\widetilde{E}_i\widetilde{\mu}$ is a projective right $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$-module, and therefore $\mathrm{p\,dim}_{\widetilde{\Lambda}}\, \widetilde{E}_i < \infty$ by Theorem 29. Indecomposabilty of $\widetilde{E}_i\widetilde{\mu}$ is clear, because $\widetilde{e}_i\widetilde{\mu} = \widetilde{e}_i$, whence the socle of $\widetilde{E}_i\widetilde{\mu}$ is simple. $\square$

Remarks 43. (1) Identifying the quiver Q with its associated path category, i.e., the category whose objects are the vertices of Q and whose morphisms are the paths, we find the full subcategory consisting of the critical vertices to be convex, meaning: if $e_i \to e_j \to e_k$ is a composition of two paths in Q such that e_i and e_k are critical, then e_j is critical as well. As an immediate consequence, one obtains that, whenever the critical corner $\mu\Lambda\mu$ is nontrivial, it is again a truncated path algebra which has the same Loewy length as Λ. Note, however, that passage to the critical corner does not preserve connectedness in general.

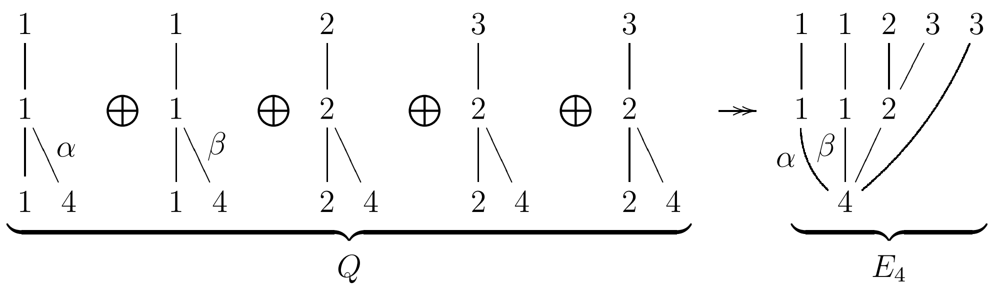
(2) Any self-injective truncated path algebra Λ is isomorphic to a finite product of Nakayama algebras (see [6] for the definition). In light of Remark 1, this shows: If the critical corner $\mu\Lambda\mu$ of Λ is self-injective, its quiver $\mu Q\mu$ is a disjoint union of simple oriented cycles (= oriented cycles which are not products of strictly shorter cycles). This applies to the critical corner in the reference example of Section 6.A, which will be revisited in depth in the next section.

8. Algorithmic aspects of the reference example

In this section we focus on the algebra $\Lambda = KQ/\langle \text{the paths of length 3}\rangle$ of the reference example in 6.A. Our goal is to verify the claims we made throughout Section 6. Again $\widetilde{\Lambda} = K\widetilde{Q}/\widetilde{I}$ denotes the algebra that results from strongly tilting Λ-mod. First, we will use Theorems 25 and 26 to determine the basic strong tilting module $T = \bigoplus_{1 \le i \le 4} T_i \in \Lambda$-mod, and next we will find the quiver Q^{op} of $\widetilde{\Lambda}^{op} = \mathrm{End}_\Lambda(T)$, as well as a set of generators for the ideal of relations. Finally, we will apply the recipes of the previous section towards identifying the basic strong tilting module $\widetilde{T}$ in mod-$\widetilde{\Lambda}$. As already mentioned in Section 6, the precyclic vertices of Q are e_1, e_2, and the postcyclic ones are e_1, e_2, e_4, whence e_1 and e_2 are the critical vertices. The idempotent ε of Λ introduced at the outset of Section 6 equals e_4.

Recall that a *top element* of a left Λ-module M is any element $x \in M \setminus JM$ such that $e_i x = x$ for some $j \le 4$; a *full sequence of top elements* of M is a collection of top elements which are linearly independent modulo JM and generate M.

8.A. The indecomposable direct summands of the strong tilting module ${}_\Lambda T$. According to Theorem 26, we have $T_i = \Lambda e_i/\varepsilon\Lambda e_i = \Lambda e_i/e_4\Lambda e_i$ for $i = 1, 2, 3$, while $T_4 = \mathcal{A}(E_4)$ is the minimal $\mathcal{P}^\infty$(mod-Λ)-approximation of $E_4 = E(S_4)$. To calculate T_4 by way of Theorem 25, let $\pi : P = \Lambda e_1^2 \oplus \Lambda e_2 \oplus \Lambda e_3^2 \twoheadrightarrow E_4$ be the projective cover of E_4, which sends the top elements e_i of the indecomposable projective modules Λe_i (see the graphs in Section 6.A) to the obvious full sequence of top elements of E_4 as shown below. To compute $P/\varepsilon \operatorname{Ker}(\pi) = P/e_4 \operatorname{Ker}(\pi)$, we may simplify P to the (nonprojective) module Q in our display, because π vanishes on the additional copies of S_4 in the socle of P. Clearly, $\mathcal{A}(E_4) \cong Q/e_4\operatorname{Ker}(q)$, and $e_4\operatorname{Ker}(q) \cong S_4^4$ is the direct sum of the simple submodules of Q generated by $(\alpha u_1, -\beta u_1, 0, 0, 0)$, $(0, \beta u_1, -\varepsilon u_2, 0, 0)$, $(0, 0, \varepsilon u_2, -\varepsilon\delta, 0)$ and $(0, 0, 0, \varepsilon\delta, -\gamma)$.



