GENERIC REPRESENTATION THEORY
OF QUIVERS WITH RELATIONS

E. Babson, B. Huisgen-Zimmermann, and R. Thomas

ABSTRACT. The irreducible components of varieties parametrizing the finite dimensional representations of a finite dimensional algebra Λ are explored, in terms of both their geometry and the structure of the modules they encode; as is to be expected, close connections between the two aspects surface. In particular, we establish the existence and uniqueness (not up to isomorphism, but in a strong sense to be specified) of a *generic module* for a given irreducible component $\mathcal{C}$, that is, of a module which displays <u>all</u> categorically defined generic properties of the modules parametrized by $\mathcal{C}$; the crucial point of the existence statement – a priori almost obvious – lies in the description of such a module in a format accessible to representation-theoretic techniques. Our approach, by way of minimal projective resolutions, is largely constructive. It is explicit for large classes of algebras of wild type. We follow with an investigation of the properties of such generic modules with regard to quiver and relations of Λ. The sharpest specific results on all fronts are obtained for truncated path algebras, that is, path algebras of quivers modulo ideals generated by all paths of a fixed length.

1. INTRODUCTION

Let Λ be a basic finite dimensional algebra over an algebraically closed field K. Tameness of the representation type of Λ – the only situation in which one can, at least in principle, meaningfully classify all finite dimensional representations of Λ – is a borderline phenomenon. However, for wild algebras, it is often possible to obtain a good grasp of the "bulk" of d-dimensional representations, for any dimension d, by understanding finitely many individual candidates. The underlying approach was initiated by Kac in 1982 for the hereditary case, refined by Schofield in 1992, and extended to arbitrary finitely generated K-algebras by Crawley-Boevey-Schröer in 2002 ([**15**], [**16**], [**6**]). The idea is to explore the "generic behavior" of the modules represented by the irreducible components of the affine variety, $\mathbf{Mod}_d(\Lambda)$, which parametrizes the d-dimensional left Λ-modules.

More precisely: Suppose that $\mathcal{C}$ is a (locally closed) irreducible subvariety of $\mathbf{Mod}_d(\Lambda)$; for instance, take $\mathcal{C}$ to be an irreducible component of $\mathbf{Mod}_d(\Lambda)$. Then a property $(*)$ of modules is said to be $\mathcal{C}$-*generic* in case there exists a dense open subset U of $\mathcal{C}$ such that all Λ-modules parametrized by points in U have property $(*)$. As is common, we will, more briefly, refer to the modules *in U*.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

For instance, due to [**15**] and [**6**], the number of indecomposable summands of a module is $\mathcal{C}$-generic for any irreducible component $\mathcal{C}$ of $\mathbf{Mod}_d(\Lambda)$, as is the family of corresponding dimension vectors of the indecomposable summands. To date, these numerical data, next to Ext- and Hom-space dimensions, have been the main objects of study along this line.

The primary purpose of this paper is to more broadly study generic properties of the modules in the irreducible components of $\mathbf{Mod}_d(\Lambda)$ – these properties include the generic behavior of their syzygies – first in general (Sections 2-4), then in a more specialized setting (Section 5). In intuitive terms, the goal is to obtain structural information for the majority of d-dimensional representations, irrespective of the representation type of the underlying algebra. Here "majority" means "a Zariski-dense open set's worth".

The foundation consists of an existence and uniqueness result (Theorem 4.3). Roughly, the existence part says that, provided the base field K has infinite transcendence degree over its prime field, *all* categorically defined generic properties of an irreducible component $\mathcal{C}$ of $\mathbf{Mod}_d(\Lambda)$ are on display in a single module $G = G(\mathcal{C})$ in $\mathcal{C}$; we will explain in a moment what we mean by "categorically defined". Existence of G is essentially obvious, unless one insists (as we do) on a specific format permitting structural and homological evaluation. Our construction provides a minimal projective presentation of G, which explicitly depends on a representation-theoretically defined dense subvariety of $\mathcal{C}$ (not necessarily open). In somewhat more detail: We start small, departing from irreducible subvarieties $\mathcal{C}$ of $\mathbf{Mod}_d(\Lambda)$ corresponding to modules that share a type of "path basis"; these varieties $\mathcal{C}$ are readily obtained from the relations of Λ and allow for an effective back and forth between their points and the first syzygies of the modules they encode. For any such $\mathcal{C}$, we show the existence of a module G in $\mathcal{C}$ which has all $\mathcal{C}$-generic properties that are invariant under Morita self-equivalences of the following restricted type: Suppose that K_0 is an algebraically closed subfield of K with $\mathrm{trdeg}(K : K_0) \geq \aleph_0$ such that Λ is defined over K_0, i.e., $\Lambda = KQ/I$, where Q is a quiver and I an ideal defined over K_0. The Morita self-equivalences of Λ-mod under consideration are those which are induced by the K_0-automorphisms of K; we dub them $\mathrm{Gal}(K/K_0)$-*equivalences*, and refer to a module property which is invariant under such equivalences as $\mathrm{Gal}(K/K_0)$-*invariant*. Calling a module G in $\mathcal{C}$ *generic for* $\mathcal{C}$ if it has all $\mathrm{Gal}(K/K_0)$-invariant generic properties of $\mathcal{C}$, we moreover prove the generic modules for $\mathcal{C}$ to be unique up to $\mathrm{Gal}(K/K_0)$-equivalence. More generally, we establish a uniqueness result for generic orbits of K-varieties which are defined over K_0 and which carry, next to an algebraic group action, a suitably compatible action of $\mathrm{Gal}(K/K_0)$. (We point out that the concept "generic for $\mathcal{C}$" is unrelated to the generic modules introduced by Crawley-Boevey in [**5**]; in particular, the generic objects considered here are finite dimensional over K.)

The general description of a $\mathcal{C}$-generic module, in terms of a minimal projective presentation, is essentially constructive, provided that Λ is given by quiver and relations. Not surprisingly, this presentation is particularly manageable when $\mathcal{C}$ is a rational variety.

Here is a bit more information on our reduction strategy in constructing generic modules that lend themselves to a systematic exploraton of generic porperties: Fixing a sequence $\mathbb{S}$ of semisimple modules and denoting by $\mathbf{Mod}(\mathbb{S}) \subseteq \mathbf{Mod}_d(\Lambda)$ the locally closed subvariety which parametrizes the modules with sequence $\mathbb{S}$ of radical layers, we start out with a roster

of subvarieties $\mathcal{D}$ covering $\mathbf{Mod}(\mathbb{S})$. Each of them is open and dense in some irreducible component of $\mathbf{Mod}(\mathbb{S})$. One constructs generic modules for the varieties $\mathcal{D}$ by shifting to a projective geometric setting, as is outlined below (under the caption "the general case"). The resulting generic modules for the irreducible components of the $\mathbf{Mod}(\mathbb{S})$ obtained in this way include the generics for the irreducible components of $\mathbf{Mod}_d(\Lambda)$. The "redundant" generic modules on the resulting list – those that are generic for irreducible components of the $\mathbf{Mod}(\mathbb{S})$, but not generic for irreducible components of the larger varieties $\mathbf{Mod}_d(\Lambda)$ – are of interest in their own right. They yield a more complete generic picture of the representation theory of Λ than restriction to the components of $\mathbf{Mod}_d(\Lambda)$ would. On the other hand, the sifting required to establish the pertinent hierarchy among the generic objects on the $\mathbf{Mod}(\mathbb{S})$-level is not be constructive, in general. In the present paper, we only carry it out in examples, but will address it more systmatically in a sequel.

The general background suggests a program: Namely, to gain a better understanding of the geometry of the irreducible components $\mathcal{C}$ of $\mathbf{Mod}_d(\Lambda)$ for specific classes of algebras Λ, in order to then effectively explore the representation-theoretic properties of their generic modules. A sampling of the categorically defined generic properties to be addressed can be found in Corollary 4.7; in particular, we re-encounter results from [**6**] mentioned above, addressing generic direct sum decompositions of modules. For our approach to generic modules, it is crucial that the list of $\mathrm{Gal}(K/K_0)$-stable generic properties includes *skeleta* of modules (see Definitions 3.1). These are preferred bases reflecting the radical layering and the KQ-structure of a module M. In light of the key role they play towards useful projective presentations of generic modules, we precede their formal introduction with a preliminary one: Suppose that $\Lambda = KQ/I$, and let $M = P/C$ be a Λ-module with projective cover P; moreover, fix elements $z_1, \ldots, z_t$ of P which induce a basis for P/JP. A skeleton of M is a set σ of elements of the form $y_{p,r} = (p + I)z_r \in P$, where p runs through certain paths in KQ. Keeping track of the lengths of these paths, we require that the following two conditions be satisfied: • the residue classes $y_{p,r} + C$ in M, corresponding to paths p of any fixed length l, induce a basis for $J^l M/J^{l+1} M$; moreover, • the set σ is closed under initial subpaths, that is, whenever $y_{p,r}$ belongs to σ and p' is an initial subpath of p, then also $y_{p',r}$ belongs to σ. The purpose of the second requirement might not be immediately apparent. It is pivotal in constructing the affine varieties $\mathcal{D}$ from which our our construction is launched; their points encode the modules with skeleton σ. These are the varieties most transparently linked to the relations in I. We note that every Λ-module has a nonempty set of skeleta, and that the set of all skeleta of the modules sharing a radical layering $\mathbb{S}$ is finite (as long as P and the z_r are fixed).

Our second major goal (Section 5) is to carry out the suggested program for *truncated path algebras*, that is, for the algebras of the form KQ/I, where Q is a quiver and I the ideal generated by all paths of a fixed length. They include the algebras with vanishing radical square and the hereditary algebras. Clearly, any basic finite dimensional algebra is isomorphic to a factor algebra of a truncated one. Suppose $\Lambda = KQ/I$ has Loewy length $L + 1$. If $\Lambda' = KQ/\langle$all paths of length $L + 1\rangle$, then the parametrizing varieties for Λ-mod are closed subvarieties of the analogous varieties for Λ'-mod.

To provide better orientation, we outline the reduction steps underlying the general

construction of generic objects in Section 4. Through Section 4, we do not impose any condition on Λ, beyond the above assumption on the transcendence degree of K. We the preview some results from Section 5, which flesh out the general theory for truncated path algebras.

The general case.

Fix an irreducible component $\mathcal{C}$ of some variety $\mathbf{Mod}_d(\Lambda)$. Our strategy for accessing $\mathcal{C}$-generic modules involves reductions of the problem to successively smaller subvarieties of $\mathcal{C}$. That the subvarieties $\mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq C_3$ we choose are better adapted to our purpose is only to a minor degree due to the reduction in size. Their main benefit lies in the availability of alternate, more helpful varieties parametrizing the classes of modules corresponding to the points of the $\mathcal{C}_i$. The largest, $\mathcal{D}_1$, of the alternates, $\mathcal{D}_1 \supseteq \mathcal{D}_2 \supseteq \mathcal{D}_3$, is a projective variety. All of the reductions are in representation-theoretic terms, that is, they are based on successively finer isomorphism invariants of the modules M they encode: First the *top* $T(M) = M/JM$, then the *radical layering* $\mathbb{S}(M) = \left(J^l M/J^{l+1}M\right)_{0 \le l \le L}$, where L is maximal with $J^L \ne 0$ (we identify the semisimple modules in this sequence with their isomorphism classes), and finally the *skeleta* σ of M.

Reduction Step 1 (Subsection 2.A). Given a semisimple module T, we let $\mathbf{Mod}_d^T$ be the locally closed subvariety of $\mathbf{Mod}_d(\Lambda)$ consisting of the points which correspond to the modules with top (isomorphic to) T. Clearly, $\mathbf{Mod}_d(\Lambda)$ is the disjoint union of the subvarieties $\mathbf{Mod}_d^T$, where T runs through finitely many choices. Since, generically, the modules in $\mathcal{C}$ have fixed top, there exist precisely one semisimple T and precisely one irreducible component $\mathcal{C}_1$ of $\mathbf{Mod}_d^T$ such that $\mathcal{C}$ is the closure of $\mathcal{C}_1$ in $\mathbf{Mod}_d(\Lambda)$. In particular, any $\mathcal{C}_1$-generic module is also $\mathcal{C}$-generic.

The projective counterpart of the variety $\mathbf{Mod}_d^T$ (presented at the end of 2.A) is denoted by $\mathfrak{Grass}_d^T$. It is the obvious closed subvariety of the classical Grassmannian of all $(\dim P - d)$-dimensional K-subspaces of JP, where P is a fixed projective cover of T; this variety was introduced and geometrically related to $\mathbf{Mod}_d^T$ by Bongartz and the author in [**3**] and [**4**]. Since the irreducible components of $\mathfrak{Grass}_d^T$ are in natural bijection with those of $\mathbf{Mod}_d^T$ – see Proposition 2.2 below – this means that, in studying the $\mathcal{C}$-generic modules, we may restrict our focus to an irreducible component $\mathcal{D}_1$ of $\mathfrak{Grass}_d^T$.

Reduction Step 2 (Subsection 2.B). Given a sequence $\mathbb{S} = (\mathbb{S}_0, \mathbb{S}_1, \ldots, \mathbb{S}_L)$ of semisimple modules that has total dimension d, we let $\mathbf{Mod}(\mathbb{S})$ be the subvariety of $\mathbf{Mod}_d(\Lambda)$ consisting of the points corresponding to the modules with radical layering $\mathbb{S}$. Clearly, each of the varieties $\mathbf{Mod}_d^T$ is the disjoint union of the subvarieties $\mathbf{Mod}(\mathbb{S})$, where $\mathbb{S}$ traces those sequences $\mathbb{S}$ for which $\mathbb{S}_0 = T$. Again, it is readily seen that, generically, the modules in $\mathcal{C}$ have fixed radical layering, say $\mathbb{S}$. As a consequence, there exists precisely one irreducible component $\mathcal{C}_2$ of $\mathbf{Mod}(\mathbb{S})$ with the property that the closure of $\mathcal{C}_2$ in $\mathbf{Mod}_d^T$ coincides with $\mathcal{C}_1$. Consequently, the closure of $\mathcal{C}_2$ in $\mathbf{Mod}_d(\Lambda)$ equals $\mathcal{C}$, whence any $\mathcal{C}_2$-generic module is also $\mathcal{C}$-generic.

The counterpart of the subvariety $\mathbf{Mod}(\mathbb{S}) \subseteq \mathbf{Mod}_d^T$ is the subvariety $\mathfrak{Grass}(\mathbb{S})$ of $\mathfrak{Grass}_d^T$ consisting of the projective points which parametrize the modules with radical layering $\mathbb{S}$. Since the irreducible components of $\mathbf{Mod}(\mathbb{S})$ are in natural bijection with

those of $\mathfrak{Grass}(\mathbb{S})$, our task is thus reduced to the study of $\mathcal{D}_2$-generic modules, where $\mathcal{D}_2$ is the irreducible component of $\mathfrak{Grass}(\mathbb{S})$ corresponding to $\mathcal{C}_2$.

Reduction Step 3 (Section 3). This step relies on a presentation $\Lambda = KQ/I$, where Q is a quiver and $I \subset KQ$ an admissible ideal. For any sequence $\mathbb{S}$ of semisimple modules, the variety $\mathfrak{Grass}(\mathbb{S})$ is a finite, but not necessarily disjoint, union of *open* affine varieties $\mathfrak{Grass}(\sigma)$, where σ runs through the skeleta "compatible with $\mathbb{S}$" in an obvious sense, and $\mathfrak{Grass}(\sigma)$ consists of the points that correspond to the modules with skeleton σ. Thus the closures of the irreducible components of the $\mathfrak{Grass}(\sigma)$ in $\mathfrak{Grass}(\mathbb{S})$ are precisely the irreducible components of $\mathfrak{Grass}(\mathbb{S})$. In other words, the generic modules for the irreducible components of the varieties $\mathfrak{Grass}(\mathbb{S})$ coincide with the generic modules for the irreducible components of the $\mathfrak{Grass}(\sigma)$. So we are left with the task of constructing generic modules for a suitable irreducible compenent $\mathcal{D}_3$ of some $\mathfrak{Grass}(\sigma)$. On this level, useful constructions are within reach.

Truncated path algebras.

In the truncated situation, we take K_0 to be the algebraic closure of the prime field of K. The following geometric information paves the way to construction and analysis of the generic modules for the irreducible components $\mathcal{C}$ of the varieties $\mathbf{Mod}_d(\Lambda)$.

Theorem A. *Suppose $\Lambda = KQ/I$ is a truncated path algebra of Loewy length $L+1$. For each sequence $\mathbb{S}$ of semisimple Λ-modules, $\mathfrak{Grass}(\mathbb{S})$ is covered by dense open subsets (the $\mathfrak{Grass}(\sigma)$) which are isomorphic copies of affine N-space $\mathbb{A}^N$, where N depends only on $\mathbb{S}$. In particular, $\mathfrak{Grass}(\mathbb{S})$ is irreducible, rational and smooth.*

In more detail, $\mathfrak{Grass}(\mathbb{S})$ is an affine bundle with fibre $\mathbb{A}^{N_1}$ over $\mathrm{Gr}\text{-}\mathfrak{Grass}(\mathbb{S})$, where $\mathrm{Gr}\text{-}\mathfrak{Grass}(\mathbb{S})$ is the subvariety of $\mathfrak{Grass}(\mathbb{S})$ consisting of the points that correspond to graded modules generated in degree zero; both N and N_1 can be determined from the quiver Q and the integer L by a simple count. Moreover, the variety $\mathrm{Gr}\text{-}\mathfrak{Grass}(\mathbb{S})$ of graded objects is projective and, in turn, smooth and rational: It is an iterated Grassmann bundle over a finite direct product of classical Grassmannians. (For terminology, consult Theorems 5.3, 5.8, and the paragraph preceding the latter.)

We follow with a slice of the representation-theoretic side of the picture, stated somewhat informally. For more precision, we refer to Theorem 5.11, which addresses the *generic graded modules* in tandem with the generic ones as defined above.

Theorem B. *Keep the hypotheses of Theorem A.*

The generic modules $G = G(\mathbb{S})$ for $\mathfrak{Grass}(\mathbb{S})$, alias $\mathbf{Mod}(\mathbb{S})$, can be read off the quiver Q, as can be several of their algebraic invariants.

In particular, the (generic) skeleta of G are available at a glance, and from those the generic syzygies. The syzygies $\Omega^k(G)$, for $k \geq 1$, are direct sums of cyclic modules, which are determined by $\mathbb{S}$ up to isomorphism (not only up to $\mathrm{Gal}(K/K_0)$-equivalence).

By means of this theorem, the generic homological dimensions of the modules with fixed radical layering have been determined, in terms of the quiver Q and the cutoff length L (see [**7**]). A suitable simultaneous choice of the modules $G(\mathbb{S})$, where $\mathbb{S}$ runs through the eligible sequences of semisimple modules, additionally yields the following "relative" generic

behavior: Given any two semisimple sequences $\mathbb{S}$ and $\mathbb{S}'$, the pair $(G(\mathbb{S}), G(\mathbb{S}'))$ possesses any $\mathrm{Gal}(K/K_0)$-stable generic property of pairs of modules in $\mathfrak{Grass}(\mathbb{S}) \times \mathfrak{Grass}\,\mathbb{S}'$. More concretely, this entails for instance, that

$$\dim \mathrm{Ext}_\Lambda^i\big(G(\mathbb{S}), G(\mathbb{S}')\big) = \min\{\dim \mathrm{Ext}_\Lambda^i(M, N) \mid \mathbb{S}(M) = \mathbb{S},\ \mathbb{S}(N) = \mathbb{S}'\} \quad \text{for } i \geq 0$$

in case $\mathbb{S} \neq \mathbb{S}'$, a slight adjustment being required in case $\mathbb{S} = \mathbb{S}'$; see Corollary 4.7.

For illustrations of the more general Theorem 4.3 providing the backdrop for Theorem B, see Example 4.8.

The crucial concepts can be found in Definitions 3.1 (skeleta), 3.7 (critical paths and affine coordinates of the $\mathfrak{Grass}(\sigma)$), 3.9 (hypergraphs of modules), 4.1 and 4.2 (generic modules for the irreducible components of the parametrizing varieties). Moreover, Theorem 3.8 summarizes background from [**14**].

2. Playing irreducible components back and forth among subvarieties of $\mathbf{Mod}_d(\Lambda)$ and their projective counterparts

We briefly review some of the constructions and results from [**12**], [**13**], and [**14**] needed in the sequel. Along the way, we line up observations aimed at reducing the problem of finding the irreducible components of $\mathbf{Mod}_d(\Lambda)$ to finding the irreducible components of successively smaller varieties, first $\mathbf{Mod}_d^T$, next $\mathbf{Mod}(\mathbb{S})$. The full collections of irreducible components of $\mathbf{Mod}_d^T$ and $\mathbf{Mod}(\mathbb{S})$ are of interest in themselves towards refined pictures of the generic representation theory of Λ.