We conclude that $T_4 \in \Lambda$-mod is indeed the module displayed in Section 6.C.

To facilitate the verification of our subsequent computations, we translate the graphs of the T_i into generators and relators; i.e., we present the T_i as quotients $P_{T_i}/\Omega\, T_i$, where P_{T_i} is the projective cover of T_i suggested by the pertinent graphs (in particular $P_{T_4} = P$ as above), and $\Omega\, T_i$ is the corresponding syzygy of T_i.

Proposition 44. *The following is the list of indecomposable direct summands of the strong tilting left Λ-module T:*

(1) $T_1 = \Lambda e_1/\Omega\, T_1$, *where* $\Omega\, T_1$ *is generated by* αu_1, βu_1, α, β;
(2) $T_2 = \Lambda e_2/\Omega\, T_2$, *where* $\Omega\, T_2$ *is generated by* εu_2 *and* ε;
(3) $T_3 = \Lambda e_3/\Omega\, T_3$, *where* $\Omega\, T_3$ *is generated by* $\varepsilon\delta$ *and* γ;
(4) $T_4 = P_{T_4}/\Omega\, T_4$, *where* $P_{T_4} = \Lambda e_1^2 \oplus \Lambda e_2 \oplus \Lambda e_3^2$ *and* $\Omega\, T_4$ *is the submodule of* P_{T_4} *generated by*
$(\beta u_1, 0, 0, 0, 0)$, $(\alpha, 0, 0, 0, 0)$, $(\beta, 0, 0, 0, 0)$, $(0, \alpha u_1, 0, 0, 0)$, $(0, \alpha, 0, 0, 0)$, $(0, \beta, 0, 0, 0)$, $(0, 0, \epsilon, 0, 0)$, $(0, 0, 0, \gamma, 0)$, $(0, 0, 0, 0, \varepsilon\delta)$, $(\alpha u_1, -\beta u_1, 0, 0, 0)$, $(\alpha u_1, 0, -\varepsilon u_2, 0, 0)$, $(\alpha u_1, 0, 0, -\varepsilon\delta, 0)$, *and* $(\alpha u_1, 0, 0, 0, -\gamma)$.

8.B. Quiver and relations of the endomorphism algebra of ${}_\Lambda T$. We will display the quiver of $\operatorname{End}_\Lambda(T)$, rather than that of $\widetilde{\Lambda}$; in our previous notation, this means that we will determine the quiver $\widetilde{Q}^{\mathrm{op}}$. Passing to the opposite of $\widetilde{\Lambda} = K\widetilde{Q}/\widetilde{I}$ proves convenient towards graphing the indecomposable projective right $\widetilde{\Lambda}$-modules. Evidently, we may think of $\widetilde{Q}^{\mathrm{op}}$ as the quiver having the four vertices $T_1, \dots, T_4$; the set of arrows from T_i

to T_j consists of a K-basis for $\mathrm{rad}(T_i, T_j)$ modulo $\mathrm{rad}^2(T_i, T_j)$. To describe our choice of arrows, we organize the non-isomorphisms among the T_i's into three groups, π_{ij}'s, ρ_{ij}'s and s_{ij}'s, with indices i, j pinning down the domain T_i and codomain T_j in each case; when $i = j$ we cut down to a single index. A homomorphism $T_i \to T_j$ belongs to the π-group if its image is not contained in the radical JT_j; it belongs to the ρ-group, if its image is contained in JT_j, but not in J^2T_j, and to the s-group if its image is contained in $J^2T_i = \mathrm{soc}(T_i)$ for each $i \le 4$. The notation P_{T_i} and $\Omega\, T_i$ is carried over from 8.A. Based on these conventions, we next describe a convenient K-basis for the radical of $\mathrm{End}_\Lambda(T)$ in Lemma 45. The proof of this lemma, as well as that of the subsequent one, resides on somewhat cumbersome computations. We leave it to the interested reader to duplicate them.