As already stated, we let Λ be a basic finite dimensional algebra over an algebraically closed field K. Hence, we may assume, without loss of generality, that $\Lambda = KQ/I$, where Q is a quiver and I an admissible ideal in the path algebra KQ. The vertices $e_1, \ldots, e_n$ of Q will be identified with the primitive idempotents of Λ corresponding to the paths of length zero. As is well-known, the left ideals Λe_i then represent all indecomposable projective (left) Λ-modules, up to isomorphism, and the factors $S_i = \Lambda e_i / J e_i$, where J is the Jacobson radical of Λ, form a set of representatives for the simple left Λ-modules. By $L + 1$ we will denote the *Loewy length* of J, that is, L is maximal with $J^L \neq 0$. Moreover, we will observe the following conventions: The product pq of two paths p and q in KQ stands for "first q, then p"; in particular, pq is zero unless the end point of q coincides with the starting point of p. In accordance with this convention, we call a path p_1 an *initial subpath* of p if $p = p_2 p_1$ for some path p_2. A *path in* Λ is a residue class of the form $p + I$, where p is a path in $KQ \setminus I$; we will suppress the residue notation, provided there is no risk of ambiguity. Further, we will gloss over the distinction between the left Λ-structure of a module $M \in \Lambda$-Mod and its induced KQ-module structure when there is no danger of confusion. An element x of M will be called a *top element* of M if $x \notin JM$ and x is *normed* by some e_i, meaning that $x = e_i x$. Any collection $x_1, \ldots, x_m$ of top elements of M generating M and linearly independent modulo JM will be called a *full sequence of top elements of M*.

The two isomorphism invariants of a Λ-module M which will be pivotal here are the *top* and the *radical filtration* of M, the latter being a refinement of the former. The *top* of M is

defined as M/JM, and the *radical filtration* as the sequence $\mathbb{S}(M) = \left(J^l M / J^{l+1} M\right)_{1 \leq l \leq L}$ of semisimple modules. We will identify isomorphic semisimple modules and, in particular, call any module M with $M/JM \cong T$ a *module with top T*.

The following choices and notation will be observed throughout: We fix a semisimple module T, say

$$T = \bigoplus_{1 \leq i \leq n} S_i^{t_i},$$

set $\mathfrak{t} = \sum_i t_i = \dim T$, and denote by

$$P = \bigoplus_{1 \leq i \leq n} (\Lambda e_i)^{t_i}$$

"the" projective cover of T. Clearly, P is a projective cover of any module with top T; in other words, the modules with top T are precisely the quotients P/C with $C \subseteq JP$, up to isomorphism. Next, we fix a full sequence $z_1, \ldots, z_\mathfrak{t}$ of top elements of P. This means $P = \bigoplus_{1 \leq r \leq \mathfrak{t}} \Lambda z_r$ with $\Lambda z_r \cong \Lambda e(r)$, where $e(r)$ is the idempotent in $\{e_1, \ldots, e_n\}$ norming z_r. A natural choice of such top elements of P is to take the z_r to be the primitive idempotents e_i, each with multiplicity t_i, distinguished by their 'slots' in the above decomposition of P. Finally, we fix a positive integer $d \geq \mathfrak{t}$.

In the following, we will refer to P as *the distinguished projective cover of T* with *distinguished sequence $z_1, \ldots, z_\mathfrak{t}$ of top elements*.

2.A. From the affine variety $\mathbf{Mod}_d(\Lambda)$ to the quasi-affine variety $\mathbf{Mod}_d^T$ and the projective variety $\mathfrak{Grass}_d^T$.

Let $a_1, \ldots, a_r$ be a set of algebra generators for Λ over K. A convenient set of such generators consists of the primitive idempotents (= vertices) $e_1, \ldots, e_n$ together with the (residue classes in Λ of the) arrows in Q. Recall that, for $d \in \mathbb{N}$, the classical affine variety of d-dimensional representations of Λ can be described in the form

$$\mathbf{Mod}_d(\Lambda) = \{(x_i) \in \prod_{1 \leq i \leq r} \mathrm{End}_K(K^d) \mid \text{ the } x_i \text{ satisfy all relations satisfied by the } a_i\}.$$

As is well-known, the isomorphism classes of d-dimensional (left) Λ-modules are in one-to-one correspondence with the orbits of $\mathbf{Mod}_d(\Lambda)$ under the GL_d-conjugation action. Moreover, the connected components of $\mathbf{Mod}_d(\Lambda)$ are in natural one-to-one correspondence with the dimension vectors $(d_1, \ldots, d_n)$ such that $\sum_i d_i = d$. If $I = 0$, that is, if Λ is hereditary, the connected components coincide with the irreducible components, but this fails already in small non-hereditary examples, e.g. for $d = 2$ and $\Lambda = KQ/I$, where Q is the quiver $1 \rightleftarrows 2$ and I is generated by the paths of length 2.

The tops lead to a first rough subdivision of $\mathbf{Mod}_d(\Lambda)$: By $\mathbf{Mod}_d^T$ we will denote the locally closed subvariety of $\mathbf{Mod}_d(\Lambda)$ which consists of the points representing the modules with top T. As is easily seen, $\mathbf{Mod}_d^T$ is nonempty if and only if $\dim T \leq d$ and the projective cover P of T has dimension at least d. In light of our identification of

isomorphic semisimple modules, the set of all finite dimensional semisimple modules T is partially ordered under inclusion. Specifically, we write $T \leq T'$ to denote that a semisimple module T is (isomorphic to) a submodule of a semisimple module T'.

The first part of the following observation is due to the fact that $\mathbf{Mod}_d(\Lambda)$ is the finite disjoint union of the locally closed subvarieties $\mathbf{Mod}_d^T$, where T runs through the semisimple modules that arise as tops of d-dimensional Λ-modules. The second part is an immediate consequence of upper semicontinuity of the functions $\mathbf{Mod}_d(\Lambda) \to \mathbb{N}$ given by $X \mapsto \dim \operatorname{Hom}_\Lambda(X, S_i)$ for $1 \leq i \leq n$.

Observation 2.1 and Terminology.

(1) *For every irreducible component $\mathcal{C}$ of $\mathbf{Mod}_d(\Lambda)$, there exist precisely one semisimple module T and precisely one irreducible component $\mathcal{D}$ of $\mathbf{Mod}_d^T$ such that $\mathcal{C}$ is the closure of $\mathcal{D}$ in $\mathbf{Mod}_d(\Lambda)$. In particular: Generically, the modules in $\mathcal{C}$ have top T.*

(2) *If $\mathbf{Mod}_d^T \neq \varnothing$ and T is minimal among the semisimple modules T' that give rise to nonempty sets $\mathbf{Mod}_d^{T'}$, then $\mathbf{Mod}_d^T$ is open in $\mathbf{Mod}_d(\Lambda)$. Consequently: If the minimal elements in the poset of T' with $\mathbf{Mod}_d^{T'} \neq \varnothing$ are $T^{(0,1)}, \ldots, T^{(0,m_0)}$, then the closures of the varieties $\mathbf{Mod}_d^{T^{(0,i)}}$, $1 \leq i \leq m_0$, are unions of irreducible components of $\mathbf{Mod}_d(\Lambda)$. In fact, the irreducible components of the closures $\overline{\mathbf{Mod}_d^{T^{(0,i)}}}$ – they are called the* irreducible components of class 0 *of $\mathbf{Mod}_d(\Lambda)$ – are in bijective correspondence with the irreducible components of $\mathbf{Mod}_d^{T^{(0,i)}}$.*

To continue recursively, suppose $T^{(h,1)}, \ldots, T^{(h,m_h)}$ are the distinct semisimple modules which are minimal in the poset of those T' for which

$$\mathbf{Mod}_d^{T'} \not\subseteq \bigcup_{k<h \text{ and } i \leq m_k} \overline{\mathbf{Mod}_d^{T^{(k,i)}}}.$$

Then each $\mathbf{Mod}_d^{T^{(h,i)}}$ intersects the closed subvariety $\mathbf{Mod}_d(\Lambda) \setminus \bigcup_{k<h, i \leq m_k} \mathbf{Mod}_d^{T^{(k,i)}}$ in a nonempty open set. In particular, the irreducible components of the various subvarieties $\mathbf{Mod}_d^{T^{(h,j)}}$ which are not contained in the closure of $\bigcup_{k<h, i \leq m_k} \mathbf{Mod}_d^{T^{(k,i)}}$ yield distinct irreducible components of $\mathbf{Mod}_d(\Lambda)$, via passage to closures. These are called the irreducible components of class h *of $\mathbf{Mod}_d(\Lambda)$.* □

Observation 2.1 says that the problem of identifying the irreducible components of $\mathbf{Mod}_d(\Lambda)$ and exploring their general modules can be played back to the components of the subvarieties $\mathbf{Mod}_d^T$. So, in studying the components of $\mathbf{Mod}_d(\Lambda)$, one of the first questions addresses the semisimple modules that actually arise as "generic tops". (To give an easy example: if $\Lambda = KQ$ where Q is the quiver $1 \to 2$, then S_1 is a generic top for the 2-dimensional modules – it gives rise to an irreducible component of class 0 – as is $S_1 \oplus S_1$ – the latter gives rise to a component of class 1 in our partial order – whereas $S_1 \oplus S_2$ fails to be a generic top for $\mathbf{Mod}_2(\Lambda)$.)

In the next subsection, the partial order on tops will be extended to one on sequences of semisimple modules which, again, are generic invariants of the irreducible components of $\mathbf{Mod}_d(\Lambda)$.

Our principal tool will be an alternate variety parametrizing the same class of representations as $\mathbf{Mod}_d^T$. Namely, we consider the following closed subvariety of the classical Grassmannian $\mathfrak{Gr}(\tilde{d}, JP)$ of $\tilde{d}$-dimensional subspaces of the K-space JP, where $\tilde{d} = \dim_K P - d$:

$$\mathfrak{Grass}_d^T = \{C \in \mathfrak{Gr}(\tilde{d}, JP) \mid C \text{ is a } \Lambda\text{-submodule of } JP\}.$$

This variety comes with an obvious surjection

$$\mathfrak{Grass}_d^T \longrightarrow \{\text{isomorphism classes of } d\text{-dimensional modules with top } T\},$$

sending C to the class of P/C. Clearly, the fibres of this map coincide with the orbits of the natural $\mathrm{Aut}_\Lambda(P)$-action on $\mathfrak{Grass}_d^T$. While the global geometry of the projective variety $\mathfrak{Grass}_d^T$ cannot be reasonably compared with that of the quasi-affine variety $\mathbf{Mod}_d^T$, the "relative geometry" of the $\mathrm{Aut}_\Lambda(P)$-stable subsets of $\mathfrak{Grass}_d^T$ is tightly linked to that of the GL_d-stable subsets of $\mathbf{Mod}_d^T$ in the following sense:

Proposition 2.2. (*See* [**4**, Proposition C].) *The assignment* $\mathrm{Aut}_\Lambda(P).C \mapsto \mathrm{GL}_d.x$, *which pairs orbits* $\mathrm{Aut}_\Lambda(P).C \subseteq \mathfrak{Grass}_d^T$ *and* $\mathrm{GL}_d.x \subseteq \mathbf{Mod}_d^T$ *representing the same Λ-module up to isomorphism, induces an inclusion-preserving bijection*

$$\Psi : \{\mathrm{Aut}_\Lambda(P)\text{-stable subsets of } \mathfrak{Grass}_d^T\} \to \{\mathrm{GL}_d\text{-stable subsets of } \mathbf{Mod}_d^T\}$$

which preserves openness, closures, connectedness, irreducibility, and types of singularities. $\square$

This correspondence permits transfer of information concerning the irreducible components of any locally closed GL_d-stable subvariety of $\mathbf{Mod}_d^T$ to the irreducible components of the corresponding $\mathrm{Aut}_\Lambda(P)$-stable subvariety of $\mathfrak{Grass}_d^T$, and vice versa. Indeed, given that the acting groups, GL_d and $\mathrm{Aut}_\Lambda(P)$, are connected, all of their orbits are irreducible. Therefore all such irreducible components are again stable under the respective actions. In particular, we obtain:

Observation 2.3. *The irreducible components of* $\mathbf{Mod}_d^T$ *are in natural one-to-one correspondence with the irreducible components of* $\mathfrak{Grass}_d^T$. *Moreover, if* $\underline{d} = (d_1, \ldots, d_n)$ *is a partition of d and* $\mathfrak{Grass}_{\underline{d}}^T$ *the closed subvariety of* $\mathfrak{Grass}_d^T$ *consisting of the points that represent modules with dimension vector* $\underline{d}$, *then* $\mathfrak{Grass}_{\underline{d}}^T$ *is a union of irreducible components of* $\mathfrak{Grass}_d^T$. $\square$

Finding the irreducible components of $\mathbf{Mod}_d^T$ by way of the alternate projective setting offers significant advantages, as we will see.

2.B. From $\mathbf{Mod}_d^T$ and $\mathfrak{Grass}_d^T$ to $\mathbf{Mod}(\mathbb{S})$ and $\mathfrak{Grass}(\mathbb{S})$.

The radical layering of modules provides us with a further partition of $\mathfrak{Grass}_d^T$ into pairwise disjoint locally closed subvarieties. (In general, this partition fails to be a stratification in the technical sense, however, even when Λ is hereditary.)

A *d-dimensional semisimple sequence* $\mathbb{S}$ *with top* T is any sequence $(\mathbb{S}_0, \ldots, \mathbb{S}_L)$ of semisimple modules such that $\mathbb{S}_0 = T$ and $\sum_{0 \leq l \leq L} \dim \mathbb{S}_l = d$. Since we identify semisimple modules with their isomorphism classes, a semisimple sequence amounts to a matrix of discrete invariants keeping count of the multiplicities of the simple modules in the semisimples occurring in the slots of $\mathbb{S}$.

Accordingly, we consider the following action-stable locally closed subvarieties of $\mathfrak{Grass}_d^T$ and $\mathbf{Mod}_d^T$, respectively:

$$\mathfrak{Grass}(\mathbb{S}) = \{C \in \mathfrak{Grass}_d^T \mid \mathbb{S}(P/C) = \mathbb{S}\},$$

while $\mathbf{Mod}(\mathbb{S}) \subseteq \mathbf{Mod}_d^T$ consists of those points in $\mathbf{Mod}_d^T$ that parametrize the modules with radical layering $\mathbb{S}$. Clearly, the above one-to-one correspondence between the $\mathrm{Aut}_\Lambda(P)$-stable subsets of $\mathfrak{Grass}_d^T$ and the GL_d-stable subsets of $\mathbf{Mod}_d^T$ restricts to a correspondence between the $\mathrm{Aut}_\Lambda(P)$-stable subsets of $\mathfrak{Grass}(\mathbb{S})$ and the GL_d-stable subsets of $\mathbf{Mod}(\mathbb{S})$. Thus we obtain the following counterpart to Observation 2.1(1):

Observation 2.4. *For each irreducible component* $\mathcal{C}$ *of* $\mathfrak{Grass}_d^T$ *(resp.,* $\mathbf{Mod}_d^T$*), there exist precisely one d-dimensional semisimple sequence* $\mathbb{S}$ *and precisely one irreducible component* $\mathcal{D}$ *of* $\mathfrak{Grass}(\mathbb{S})$ *(resp.,* $\mathbf{Mod}(\mathbb{S})$*) such that* $\mathcal{C}$ *equals the closure of* $\mathcal{D}$ *in* $\mathfrak{Grass}_d^T$ *(resp.,* $\mathbf{Mod}_d^T$*). In particular: Generically, the modules in* $\mathcal{C}$ *have radical layering* $\mathbb{S}$. $\square$

In parallel to Section 2.A, we will next introduce a partial order on the (finite) set of d-dimensional semisimple sequences. It refines the partial order for semisimple modules in that $\mathbb{S} \leq \mathbb{S}'$ implies that $\mathbb{S}_0 \leq \mathbb{S}_0'$.

Definition 2.5. Let $\mathbb{S}$ and $\mathbb{S}'$ be two semisimple sequences with the same dimension. We say that $\mathbb{S}'$ *dominates* $\mathbb{S}$ and write $\mathbb{S} \leq \mathbb{S}'$ if and only if $\bigoplus_{l \leq r} \mathbb{S}_l \leq \bigoplus_{l \leq r} \mathbb{S}_l'$ for all $r \geq 0$.

Roughly speaking, $\mathbb{S}'$ dominates $\mathbb{S}$ if and only if $\mathbb{S}'$ results from $\mathbb{S}$ through a finite sequence of leftward shifts of simple summands of $\bigoplus_{0 \leq l \leq L} \mathbb{S}_l$ in the layering provided by the $\mathbb{S}_l$.

In intuitive terms, the next observation says that the simple summands in the radical layers of the modules represented by $\mathfrak{Grass}(\mathbb{S})$ are only "upwardly mobile" as one passes to modules in the boundary of the closure $\overline{\mathfrak{Grass}(\mathbb{S})}$ of $\mathfrak{Grass}(\mathbb{S})$ in $\mathfrak{Grass}_d^T$. All of the semisimple sequences are tacitly assumed to be d-dimensional.

Observation 2.6. (*For a proof, see* [**13**, Observation 3.4]*.) Suppose that* $\mathbb{S}$ *is a semisimple sequence with top* T. *Then the union* $\bigcup_{\mathbb{S}' \geq \mathbb{S},\, \mathbb{S}_0' = T} \mathfrak{Grass}(\mathbb{S}')$ *is closed in* $\mathfrak{Grass}_d^T$. *(As before* $\mathbb{S}_0'$ *is the semisimple module in the 0-th slot of the sequence* $\mathbb{S}'$.*)* $\square$

The following corollary is the counterpart to Observation 2.1(2).

Corollary 2.7 and Terminology. *All semisimple sequences are assumed to be d-dimensional with top* T.

If $\mathbb{S}$ *is minimal among the semisimple sequences* $\mathbb{S}'$ *with top* T *that give rise to nonempty sets* $\mathfrak{Grass}(\mathbb{S}')$*, then* $\mathfrak{Grass}(\mathbb{S})$ *is open in* $\mathfrak{Grass}_d^T$. *Thus: If the minimal elements in the poset of such sequences* $\mathbb{S}'$ *are* $\mathbb{S}^{(0,1)}, \ldots, \mathbb{S}^{(0,n_0)}$*, then the closures of the varieties*

$\mathfrak{Grass}\,\mathbb{S}^{(0,i)}$ in $\mathfrak{Grass}_d^T$, $1 \leq i \leq n_0$, *are unions of irreducible components of* $\mathfrak{Grass}_d^T$. *In fact, the irreducible components of* $\overline{\mathfrak{Grass}\,\mathbb{S}^{(0,i)}}$ *– they are called* the irreducible components of class 0 of $\mathfrak{Grass}_d^T$ *– are in bijective correspondence with those of* $\mathfrak{Grass}\,\mathbb{S}^{(0,i)}$.

To continue recursively, suppose $\mathbb{S}^{(h,1)}, \ldots, \mathbb{S}^{(h,n_h)}$ *are the distinct semisimple sequences which are minimal in the poset of semisimple sequences* $\mathbb{S}'$ *with*

$$\mathfrak{Grass}\,(\mathbb{S}') \;\not\subseteq\; \bigcup_{k<h,\,i\leq n_k} \overline{\mathfrak{Grass}\,\mathbb{S}^{(k,i)}}.$$

Then each $\mathfrak{Grass}\,\mathbb{S}^{(h,i)}$ *intersects the closed subvariety* $\mathfrak{Grass}_d^T \setminus \bigcup_{k<h,\,i\leq n_k} \mathfrak{Grass}\,\mathbb{S}^{(k,i)}$ *in a nonempty open set. In particular, the irreducible components of the various subvarieties* $\mathfrak{Grass}\,\mathbb{S}^{(h,j)}$ *which are not contained in the closure of* $\bigcup_{k<h,\,i\leq n_k} \mathfrak{Grass}\,\mathbb{S}^{(k,i)}$ *yield distinct irreducible components of* $\mathfrak{Grass}_d^T$, *via passage to closures. They are called* the irreducible components of class h of $\mathfrak{Grass}_d^T$.

Proof. For the first assertion, suppose that $\mathbb{S}$ is minimal among the semisimple sequences $\mathbb{S}'$ with $\mathfrak{Grass}\,(\mathbb{S}') \neq \varnothing$. Let $\mathcal{A}$ be the collection of semisimple sequences that lie minimally above $\mathbb{S}$, and $\mathcal{B}$ the collection of those sequences which are not comparable with $\mathbb{S}$. Moreover, let $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ be the union of all $\mathfrak{Grass}(\mathbb{S}')$ for those semisimple sequences $\mathbb{S}'$ that are larger than or equal to some member of $\mathcal{A}$ and $\mathcal{B}$, respectively. Then $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ are closed, since the set of semisimple sequences (with top T) is finite. The claim thus follows from the equality $\mathfrak{Grass}_d^T \setminus \mathfrak{Grass}(\mathbb{S}) = \widetilde{\mathcal{A}} \cup \widetilde{\mathcal{B}}$.