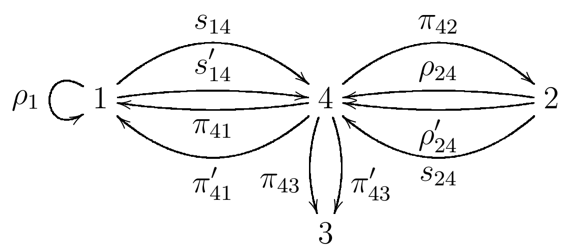
Lemma 45. *The Jacobson radical of* $\mathrm{End}({}_\Lambda T)$ *has a K-basis consisting of the morphisms listed below. Regarding the format of the list: Each map $T_i \longrightarrow T_j$ is communicated via a map $f : P_{T_i} \longrightarrow P_{T_j}$ such that $f(\Omega\, T_i) \subseteq \Omega\, T_j$. Moreover, for each of the listed maps, precisely one of the distinguished top elements of its domain P_{T_i} is not mapped to zero, whence f is pinned down by its value on this top element:*

(1) $\pi_{41} : T_4 \longrightarrow T_1,\ (e_1, 0, 0, 0, 0) \mapsto e_1;$
(2) $\pi'_{41} : T_4 \longrightarrow T_1,\ (0, e_1, 0, 0, 0) \mapsto e_1;$
(3) $\pi_{42} : T_4 \longrightarrow T_2,\ (0, 0, e_2, 0, 0) \mapsto e_2;$
(4) $\pi_{43} : T_4 \longrightarrow T_3,\ (0, 0, 0, e_3, 0) \mapsto e_3;$
(5) $\pi'_{43} : T_4 \longrightarrow T_3,\ (0, 0, 0, 0, e_3) \mapsto -e_3;$
(6) $\rho_1 : T_1 \longrightarrow T_1,\ e_1 \mapsto u_1;$
(7) $\rho_2 : T_2 \longrightarrow T_2,\ e_2 \mapsto u_2;$
(8) $\rho_{23} : T_2 \longrightarrow T_3,\ e_2 \mapsto \delta;$
(9) $\rho_{24} : T_2 \longrightarrow T_4,\ e_2 \mapsto (0, 0, 0, 0, \delta);$
(10) $\rho'_{24} : T_2 \longrightarrow T_4,\ e_2 \mapsto (0, 0, u_2, -\delta, 0);$
(11) $\rho_{41} : T_4 \longrightarrow T_1,\ (e_1, 0, 0, 0, 0) \mapsto u_1;$
(12) $\rho'_{41} : T_4 \longrightarrow T_1,\ (0, e_1, 0, 0, 0) \mapsto u_1;$
(13) $\rho_{42} : T_4 \longrightarrow T_2,\ (0, 0, e_2, 0, 0) \mapsto u_2;$
(14) $\rho_{43} : T_4 \longrightarrow T_3,\ (0, 0, e_2, 0, 0) \mapsto \delta;$
(15) $\rho_4 : T_4 \longrightarrow T_4,\ (0, 0, e_2, 0, 0) \mapsto (0, 0, 0, 0, \delta);$
(16) $\rho'_4 : T_4 \longrightarrow T_4,\ (0, 0, e_2, 0, 0) \mapsto (0, 0, u_2, -\delta, 0)$
(17) $s_1 : T_1 \longrightarrow T_1;\ e_1 \mapsto u_1^2;$
(18) $s_{14} : T_1 \longrightarrow T_4,\ e_1 \mapsto (u_1^2, 0, 0, 0, 0);$
(19) $s'_{14} : T_1 \longrightarrow T_4,\ e_1 \mapsto (0, u_1^2, 0, 0, 0);$
(20) $s_2 : T_2 \longrightarrow T_2;\ e_2 \mapsto u_2^2;$
(21) $s_{23} : T_2 \longrightarrow T_3,\ e_2 \mapsto u_2\delta;$
(22) $s_{24} : T_2 \longrightarrow T_4,\ e_2 \mapsto (0, 0, u_2^2, 0, 0);$
(23) $s'_{24} : T_2 \longrightarrow T_4,\ e_2 \mapsto (0, 0, 0, u_2\delta, 0);$
(24) $s''_{24} : T_2 \longrightarrow T_4,\ e_2 \mapsto (0, 0, 0, 0, u_2\delta);$
(25) $s_{41} : T_4 \longrightarrow T_1,\ (e_1, 0, 0, 0, 0) \mapsto u_1^2;$
(26) $s'_{41} : T_4 \longrightarrow T_1,\ (0, e_1, 0, 0, 0) \mapsto u_1^2;$
(27) $s_{42} : T_4 \longrightarrow T_2,\ (0, 0, e_2, 0, 0) \mapsto u_2^2;$
(28) $s_{43} : T_4 \longrightarrow T_3,\ (0, 0, 0, e_3, 0) \mapsto u_2\delta;$

(29) $s_4 : T_4 \longrightarrow T_4$, $(e_1, 0, 0, 0, 0) \mapsto (u_1^2, 0, 0, 0, 0)$;
(30) $s_4' : T_4 \longrightarrow T_4$, $(e_1, 0, 0, 0) \mapsto (0, u_1^2, 0, 0, 0)$;
(31) $s_4'' : T_4 \longrightarrow T_4$, $(0, e_1, 0, 0, 0) \mapsto (u_1^2, 0, 0, 0)$;
(32) $s_4''' : T_4 \longrightarrow T_4$, $(0, e_1, 0, 0, 0) \mapsto (0, u_1^2, 0, 0, 0)$;
(33) $s_4^{iv} : T_4 \longrightarrow T_4$, $(0, 0, e_2, 0, 0) \mapsto (0, 0, u_2^2, 0, 0)$;
(34) $s_4^{v} : T_4 \longrightarrow T_4$, $(0, 0, e_2, 0, 0) \mapsto (0, 0, 0, u_2\delta, 0)$;
(35) $s_4^{vi} : T_4 \longrightarrow T_4$, $(0, 0, e_2, 0, 0) \mapsto (0, 0, 0, 0, u_2\delta)$.