The other assertions are proved similarly. $\square$

Provided we understand the irreducible components of the $\mathfrak{Grass}(\mathbb{S})$ and their closures in $\mathfrak{Grass}_d^T$, we can therefore find the irreducible components of the latter variety. Hence, the problem of finding the generic semisimple sequences for $\mathfrak{Grass}_d^T$ imposes itself, namely of finding those $\mathbb{S}$ which generically arise as radical layerings of the modules in the various irreducible components of $\mathfrak{Grass}_d^T$.

Finally, we point out that our findings in this section readily translate into the classical affine setting with the aid of Proposition 2.2.

3. SKELETA AND IRREDUCIBLE COMPONENTS OF $\mathfrak{Grass}(\mathbb{S})$ IN AFFINE COORDINATES

In a final transition, we pass to still smaller parametrizing varieties, which retain *stability under the action of the unipotent radical of* $\mathrm{Aut}_\Lambda(P)$ and are endowed with particularly convenient affine coordinates. For that purpose, we will reduce the problem of exploring generic properties of (the modules represented by) a fixed irreducible component of $\mathfrak{Grass}(\mathbb{S})$ – or $\mathfrak{Grass}_d^T$, or $\mathbf{Mod}_d(\Lambda)$ – to finding those of varieties parametrizing the modules with fixed "skeleton". The latter varieties are readily accessible, combinatorially, from quiver and relations of Λ (see the remarks following Theorem 3.8). As they constitute an *open* cover of $\mathfrak{Grass}(\mathbb{S})$, they will give us a good handle on the generic modules representing the components of $\mathfrak{Grass}(\mathbb{S})$.

Let $\mathbb{S}$ be a d-dimensional semisimple sequence with top $\mathbb{S}_0 = T$. Again, P denotes a projective cover of T with distinguished sequence of top elements, $z_1, \ldots, z_t$, such that $z_r =$

$e(r)z_r$ (cf. beginning of Section 2), our non-canonical input for specifying coordinates. As announced, the affine subvarieties to be described will turn out to be stable under the action of the unipotent radical $\big(\mathrm{Aut}_\Lambda(P)\big)_u$ of $\mathrm{Aut}_\Lambda(P)$. We can do a little better in fact, as we will see: Recall that $\mathrm{Aut}_\Lambda(P)$ is isomorphic to a semidirect product $\big(\mathrm{Aut}_\Lambda(P)\big)_u \rtimes \mathrm{Aut}_\Lambda(T)$, and let $\mathcal{T}$ be the following incarnation of a maximal torus in $\mathrm{Aut}_\Lambda(P)$: namely $\mathcal{T} \cong (K^*)^{\mathfrak{t}}$, where $(a_1, \ldots, a_{\mathfrak{t}})$ represents the automorphism of P determined by $z_i \mapsto a_i z_i$. If G is the subgroup $\big(\mathrm{Aut}_\Lambda(P)\big)_u \rtimes \mathcal{T}$ of $\mathrm{Aut}_\Lambda(P)$, the patches of our affine cover will in fact be G-stable. If T is squarefree (that is, if T does not contain any simple summands with multiplicity ≥ 2), G equals $\mathrm{Aut}_\Lambda(P)$, whence our affine cover is $\mathrm{Aut}_\Lambda(P)$-stable in that case.

Here we make no assumptions on T, and $\mathfrak{Grass}_d^T$ will in general not possess any finite cover consisting of $\mathrm{Aut}_\Lambda(P)$-stable affine charts, since the $\mathrm{Aut}_\Lambda(P)$-orbits need not be quasi-affine in general. (In fact, examples attest to the fact that projective $\mathrm{Aut}_\Lambda(P)$-orbits of positive dimension may arise if T has repeated simple summands.)

3.A. Skeleta.

As before, $\Lambda = KQ/I$, and $T \in \Lambda$-mod is semisimple. Roughly speaking, skeleta allow us to carry over some of the benefits of the path-length grading of projective KQ-modules to arbitrary Λ-modules. Next to P – the distinguished Λ-projective cover of T with top elements $z_1, \ldots, z_{\mathfrak{t}}$ – we therefore consider the projective KQ-module $\widehat{P} = \bigoplus_{1 \leq r \leq \mathfrak{t}} (KQ)\widehat{z}_r$, with corresponding sequence of top elements $\widehat{z}_1, \ldots, \widehat{z}_{\mathfrak{t}}$ so that the class of $\widehat{z}_r$ modulo $I\widehat{P}$ coincides with z_r; in particular, $\widehat{z}_r = e(r)\widehat{z}_r$. By a *path in $\widehat{P}$ starting in $\widehat{z}_r$* we mean any element $\widehat{p} = p\widehat{z}_r \in \widehat{P}$, where p is a path in KQ starting in the vertex $e(r)$. The *length* of $\widehat{p}$ is defined to be that of p; ditto for the *endpoint* of $\widehat{p}$. If p_1 is an initial subpath of p, meaning that $p = p_2 p_1$ for paths p_1, p_2, we call $p_1\widehat{z}_r$ an *initial subpath of $p\widehat{z}_r$*. So, in particular, $\widehat{z}_r = e(r)\widehat{z}_r$ is an initial subpath of length 0 of any path $p\widehat{z}_r$ in $\widehat{P}$. The reason why, a priori, we do not identify $p\widehat{z}_r \in \widehat{P}$ with $pz_r \in P = \widehat{P}/I\widehat{P}$ lies in the fact that we require an unambiguous notion of path length, not guaranteed for paths pz_r in P. However, in the sequel, we will often not make a notational distinction between $p\widehat{z}_r$ and pz_r, unless there is need to emphasize well-definedness of path lengths. Moreover, we emphasize that the label r is a crucial attribute of a path in $\widehat{P}$: given a path $p \in KQ$ starting in a vertex $e(r) = e(s)$, the elements $p\widehat{z}_r = p\widehat{z}_s$ of P are distinct unless $r = s$.

Definition 3.1 and Further Conventions.

(1) An (*abstract*) *d-dimensional skeleton with top T* is a set σ of paths of lengths at most L in $\widehat{P}$, which has cardinality d, contains the paths $\widehat{z}_1, \ldots, \widehat{z}_{\mathfrak{t}}$ of length zero (i.e., contains the full sequence of distinguished top elements of $\widehat{P}$), and has the following closure property: namely, σ is closed under initial subpaths; that is, whenever $p_2 p_1 \widehat{z}_r \in \sigma$, then $p_1 \widehat{z}_r \in \sigma$.

Usually, we view a skeleton σ as a forest of $\mathfrak{t}$ tree graphs each consisting of the paths in σ that start in a fixed top element $\widehat{z}_r$ of $\widehat{P}$; see Example 3.3 below and the remarks preceding it. Further conventions:
• By σ_l we denote the set of paths of length l in σ.
• When we pass from the KQ-module $\widehat{P}$ to the Λ-module P by modding out $I\widehat{P}$, we often

identify $\widehat{z}_r$ with z_r and view σ as the set

$$\{pz_r \in P \mid p\widehat{z}_r \in \sigma, \, r \leq \mathfrak{t}\} \subseteq P.$$

(2) Let σ be an abstract d-dimensional skeleton with top T, and M a Λ-module.

We call σ a *skeleton of M* if there exist top elements $m_1, \ldots, m_\mathfrak{t}$ of M and a KQ-epimorphism $f : \widehat{P} \to M$ satisfying $f(\widehat{z}_r) = m_r$ for all r such that, for each $l \in \{0, \ldots, L\}$, the subset

$$\{f(p\widehat{z}_r) \mid p\widehat{z}_r \in \sigma_l\} = \{pm_r \mid r \leq \mathfrak{t}, \, p\widehat{z}_r \in \sigma_l\}$$

of M induces a K-basis for the radical layer $J^l M / J^{l+1} M$ (here we identify the Λ-multiplication on M with the induced KQ-multiplication). In this situation, we also say that σ is a *skeleton of M relative to the sequence $m_1, \ldots, m_\mathfrak{t}$ of top elements*, and observe that the union over l of the above sets, namely $\{pm_r \mid r \leq \mathfrak{t}, \, p\widehat{z}_r \in \sigma\}$, is a K-basis for M. As a consequence, we recognize skeleta of M as special K-bases which respect the radical layering of M and are closely tied to its KQ-structure.

If $M = P/C$, we call σ a *distinguished* skeleton of M provided that σ is a skeleton of M relative to the distinguished sequence $z_1 + C, \ldots, z_\mathfrak{t} + C$ of top elements. (Note: If $P/C \cong P/C'$, any distinguished skeleton of P/C is a skeleton of P/C', but not vice versa in general, as the top elements are shuffled under the $\mathrm{Aut}_\Lambda(T)$-action; see Example 3.3 below.)

(3) Finally, we define

$$\mathfrak{Grass}(\sigma) = \{C \in \mathfrak{Grass}_d^T \mid \sigma \text{ is a distinguished skeleton of } P/C\}.$$

The set of *all* skeleta of M is clearly an isomorphism invariant of M. It contains at least one distinguished candidate when $M = P/C$. Our definition of a skeleton coincides in essence with that given in [**12**] for the situation of a squarefree top $T = P/JP$. However, in that special case, it is unnecessary to hook up the elements of an abstract skeleton σ with specific top elements of the KQ-module $\widehat{P}$, since the dependence on specific sequences of top elements disappears. In particular, every skeleton of P/C is distinguished in the case of squarefree T.

We return to the general case: Again suppose that $M = P/C$, and let σ be an abstract skeleton. Then $M \cong P/D$ for *some* point D in $\mathfrak{Grass}(\sigma)$ if and only if σ is a skeleton of M. Yet, $\mathrm{Aut}_\Lambda(P).C \cap \mathfrak{Grass}(\sigma) \neq \varnothing$ need not imply $C \in \mathfrak{Grass}(\sigma)$. On the other hand, the $\mathfrak{Grass}(\sigma)$ do enjoy the following partial stability under the $\mathrm{Aut}_\Lambda(P)$-action:

Observation 3.2. *The sets $\mathfrak{Grass}(\sigma)$ are locally closed subvarieties of $\mathfrak{Grass}_d^T$, which are stable under the action of $G = \left(\mathrm{Aut}_\Lambda(P)\right)_u \rtimes \mathcal{T}$. Moreover, $\mathfrak{Grass}_d^T$ is the union of the $\mathfrak{Grass}(\sigma)$.*

Proof. Local closedness will follow from Observation 3.5 below, since the $\mathfrak{Grass}(\mathbb{S})$ are known to be locally closed in $\mathfrak{Grass}_d^T$.