Lemma 46. *The morphisms* π_{41}, π_{41}', π_{42}, π_{43}, π_{43}', ρ_1, ρ_{24}, ρ_{24}', s_{14}, s_{14}' *and* s_{24} *form a basis for* $\mathrm{rad}(\mathrm{End}_\Lambda(\mathrm{T}))$ *modulo* $\mathrm{rad}(\mathrm{End}_\Lambda(\mathrm{T}))^2$.

Theorem 47. (1) *The quiver* $\widetilde{Q}^{\mathrm{op}}$ *of* $\mathrm{End}_\Lambda(T)$ *is*



(2) *The following is a set of relations for* $\mathrm{End}_\Lambda(T)$ *in* $K\widetilde{Q}^{\mathrm{op}}$ *:*

- *Monomial relations:* ρ_1^2, $\rho_1 s_{14}$, $\rho_1 s_{14}'$, $s_{14}\pi_{41}'$, $s_{14}\pi_{42}$, $s_{14}\pi_{43}$, $s_{14}\pi_{43}'$, $s_{14}'\pi_{41}$, $s_{14}'\pi_{42}$, $s_{14}'\pi_{43}$, $s_{14}'\pi_{43}'$, $s_{24}\pi_{41}$, $s_{24}\pi_{41}'$, $s_{24}\pi_{43}$, $s_{24}\pi_{43}'$, $s_{24}\pi_{42}\rho_{24}$, $s_{24}\pi_{42}\rho_{24}'$, $s_{24}\pi_{42}s_{24}$, $\rho_{24}\pi_{41}$, $\rho_{24}\pi_{41}'$, $\rho_{24}\pi_{42}$, $\rho_{24}\pi_{43}$, $\rho_{24}'\pi_{41}$, $\rho_{24}'\pi_{41}'$, $\rho_{24}'\pi_{43}'$, $\rho_{24}'\pi_{42}s_{24}$
- *Non-monomial relations:* $s_{14}\pi_{41} - \rho_1^2$, $s_{14}'\pi_{41}' - \rho_1^2$, $s_{24}\pi_{42} - \rho_{24}'\pi_{42}\rho_{24}'\pi_{42}$, $\rho_{24}\pi_{43}' - \rho_{24}'\pi_{43}$.

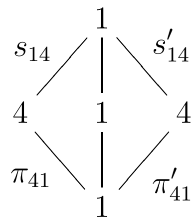
Proof. The first assertion is immediate from Lemma 46.

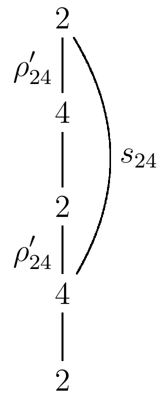
We denote by $\mathcal{R}$ the set of relations listed in (2). Part (1) provides us with a surjective algebra homomorphism $\Psi : K\widetilde{Q}^{\mathrm{op}} \twoheadrightarrow \mathrm{End}({}_\Lambda T)$. The verification that $K\widetilde{Q}^{\mathrm{op}}/\langle\mathcal{R}\rangle = K\widetilde{Q}^{\mathrm{op}}/\mathrm{Ker}(\Psi) \cong \mathrm{End}_\Lambda(T)$ is best carried out by comparing the relations in $\mathcal{R}$ with the graphs of the indecomposable projective left $\mathrm{End}_\Lambda(T)$-modules (= the indecomposable projective right $\widetilde{\Lambda}$-modules) shown in 6.D. $\square$

8.C. The indecomposable direct summands of the strong tilting module $\widetilde{T}_{\widetilde{\Lambda}}$. From Theorems 4 and 32, we know that $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ contains a basic strong tiliting module $\widetilde{T} = \bigoplus_{1 \le i \le 4} \widetilde{T}_i$. Moreover, Proposition 6 tells us that the $\widetilde{T}_i$ coincide with the indecomposable direct summands of $\bigoplus_{i=1}^4 \widetilde{\mathcal{A}}(\widetilde{E}_i)$, where $\widetilde{\mathcal{A}}(\widetilde{E}_i)$ is the minimal $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation of the indecomposable injective module $\widetilde{E}_i$. Determining $\widetilde{\mathcal{A}}(\widetilde{E}_i)$ for $i = 1, 2$ is effortless, since the critical corner $\mu\Lambda\mu = (e_1 + e_2)\Lambda(e_1 + e_2)$ of Λ is a self-injective algebra. Indeed, as a consequence of Proposition 42, we obtain:

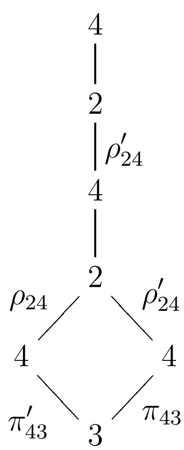
Corollary 48. *In our reference example,* $\widetilde{T}_1 = \widetilde{E}_1$ *and* $\widetilde{T}_2 = \widetilde{E}_2$. *These modules are pinned down, up to isomorphism, by the graphs shown at the end of* Section 6.G.

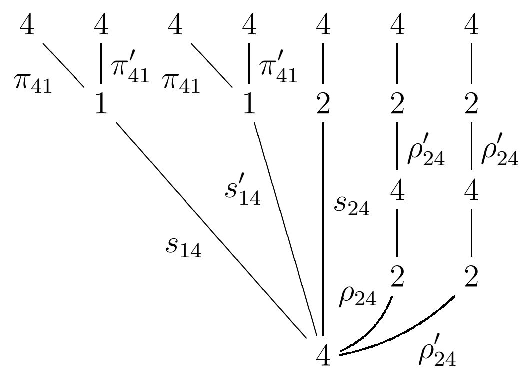
To construct $\widetilde{T}_3$ and $\widetilde{T}_4$, we follow the recipe developed in the previous section. Since this will require finding the $(\widetilde{\Lambda}/\widetilde{\Delta})$-projective covers of certain $\widetilde{\Lambda}/\widetilde{\Delta}$-modules with tops of the form $\widetilde{S}_1^{m_1} \oplus \widetilde{S}_2^{m_2}$, it will be convenient to have at hand the graphs of the indecomposable projective right $\widetilde{\Lambda}/\widetilde{\Delta}$-modules $\widetilde{e}_i\widetilde{\Lambda}/\widetilde{e}_i\widetilde{\Delta}$ for $i = 1, 2$. We denote these modules by $\hat{P}_1$ and $\hat{P}_2$, respectively. They are

$\hat{P}_1$


$\hat{P}_2$


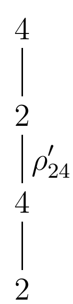
Moreover, it is readily checked that the indecomposable injectives $\widetilde{E}_3$ and $\widetilde{E}_4$ are determined by the graphs

$\widetilde{E}_3$


$\widetilde{E}_4$


Note that the left-hand graph pins down $\widetilde{E}_3$ up to isomorphism, even though this graph contains a closed edge path; the (unique) scalar involved in the corresponding module comes from the relation $\rho_{24}\pi'_{43} - \rho'_{24}\pi_{43}$.

Clearly, $\widetilde{F}_3 := \widetilde{E}_3/\Delta(\widetilde{E}_3)$ is the uniserial module

$\widetilde{F}_3$


In particular, $\nabla(\widetilde{F}_3)$ is the submodule with composition factors $(\widetilde{S}_2, \widetilde{S}_4, \widetilde{S}_2)$. Its projective cover in mod-$(\widetilde{\Lambda}/\widetilde{\Delta})$ is the obvious epimorphism $q : \hat{P}_2 \twoheadrightarrow \nabla(\widetilde{F}_3)$. Since the injective envelope $\widetilde{E}(\hat{P}_2)$ equals $\widetilde{E}_2$ (see 6.G), we find that $\widetilde{E}_2/\hat{P}_2 \cong \widetilde{S}_4$ has no critical composition factors; moreover, there is a unique morphism $p : \widetilde{E}_2 \longrightarrow \widetilde{F}_3$ with the property that $p_{|\hat{P}_2}$ is the composition $\hat{P}_2 \xrightarrow{q} \nabla(\widetilde{F}_3) \hookrightarrow \widetilde{F}_3 = \widetilde{E}_3/\Delta(\widetilde{E}_3)$. Proposition 40 thus guarantees that p is the minimal $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation of $\widetilde{F}_3$. By Proposition 39 the minimal approximation $\rho : \widetilde{\mathcal{A}}(\widetilde{E}_3) \longrightarrow \widetilde{E}_3$ is therefore the pullback of the pair (π_3, p), where $\pi_3 : \widetilde{E}_3 \twoheadrightarrow \widetilde{F}_3 = \widetilde{E}_3/\Delta(\widetilde{E}_3)$ is the canonical map. A routine check now allows us to ascertain that $\mathcal{A}(\widetilde{E}_3)$ is the local module $\widetilde{T}_3$ displayed at the end of Section 6.G.

To determine $\widetilde{T}_4$, we compute the minimal $\mathcal{P}^{<\infty}$-approximation $\mathcal{A}(\widetilde{E}_4) \longrightarrow \widetilde{E}_4$, once again guided by Section 7.