As for stability: Suppose $C \in \mathfrak{Grass}(\sigma)$, meaning that σ is a distinguished skeleton of P/C. Then σ remains a distinguished skeleton of P/C after passage from our given distinguished sequence $z_1, \ldots, z_t$ of top elements of P to a new distinguished sequence of the form $g.z_1, \ldots, g.z_t$ for any $g \in G$. But this is tantamount to saying that σ is a distinguished skeleton of $P/(g^{-1}.C)$ relative to the original sequence. In other words, $g^{-1}.C \in \mathfrak{Grass}(\sigma)$.

The straightforward fact that each Λ-module P/C with $C \in \mathfrak{Grass}_d^T$ has at least one distinguished skeleton yields the remaining assertion. $\square$

Any abstract skeleton σ can be communicated by means of an undirected graph which is a *forest*, that is, a finite disjoint union of tree graphs: There are t trees if σ is a skeleton with top T, one for each $r \leq t$; here $\widehat{z}_r$, identified with $e(r)$, represents the root of the r-th tree, recorded in the top row of the graph. The paths $p\widehat{z}_r$ of positive length in σ are represented by edge paths of positive length. Instead of formalizing this convention, we will illustrate it with an example to which we will return in Sections 3.C and 4.

Example 3.3. Let $\Lambda = KQ$, where Q is the quiver
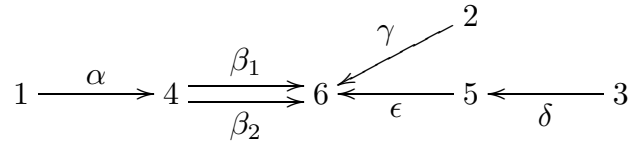
Moreover, let $T = S_1^2 \oplus S_2 \oplus S_3$ and $P = \bigoplus_{1 \leq r \leq 4} \Lambda z_r$ with distinguished top elements $z_1 = (e_1, 0, 0, 0)$, $z_2 = (0, e_1, 0, 0)$, $z_3 = (0, 0, e_2, 0)$, $z_4 = (0, 0, 0, e_3)$. Choose $d = 9$. We consider the module $M = P/C$, where C is the submodule of P generated by $\beta_2 \alpha z_1$, $\beta_1 \alpha z_2$, $\gamma z_3 - \epsilon \delta z_4$, and $\beta_1 \alpha z_1 + \beta_2 \alpha z_2 + \gamma z_3$. Then M has precisely three distinguished skeleta as follows:

(1) $\{e_1\widehat{z}_1, \alpha\widehat{z}_1, \beta_1\alpha\widehat{z}_1\} \sqcup \{e_1\widehat{z}_2, \alpha\widehat{z}_2, \beta_2\alpha\widehat{z}_2\} \sqcup \{e_2\widehat{z}_3\} \sqcup \{e_3\widehat{z}_4, \delta\widehat{z}_4\}$;
(2) $\{e_1\widehat{z}_1, \alpha\widehat{z}_1, \beta_1\alpha\widehat{z}_1\} \sqcup \{e_1\widehat{z}_2, \alpha\widehat{z}_2\} \sqcup \{e_2\widehat{z}_3\} \sqcup \{e_3\widehat{z}_4, \delta\widehat{z}_4, \epsilon\delta\widehat{z}_4\}$;
(3) $\{e_1\widehat{z}_1, \alpha\widehat{z}_1\} \sqcup \{e_1\widehat{z}_2, \alpha\widehat{z}_2, \beta_2\alpha\widehat{z}_2\} \sqcup \{e_2\widehat{z}_3\} \sqcup \{e_3\widehat{z}_4, \delta\widehat{z}_4, \epsilon\delta\widehat{z}_4\}$.

There are five additional, non-distinguished, skeleta of M. The first two of the three graphs below show skeleta of M relative to the permuted sequence of top elements $(z_2 + C, z_1 + C, z_3 + C, z_4 + C)$. The rightmost graph displays a skeleton of M relative to the sequence $(z_1 + z_2 + C, z_2 + C, z_3 + C, z_4 + C)$ of top elements.
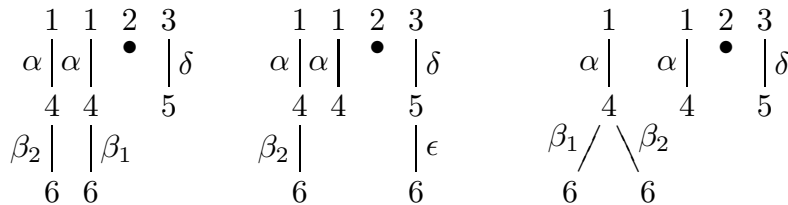
Note that no skeleton of M contains the path $\gamma\widehat{z}_3$, since $\gamma M \subseteq J^2 M$. $\square$

3.B. $\mathfrak{Grass}(\mathbb{S})$ as a union of $\mathfrak{Grass}(\sigma)$'s.

For unproven statements in this subsection and the next, we refer to [**14**].

Suppose that $\mathbb{S}$ is a d-dimensional semisimple sequence with top T. Then $\mathfrak{Grass}(\mathbb{S})$ is the union of those subvarieties $\mathfrak{Grass}(\sigma)$ which have non-empty intersection with $\mathfrak{Grass}(\mathbb{S})$; indeed, $\mathfrak{Grass}(\mathbb{S}) \cap \mathfrak{Grass}(\sigma) \neq \varnothing$ implies that $\mathfrak{Grass}(\sigma)$ is contained in $\mathfrak{Grass}(\mathbb{S})$. These are precisely those nonempty candidates among the $\mathfrak{Grass}(\sigma)$ which are based on skeleta σ compatible with $\mathbb{S}$ in the following sense:

Definition 3.4. Given a semisimple sequence $\mathbb{S}$, we call a skeleton σ *compatible with* $\mathbb{S} = (\mathbb{S}_0, \ldots, \mathbb{S}_L)$ if, for each $l \leq L$ and $i \leq n$, the number of paths in σ_l ending in the vertex e_i coincides with the multiplicity of the simple module S_i in $\mathbb{S}_l$.

Evidently, each abstract skeleton compatible with $\mathbb{S}$ shares the dimension and top with $\mathbb{S}$. In fact, given any skeleton σ of a Λ-module M, the radical layering $\mathbb{S}(M)$ of M is the only semisimple sequence with which σ is compatible.

Suppose that σ is compatible with $\mathbb{S}$. Provided that $\mathfrak{Grass}(\sigma) \neq \varnothing$, the sum of the subspaces $Kpz_r \subseteq JP$, with $p\widehat{z_r} \in \sigma$ of positive length, is direct – this follows from the definition of a skeleton – and the variety $\mathfrak{Grass}(\sigma)$ is the intersection of $\mathfrak{Grass}(\mathbb{S})$ with the following big Schubert cell $\mathrm{Schu}(\sigma)$ of $\mathfrak{Grass}_d^T$: Namely,

$$\mathrm{Schu}(\sigma) = \{ C \in \mathfrak{Grass}_d^T \mid JP = C \oplus \bigoplus_{p\widehat{z_r} \in \sigma,\, \mathrm{length}(p) > 0} Kpz_r \}.$$

Thus we obtain:

Observation 3.5 and Terminology. *The $\mathfrak{Grass}(\sigma)$, where σ runs through the skeleta compatible with $\mathbb{S}$, form an open cover of $\mathfrak{Grass}(\mathbb{S})$; in general, the $\mathfrak{Grass}(\sigma)$ fail to be open in $\mathfrak{Grass}_d^T$, however.*

Suppose $\mathcal{C}$ is an irreducible component of $\mathfrak{Grass}(\mathbb{S})$ and $\mathfrak{S}$ the (finite) set of skeleta σ with the property that $\mathfrak{Grass}(\sigma) \cap \mathcal{C} \neq \varnothing$. Then $\mathcal{C} \cap \bigcap_{\sigma \in \mathfrak{S}} \mathfrak{Grass}(\sigma)$ is a dense open subset of $\mathcal{C}$. We call $\mathfrak{S}$ the generic set of skeleta *of the modules in $\mathcal{C}$.* $\square$

Thus, finding the irreducible components of $\mathfrak{Grass}(\mathbb{S})$ can be played back to finding those of the $\mathfrak{Grass}(\sigma)$ (as mentioned, this task is computationally mastered). Indeed, we have the following correspondence; its elementary proof is left to the reader.

Observation 3.6. *Let $\mathbb{S}$ be a semisimple sequence such that $\mathfrak{Grass}(\mathbb{S}) \neq \varnothing$.*

(1) Whenever $\mathcal{C}$ is an irreducible component of $\mathfrak{Grass}(\mathbb{S})$ and σ a skeleton such that $\mathfrak{Grass}(\sigma)$ intersects $\mathcal{C}$ nontrivially, there exists a unique irreducible component $\mathcal{D}$ of $\mathfrak{Grass}(\sigma)$ whose closure in $\mathfrak{Grass}(\mathbb{S})$ equals $\mathcal{C}$. In fact, $\mathcal{D} = \mathcal{C} \cap \mathfrak{Grass}(\sigma)$.

(2) Conversely, given any irreducible component $\mathcal{D}$ of a nonempty $\mathfrak{Grass}(\sigma)$, where σ is a skeleton compatible with $\mathbb{S}$, the closure of $\mathcal{D}$ in $\mathfrak{Grass}(\mathbb{S})$ is an irreducible component of $\mathfrak{Grass}(\mathbb{S})$. $\square$

3.C. Affine coordinates and irreducible components of the $\mathfrak{Grass}(\sigma)$.

As we saw in the preceding subsection, the map assigning to each point x in an irreducible component $\mathcal{C}$ of $\mathbf{Mod}_d(\Lambda)$ the set of skeleta of the module corresponding to x is constant on a dense open subset of $\mathcal{C}$. This generic behavior of skeleta singles them out as relevant for the construction of generic modules representing the irreducible components of $\mathbf{Mod}_d(\Lambda)$.