Towards identifying the minimal approximation of $\widetilde{E}_4/\Delta(\widetilde{E}_4)$, we find that this quotient decomposes in the form $\widetilde{E}_4/\Delta(\widetilde{E}_4) \cong \widetilde{L}^2 \oplus \widetilde{M} \oplus \widetilde{N}^2$, where $\widetilde{L}$ is given by the diagram $\begin{array}{ccc} 4 & & 4 \\ & \diagdown \ \diagup & \\ & 1 & \end{array}$, the summand $\widetilde{M}$ is the uniserial module $\begin{array}{c} 4 \\ | \\ 2 \end{array}$, and $\widetilde{N} = \widetilde{F}_3$ is as introduced above. Clearly, $\nabla(\widetilde{L}) \cong \widetilde{S}_1$ and $\nabla(\widetilde{M}) \cong \widetilde{S}_2$, whence the projective covers of $\nabla(\widetilde{L})$ and $\nabla(\widetilde{M})$ in mod-$(\widetilde{\Lambda}/\widetilde{\Delta})$ are the canonical epimorphisms $f_1 : \hat{P}_1 = \widetilde{e}_1\widetilde{\Lambda}/\widetilde{e}_1\widetilde{\Delta} \twoheadrightarrow \nabla(\widetilde{L}) \cong \widetilde{S}_1$ and $f_2 : \hat{P}_2 = \widetilde{e}_2\widetilde{\Lambda}/\widetilde{e}_2\widetilde{\Delta} \twoheadrightarrow \nabla(\widetilde{M}) \cong \widetilde{S}_2$. Further, one identifies the injective envelopes of the $\hat{P}_i$ in mod-$\widetilde{\Lambda}$ as being $\widetilde{E}(\widetilde{P}_i) = \widetilde{E}_i$ for $i = 1, 2$. As inspection confirms, all composition factors of the $\widetilde{E}(\hat{P}_i)/\hat{P}_i$ are isomorphic to $\widetilde{S}_4$, whence these quotients are free of critical composition factors. Moreover, the obvious morphisms $q_1 : \widetilde{E}_1 = \widetilde{E}(\hat{P}_1) \longrightarrow \widetilde{L}$ and $q_2 : \widetilde{E}_2 = \widetilde{E}(\hat{P}_2) \longrightarrow \widetilde{M}$ restrict to the maps f_i followed by the respective inclusions $\nabla(\widetilde{L}) \hookrightarrow \widetilde{L}$ and $\nabla(\widetilde{M}) \hookrightarrow \widetilde{M}$. Consequently, another application of Proposition 40 ensures that $q_1 : \widetilde{E}_1 \longrightarrow \widetilde{L}$ and $q_2 : \widetilde{E}_2 \longrightarrow \widetilde{M}$ are the minimal $\mathcal{P}^{<\infty}(\text{mod-}\Lambda)$-approximations of $\widetilde{L}$ and $\widetilde{M}$, respectively. The corresponding information for $\widetilde{N}$ is already available from the computation of $\widetilde{T}_3$. Indeed, that argument shows that the minimal $\mathcal{P}^{<\infty}(\text{mod-}\Lambda)$-approximation of $\widetilde{N}$ is the morphism $p : \widetilde{E}_2 \longrightarrow \widetilde{F}_3 = \widetilde{N}$ specified there. Consequently, the minimal $\mathcal{P}^{<\infty}(\text{mod-}\Lambda)$-approximation of $\widetilde{E}_4/\Delta(\widetilde{E}_4)$ is the induced map $\begin{pmatrix} q_1 & q_1 & q_2 & p & p \end{pmatrix} : (\widetilde{E}_1)^2 \oplus (\widetilde{E}_2)^3 \longrightarrow \widetilde{L}^2 \oplus \widetilde{M} \oplus \widetilde{N}^2 \cong \widetilde{E}_4/\Delta(\widetilde{E}_4)$.

In order to facilitate the application of Proposition 39 towards obtaining the minimal $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation of $\widetilde{E}_4$ from that of $\widetilde{E}_4/\Delta(\widetilde{E}_4) \cong \widetilde{L}^2 \oplus \widetilde{M} \oplus \widetilde{N}^2$, we will twice make use of the upcoming "isolation of direct summands" in pullback diagrams.