Let σ be a d-dimensional skeleton with top T. Recall from 3.A that this makes σ a set of "paths" with well-defined lengths in the projective KQ-module $\widehat{P} = \bigoplus_{1 \le r \le t} KQ\widehat{z}_r$. We supplement Definition 3.1 as follows:

Definition 3.7. A σ-*critical path* is a path $\alpha p\widehat{z}_r$ of length at most L in $\widehat{P} \setminus \sigma$, where α is an arrow and $p\widehat{z}_r \in \sigma$. Moreover, for any such σ-critical path, the σ-*set of* $\alpha p\widehat{z}_r$ is the set $\sigma(\alpha p\widehat{z}_r)$ consisting of all paths $q\widehat{z}_s \in \sigma$ which are at least as long as $\alpha p\widehat{z}_r$ and end in the same vertex as $\alpha p\widehat{z}_r$.

In other words, a path in $\widehat{P}$ is σ-critical if and only if it fails to belong to σ, whereas every proper initial subpath does. If σ is Skeleton (1) in Example 3.3, the σ-critical paths are $\beta_2\alpha\widehat{z}_1$, $\beta_1\alpha\widehat{z}_2$, $\gamma\widehat{z}_3$ and $\epsilon\delta\widehat{z}_4$, with $\sigma(\beta_2\alpha\widehat{z}_1) = \{\beta_1\alpha\widehat{z}_1, \beta_2\alpha\widehat{z}_2\} = \sigma(\beta_1\alpha\widehat{z}_2)$ and $\sigma(\gamma\widehat{z}_3) = \{\beta_1\alpha\widehat{z}_1, \beta_2\alpha\widehat{z}_2\} = \sigma(\epsilon\delta\widehat{z}_4)$. If we add the arrows α giving rise to σ-critical paths $\alpha p\widehat{z}_r$ to the graph of this skeleton, marking them by broken edges, we obtain the following picture:
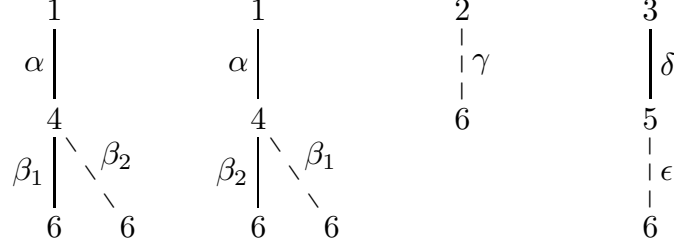
That C be a point in $\mathfrak{Grass}(\sigma)$ obviously entails the existence of unique scalars $c(\alpha p\widehat{z}_r, q\widehat{z}_s)$ with the property that

$$\alpha p z_r + C = \sum_{q\widehat{z}_s \in \sigma(\alpha p\widehat{z}_r)} c(\alpha p\widehat{z}_r, q\widehat{z}_s)\, q z_s + C,$$

in P/C, whenever $\alpha p\widehat{z}_r$ is a σ-critical path. Clearly, the isomorphism type of $M = P/C$ is completely determined by the family of these scalars. Thus we obtain a map

$$\mathfrak{Grass}(\sigma) \to \mathbb{A}^N, \quad C \mapsto c = \big(c(\alpha p\widehat{z}_r, q\widehat{z}_s)\big)_{\alpha p\widehat{z}_r \ \sigma\text{-critical}, \ q\widehat{z}_s \in \sigma(\alpha p\widehat{z}_r)}$$

where N is the (disjoint) union of the sets $\{\alpha p\widehat{z}_r\} \times \sigma(\alpha p\widehat{z}_r)$. Note: Since, a priori, the sets $\sigma(\alpha p\widehat{z}_r)$, where $\alpha p\widehat{z}_r$ traces the σ-critical paths, need not be disjoint, we have paired their elements with the pertinent σ-critical paths to force disjointness.

By [**14**], the map $\mathfrak{Grass}(\sigma) \to \mathbb{A}^N$, $C \mapsto c$, described above is an isomorphism from $\mathfrak{Grass}(\sigma)$ onto a closed subvariety of $\mathbb{A}^N$. The point $C \in \mathfrak{Grass}(\sigma)$ corresponding to a

point c in the image of this map is the submodule of JP generated over Λ by the elements $\alpha p z_r - \sum_{q\widehat{z}_s \in \sigma(\alpha p\widehat{z}_r)} c(\alpha p\widehat{z}_r, q\widehat{z}_s) \, q z_s$, where $\alpha p\widehat{z}_r$ runs through the σ-critical paths. We will identify C with c whenever convenient.

The following result from [**14**] summarizes those properties of the cover $\big(\mathfrak{Grass}(\sigma)\big)_\sigma$ of $\mathfrak{Grass}_d^T$ which will be relevant here; the first statement partially overlaps with Sections 3.A/B.

Theorem 3.8. *For every d-dimensional skeleton σ with top T, the set $\mathfrak{Grass}(\sigma)$ is a locally closed affine subvariety of $\mathfrak{Grass}_d^T$ which is stable under the action of the group $G = \big(\operatorname{Aut}_\Lambda(P)\big)_u \rtimes \mathcal{T}$. Moreover, given any d-dimensional semisimple sequence $\mathbb{S}$ with top T, the varieties $\mathfrak{Grass}(\sigma)$, where σ traces the skeleta compatible with $\mathbb{S}$, form an open affine cover of $\mathfrak{Grass}(\mathbb{S})$.*

Polynomial equations determining the $\mathfrak{Grass}(\sigma)$ in $\mathbb{A}^N$ can be algorithmically obtained from Q and generators for the admissible ideal $I \subseteq KQ$, where I is viewed as a left ideal. If K_0 is a subfield of K over which such generators for I are defined, the resulting polynomials determining $\mathfrak{Grass}(\sigma)$ are defined over K_0 as well. $\square$

The authors have implemented the mentioned algorithm for $\mathfrak{Grass}(\sigma)$ at [**2**], and combined it with a software package that computes the irreducible components.

The usefulness of the graphing technique of [**9**] and [**11**] is limited, since it calls for display of *all* relations of a module P/C, including those that are consequences of others. The following more sparing graphs will better serve our purpose of graphically representing "generic modules" for the irreducible components of various parametrizing varieties. Recall that the term *hypergraph* is commonly used to refer to an undirected graph which not only allows for conventional edges coupling pairs of vertices, but for *hyperedges* connecting more than two vertices. The hypergraphs of a module P/C considered here presuppose a fixed choice of top elements of P. Roughly speaking, they consist of a distinguished skeleton of P/C (which may be viewed as a forest of traditional tree graphs – see Subsection 3.A), combined with suitable subsets of the σ-sets of the σ-critical paths which in turn depend on C; these latter subsets provide additional (hyper)edges.

Definition 3.9. Hypergraphs of Modules. Suppose M is a d-dimensional module with top T and skeleton σ. This means that $M \cong P/C$ for some $C \in \mathfrak{Grass}(\sigma)$. Denote by

$$\big(c(\alpha p\widehat{z}_r, q\widehat{z}_s)\big)_{\alpha p\widehat{z}_r \ \sigma\text{-critical}, \, q\widehat{z}_s \in \sigma(\alpha p\widehat{z}_r)}$$

the local affine coordinates of C relative to σ. Then the following will be called a *hypergraph* of M: The skeleton σ, paired with the family $\big(M(\alpha p\widehat{z}_r)\big)$ of subsets $M(\alpha p\widehat{z}_r) \subseteq \sigma(\alpha p\widehat{z}_r)$, respectively, where $\alpha p\widehat{z}_r$ ranges through the σ-critical paths and $M(\alpha p\widehat{z}_r)$ consists of those paths $q\widehat{z}_s \in \sigma(\alpha p\widehat{z}_r)$ for which $c(\alpha p\widehat{z}_r, q\widehat{z}_s) \neq 0$; here it is important that the sets $M(\alpha p\widehat{z}_r)$ be recorded in conjunction with the σ-critical paths by which they are labeled.

In depicting such a hypergraph, we start with the graph of σ, add on edges representing the terminating arrows α of the σ-critical paths $\alpha p\widehat{z}_r$ – the latter distinguished from the edges of σ via broken lines – and finally mark the *hyperedges*, one for each $M(\alpha p\widehat{z}_r) \subseteq \sigma(\alpha p\widehat{z}_r)$, by means of closed curves including the sets of endpoints of the paths in $\{\alpha p\widehat{z}_r\} \cup$

$M(\alpha p \widehat{z}_r)$. In case $M(\alpha p \widehat{z}_r)$ is a singleton, say $\{q \widehat{z}_s\}$, we simply join the endpoints of $\alpha p \widehat{z}_r$ and $q \widehat{z}_s$. If, for the algebra Λ of Example 3.3, we take $T = \sigma_1^2$, $P = (\Lambda e_1)^2$ – meaning that z_1, z_2 are both normed by e_1 – and $C = \Lambda(\beta_1 \alpha - \beta_2 \alpha)z_1 + \Lambda[\beta_1 \alpha z_1 + (\beta_1 \alpha - \beta_2 \alpha)z_2]$, then the hypergraph of P/C with respect to the skeleton $\sigma = \{\widehat{z}_1 = e_1 \widehat{z}_1, \alpha \widehat{z}_1, \beta_1 \alpha \widehat{z}_1, \widehat{z}_2 = e_1 \widehat{z}_2, \alpha \widehat{z}_2, \beta_1 \alpha \widehat{z}_2\}$ consists of σ, together with the sets $M(\beta_2 \alpha \widehat{z}_1) = \{\beta_1 \alpha \widehat{z}_1\}$ and $M(\beta_2 \alpha \widehat{z}_2) = \{\beta_1 \alpha \widehat{z}_1, \beta_1 \alpha \widehat{z}_2\}$. It looks as follows:
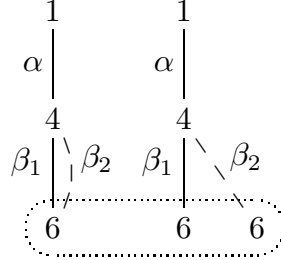
Note that this hypergraph of P/C is connected. More strongly, P/C is indecomposable.

For further illustrations, see Examples 4.7, 5.7, and 5.9.

The final observation of this section is an immediate consequence of the definitions. We leave the easy proof to the reader.

Observation 3.10 and Notational Convention. *Let $\mathbb{S}$ be a d-dimensional semisimple sequence with top T and σ a skeleton compatible with $\mathbb{S}$. Then the number*

$$N(\mathbb{S}) = \sum_{\alpha p \widehat{z}_r \ \sigma\text{-critical}} |\sigma(\alpha p \widehat{z}_r)|$$

depends only on $\mathbb{S}$, not on σ.