Lemma 49. *Let $f \in \operatorname{Hom}_{\widetilde{\Lambda}}(X, N)$, $g \in \operatorname{Hom}_{\widetilde{\Lambda}}(Y, N)$, and $h \in \operatorname{Hom}_{\widetilde{\Lambda}}(U, N)$. Suppose, moreover, that Z is the pullback (viewed as a module) of h and $(f \;\; g) \in \operatorname{Hom}_{\widetilde{\Lambda}}(X \oplus Y, N)$. If g factors through h, then $Z \cong Z' \oplus Y$, where Z' is the pullback of h and f.*

Proof. To introduce our notation, we display the pullback diagram for Z:

$$\begin{tikzcd} Z \arrow[r, "\widetilde{h}"] \arrow[d, "\widetilde{\chi}"'] & X \oplus Y \arrow[d, "\chi = (f \;\; g)"] \\ U \arrow[r, "h"] & N \end{tikzcd}$$

Moreover, we choose $\eta \in \mathrm{Hom}_{\widetilde{\Lambda}}(Y, U)$ such that $h \circ \eta = g$. To show that $Z \cong Z' \oplus Y$, consider the following square:

$$\begin{tikzcd} Z' \oplus Y \arrow[r, "\rho"] \arrow[d, "\sigma"'] & X \oplus Y \arrow[d, "\chi"] \\ U \arrow[r, "h"] & N \end{tikzcd}$$

where ρ and σ are as follows: For $(u,x) \in Z'$, we define $\rho\big((u,x),y\big) = (x,y)$ and $\sigma\big((u,x),y\big) = u + \eta(y)$. In light of the equality $h(u) = f(x)$, a straightforward check shows this diagram to be commutative, whence the universal property of Z yields a unique $\tau \in \mathrm{Hom}_{\widetilde{\Lambda}}(Z' \oplus Y, Z)$ satisfying $\sigma = \widetilde{\chi} \circ \tau$ and $\rho = \widetilde{h} \circ \tau$. Necessarily, τ assigns to an element $\big((u,x),y\big) \in Z' \oplus Y$, the element $\big(u + \eta(y), (x,y)\big) \in Z$.

Clearly, τ is injective. For surjectivity, let $\big(u, (x,y)\big) \in Z$, i.e., $h(u) = f(x) + g(y)$. Then $\big(u - \eta(y), x\big)$ belongs to Z', and $\tau\Big(\big(u - \eta(y), x\big), y\Big) = \big(u, (x,y)\big)$. Thus τ is an isomorphism. $\square$

We return to the construction of the minimal $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation of $\widetilde{E}_4$. In the first application of the lemma, the role of h is played by $\pi_4 : \widetilde{E}_4 \longrightarrow \widetilde{E}_4/\Delta(\widetilde{E}_4)$, that of f by $\begin{pmatrix} q_2 & p & p \end{pmatrix} : (\widetilde{E}_2)^3 \longrightarrow \widetilde{E}_4/\Delta(\widetilde{E}_4)$, and that of g by $\begin{pmatrix} q_1 & q_1 \end{pmatrix} : (\widetilde{E}_1)^2 \longrightarrow \widetilde{E}_4/\Delta(\widetilde{E}_4)$. In order to bring the lemma to bear, we observe that the induced morphisms $\widetilde{E}_1 \xrightarrow{q_1} \widetilde{L} \xrightarrow{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}} \widetilde{E}_4/\Delta(\widetilde{E}_4)$ and $\widetilde{E}_1 \xrightarrow{q_1} \widetilde{L} \xrightarrow{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \end{pmatrix}} \widetilde{E}_4/\Delta(\widetilde{E}_4)$ both factor through $\pi_4 : \widetilde{E}_4 \longrightarrow \widetilde{E}_4/\Delta(\widetilde{E}_4)$, whence so does the map $\begin{pmatrix} q_1 & q_1 \end{pmatrix} : (\widetilde{E}_1)^2 \longrightarrow \widetilde{E}_4/\Delta(\widetilde{E}_4)$. Thus we conclude that $\widetilde{\mathcal{A}}(\widetilde{E}_4) = (\widetilde{E}_1)^2 \oplus \hat{T}_4$, where $\hat{T}_4$ is the pullback of π_4 and $\begin{pmatrix} q_2 & p & p \end{pmatrix}$. For notational simplicity, we will write $\varphi = \begin{pmatrix} q_2 & p & p \end{pmatrix}$ in the sequel.

To prepare for another application of Lemma 49, we pass to an alternative decomposition of the domain $(\widetilde{E}_2)^3$ of φ, namely

$$(\widetilde{E}_2)^3 = (\widetilde{E}_2 \oplus 0 \oplus 0) \oplus (0 \oplus \widetilde{E}_2 \oplus 0) \oplus \widetilde{E}',$$

where $\widetilde{E}' \cong \widetilde{E}_2$ is the image of the morphism $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} : \widetilde{E}_2 \longrightarrow \widetilde{E}_2 \oplus \widetilde{E}_2 \oplus \widetilde{E}_2$. Based on the new decomposition of the domain, communicated in the form $(\widetilde{E}_2)^3 = (\widetilde{E}_2)^2 \oplus \widetilde{E}'$, the map φ takes on the form $\varphi = \begin{pmatrix} f & g \end{pmatrix} : (\widetilde{E}_2)^2 \oplus \widetilde{E}' \longrightarrow \widetilde{E}_4/\Delta(\widetilde{E}_4)$, where $f = \begin{pmatrix} q_2 & p \end{pmatrix} : (\widetilde{E}_2)^2 \longrightarrow \widetilde{E}_4/\Delta(\widetilde{E}_4)$ and $g = \varphi_{|\widetilde{E}'} : \widetilde{E}' \longrightarrow \widetilde{E}_4/\Delta(\widetilde{E}_4)$.

We check that the morphism g in turn factors through π_4. To see this, start by noting that $\widetilde{E}_4/\widetilde{E}_4\widetilde{J} \cong \widetilde{S}_4^7$. Tracing the top row of the graph of $\widetilde{E}_4$ from left to right, we number the displayed full sequence of top elements of $\widetilde{E}_4$ by $\widetilde{z}_1$, ..., $\widetilde{z}_7$. Next we observe that $\widetilde{E}' \cong \widetilde{E}_2$ is a quotient of $\widetilde{e}_4\widetilde{\Lambda}$; indeed, in view of the graph of $\widetilde{E}_2$ (see Section 6.G), $\widetilde{E}' = \widetilde{z}\widetilde{\Lambda}$, where $\widetilde{z} = \widetilde{z}\,\widetilde{e}_4$. We claim that the assignment $\widetilde{z} \mapsto \widetilde{z}_5 + \widetilde{z}_7$ yields a well-defined morphism $\eta : \widetilde{E}' \longrightarrow \widetilde{E}_4$ with the property that $\pi_4 \circ \eta = g$. It suffices to check well-definedness of η; once this is secured, we may conclude that $(\pi_4 \circ \eta)(\widetilde{z}) = \pi_4(\widetilde{z}_5) + \pi_4(\widetilde{z}_7) = g(\widetilde{z})$. In fact, we show that η defines an epimorphism from $\widetilde{E}'$ onto the submodule $\widetilde{X}$ of $\widetilde{E}_4$ which is generated by $\widetilde{z}_5 + \widetilde{z}_7$. From the graph of $\widetilde{E}_4$ it is apparent that the submodule of $\widetilde{X}$ may be visualized by way of the following diagram, which displays the itinerary of the top element $\widetilde{z}_5 + \widetilde{z}_7$ on successive multiplication by arrows:

$$\begin{tikzcd}
\widetilde{z}_5 + \widetilde{z}_7 \arrow[d, dash, "\pi_{42}"] \\
(\widetilde{z}_5 + \widetilde{z}_7)\pi_{42} \arrow[d, dash, "\rho'_{24}"] \arrow[dddd, dash, bend right, "s_{24}"'] \\
\widetilde{z}_7\pi_{42}\rho'_{24} \arrow[d, dash, "\pi_{42}"] \\
\widetilde{z}_7\pi_{42}\rho'_{24}\pi_{42} \arrow[d, dash, "\rho'_{24}"] \\
\widetilde{z}_5\pi_{42}s_{24} = \widetilde{z}_7\pi_{42}\rho'_{24}\pi_{42}\rho'_{24}
\end{tikzcd}$$

Given that $\widetilde{z}\,\pi_{42}s_{24} = \widetilde{z}\,\pi_{42}\rho'_{24}\pi_{42}\rho'_{24}$, we are thus looking at a copy of $\widetilde{E}_2/\operatorname{soc}(\widetilde{E}_2)$.

The preceding paragraph positions us for a reapplication of Lemma 49, to the effect that $\hat{T}_4 \cong \widetilde{T}'_4 \oplus \widetilde{E}' \cong \widetilde{T}'_4 \oplus \widetilde{E}_2$, where the summand $\widetilde{T}'_4$ is the pullback of $\pi_4 : \widetilde{E}_4 \longrightarrow \widetilde{E}_4/\Delta(\widetilde{E}_4)$ and $f = \begin{pmatrix} q_2 & p \end{pmatrix} : (\widetilde{E}_2)^2 \longrightarrow \widetilde{E}_4/\Delta(\widetilde{E}_4)$. Computation of this pullback finally yields $\widetilde{T}'_4$ to be a copy of the module $\widetilde{T}_4$ displayed at the end of Section 6.G. In particular, $\widetilde{T}'_4$ is indecomposable. Since $\widetilde{T}'_4$ is the only indecomposable direct summand of $\widetilde{A}(\widetilde{E}_4) = (\widetilde{E}_1)^2 \oplus \hat{T}_4 = (\widetilde{E}_1)^2 \oplus \widetilde{T}'_4 \oplus \widetilde{E}_2$ which is not among the $\widetilde{T}_i$ for $i \le 3$, we conclude that $\widetilde{T}'_4$ supplements $\bigoplus_{1\le i\le 3} \widetilde{T}_i$, so as to yield $\widetilde{T}$. Thus the label $\widetilde{T}_4$ in Section 6 is justified. In summary, we have confirmed the claims with which we concluded 6.G. Namely:

Proposition 50. *The basic strong tilting object in* mod-$\widetilde{\Lambda}$ *is* $\widetilde{T} = \oplus_{i=1}^4 \widetilde{T}_i$*, where the* $\widetilde{T}_i$ *are the modules determined* (*up to isomorphism*) *by the graphs shown in* Section 6.G.

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