Moreover, $N(\mathbb{S}) = N_0(\mathbb{S}) + N_1(\mathbb{S})$, where both of the numbers

$$N_0(\mathbb{S}) = \sum_{\alpha p \widehat{z}_r \ \sigma\text{-critical}} |\{q \widehat{z}_s \in \sigma(\alpha p \widehat{z}_r) \mid \text{length}(q) = \text{length}(\alpha p)\}|$$

and

$$N_1(\mathbb{S}) = \sum_{\alpha p \widehat{z}_r \ \sigma\text{-critical}} |\{q \widehat{z}_s \in \sigma(\alpha p \widehat{z}_r) \mid \text{length}(q) > \text{length}(\alpha p)\}|$$

depend solely on $\mathbb{S}$. $\square$

4. $\mathbb{S}$-GENERIC, (T, d)-GENERIC, AND d-GENERIC MODULES

Throughout this section, we assume the base field K to have infinite transcendence degree over its prime field. Moreover, we fix an algebraically closed subfield K_0 of K such that K has infinite transcendence degree over K_0 and $I \subseteq KQ$ is defined over K_0. The latter means that, as a KQ-ideal, I is generated by $I_0 = I \cap K_0 Q$.

The assumption on the transcendence degree of our base field will enable us to realize and study "$\mathbb{S}$-generic", "(T, d)-generic", or "d-generic" modules inside the category Λ-mod.

Since we will obtain one generic object for every irreducible component of $\mathfrak{Grass}(\mathbb{S})$ (or $\mathfrak{Grass}_d^T$, or $\mathbf{Mod}_d(\Lambda)$), unique up to suitable equivalence, there will be only finitely many generic modules to be studied in each dimension, which permits assembling essential information on the representation theory of Λ in a finite frame. (Here the attribute "generic" is unrelated to the generic modules introduced by Crawley-Boevey in [**5**]. However, it is unlikely to conflict with existing usage, as we attach specifying prefixes; moreover, Crawley-Boevey's generic modules are infinite dimensional by definition, whereas we exclusively consider finite dimensional modules.)

In the present setting, Λ is clearly isomorphic to the algebra $\Lambda_0 \otimes_{K_0} K$ via $\lambda_0 \otimes k \mapsto k\lambda_0$, where $\Lambda_0 = K_0 Q/I_0$, and all of the varieties $\mathfrak{Grass}_d^T$, $\mathfrak{Grass}(\mathbb{S})$ and $\mathfrak{Grass}(\sigma)$ are defined over K_0. For the $\mathfrak{Grass}(\sigma)$, this follows from Theorem 3.8; both $\mathfrak{Grass}_d^T$ and $\mathfrak{Grass}(\mathbb{S})$ are finite unions of $\mathfrak{Grass}(\sigma)$'s. We denote by $\left(\mathfrak{Grass}_d^T\right)_0$, $\left(\mathfrak{Grass}(\mathbb{S})\right)_0$ and $\left(\mathfrak{Grass}(\sigma)\right)_0$ the restrictions of the mentioned varieties to K_0. Hence, given an irreducible component $\mathcal{C}$ of $\mathfrak{Grass}_d^T$, $\mathfrak{Grass}(\mathbb{S})$, or $\mathfrak{Grass}(\sigma)$, it makes sense to refer to the corresponding irreducible component $\mathcal{C}_0$ of $\left(\mathfrak{Grass}_d^T\right)_0$, $\left(\mathfrak{Grass}(\mathbb{S})\right)_0$, or $\left(\mathfrak{Grass}(\sigma)\right)_0$.

Next to the described algebraic group actions on the parametrizing varieties, we will consider the following action of the Galois group $\mathrm{Gal}(K/K_0)$. It results from the obvious fact that any projective K-space carries such an action and the closed subvariety $\mathfrak{Grass}_d^T$ of $\mathbb{P}(\bigwedge^{\tilde{d}} JP)$ is defined over K_0 (again, $\tilde{d} = \dim P - d$). We briefly discuss the forms this $\mathrm{Gal}(K/K_0)$-action takes on in terms of the coordinate systems we are using. Given any skeleton σ with $\mathfrak{Grass}(\sigma) \neq \varnothing$, we view σ as a (necessarily K-linearly independent) subset of P, and supplement $\sigma \cap JP$ to a path basis for JP. This provides a homogeneous coordinatization of $\mathbb{P}(\bigwedge^{\tilde{d}} JP)$, and we let a map $\phi \in \mathrm{Gal}(K/K_0)$ act on a point in projective space by applying ϕ to the homogeneous coordinates. The action does not depend on the choice of path basis (and hence does not depend on σ), since the transition matrices from one such basis to another have coefficients in K_0. Moreover, we note: The subvarieties $\mathfrak{Grass}_d^T$, $\mathfrak{Grass}(\mathbb{S})$, and $\mathfrak{Grass}(\sigma)$ are stable under this $\mathrm{Gal}(K/K_0)$-action, and all maps $\phi \in \mathrm{Gal}(K/K_0)$ leave the restrictions $(\mathfrak{Grass}_d^T)_0$, etc., pointwise fixed. In particular, the $\mathrm{Gal}(K/K_0)$-action stabilizes the irreducible components of the considered varieties because they are all defined over K_0. Given $C \in \mathfrak{Grass}_d^T$, we will write C_ϕ for $\phi.C$ to set it off from the $\mathrm{Aut}_\Lambda(P)$-action. For a point $C \in \mathfrak{Grass}(\sigma)$, the affine coordinates $(c_\nu)_{\nu \in N}$ of Section 3.C are readily seen to be K_0-linear combinations of certain Plücker coordinates of $\mathbb{P}(\bigwedge^{\tilde{d}} JP)$ relative to the described path basis for JP; in fact, it is routine to check that C is detemined by a subcollection of the c_ν which coinicde with selected Plücker coordinates (that we do not need to keep track of all of them in order to pin down C, is due to the fact that C is not only a K-space, but a Λ-submodule of JP). Thus $C_\phi = \left(\phi(c_\nu)\right)$ in the distinguished affine coordinates of $\mathfrak{Grass}(\sigma)$. Clearly, $\mathrm{Gal}(K/K_0)$ acts on the classical affine module varieties $\mathbf{Mod}_d(\Lambda)$, $\mathbf{Mod}_d^T$ and $\mathbf{Mod}(\mathbb{S})$ as well, and these actions are compatible with the ones on their Grassmannian counterparts under the transfer maps for orbits described in Proposition 2.2.

Finally, we relate the above actions on the varieties $\mathfrak{Grass}_d^T$ to the well-known fact that $\mathrm{Gal}(K/K_0)$ also acts on the category Λ-Mod, in that each $\phi \in \mathrm{Gal}(K/K_0)$ gives rise to a

ring automorphism of Λ (actually, a K_0-algebra automorphism), also denoted ϕ, which in turn induces a self-equivalence F_ϕ of Λ-mod.

Observation 4.1 and Definition of $\mathrm{Gal}(K/K_0)$**-equivalence and stability.** *Let* $\phi \in \mathrm{Gal}(K/K_0)$ *and, for* $M \in \Lambda$-*Mod, denote by* $F_\phi(M)$ *the* Λ-*module with underlying abelian group* M *and module multiplication* $\lambda * m = \phi^{-1}(\lambda)m$. *Then the assignment* $M \mapsto F_\phi(M)$ *extends to an equivalence*

$$F_\phi : \Lambda\text{-Mod} \longrightarrow \Lambda\text{-Mod}$$

of categories. For $C \in \mathfrak{Grass}_d^T$, *this equivalence sends* P/C *to a module in the isomorphism class of* P/C_ϕ, *where* C_ϕ *is as above; in particular,* F_ϕ *sends* $P = \bigoplus_{1 \le r \le \mathfrak{t}} \Lambda z_r$ *to an isomorphic copy of itself. Moreover,* $F_\phi(C) \cong C_\phi$ *in* Λ-*Mod, under the* Λ-*isomorphism* $P \to P$ *which sends* kz_r *to* $\phi(k)z_r$ *for* $k \in K$ *and all* r.

Clearly, F_ϕ *preserves all properties of* Λ-*modules which are invariant under Morita-equivalence, fixes the isomorphism classes of the indecomposable projectives and the simple modules, and preserves dimension vectors of modules and their tops; a fortiori,* F_ϕ *preserves* K-*dimension. Moreover, for* $C, D \in \mathfrak{Grass}_d^T$, *a homomorphism* $f : P/C \to P/D$ *is an isomorphism precisely when this is true for* $f_\phi = F_\phi(f) : P/C_\phi \to P/D_\phi$. *Consequently, the action of* $\mathrm{Gal}(K/K_0)$ *permutes the* $\mathrm{Aut}_\Lambda(P)$-*orbits within each irreducible component of* $\mathfrak{Grass}_d^T$, $\mathfrak{Grass}(\mathbb{S})$, *or* $\mathfrak{Grass}(\sigma)$.

We call two Λ-modules M and M' *equivalent* or, more precisely, $\mathrm{Gal}(K/K_0)$-*equivalent*, if there exists a map $\phi \in \mathrm{Gal}(K/K_0)$ such that $M' \cong F_\phi(M)$. Moreover, two connected components $\mathcal{A}$ and $\mathcal{A}'$ of the Auslander-Reiten quiver of Λ-mod are said to be $\mathrm{Gal}(K/K_0)$-*equivalent* if there exists $\phi \in \mathrm{Gal}(K/K_0)$ with the property that the functor F_ϕ takes $\mathcal{A}$ to $\mathcal{A}'$.

A property of modules is said to be $\mathrm{Gal}(K/K_0)$-*stable* if it is passed on from a module to any equivalent module. Analogously, we define stability of properties of *pairs* of modules, calling such a property $\mathrm{Gal}(K/K_0)$-*stable* if, for all ϕ, either both pairs (M, M') and $(F_\phi(M), F_\phi(M'))$ have this property, or else neither of them does. $\square$

Next to categorically defined features (such as the standard homological properties), the $\mathrm{Gal}(K/K_0)$-stable properties of a module include its set of skeleta, since $F_\phi(P/C) = P/C_\phi$ whenever $C \in \mathfrak{Grass}(\sigma)$. The concept of $\mathrm{Gal}(K/K_0)$-stability translates into our parametrizing varieties as follows: A property (*) that pertains to d-dimensional modules with top T is $\mathrm{Gal}(K/K_0)$-stable precisely when it is preserved under module isomorphism and the set of points C in $\mathfrak{Grass}_d^T$ such that P/C has property (*) is stable under the $\mathrm{Gal}(K/K_0)$-action on $\mathfrak{Grass}_d^T$.

To illustrate the concept of $\mathrm{Gal}(K/K_0)$-equivalence, we return to Example 3.3, taking the base field to be $\mathbb{C}$ and the subfield K_0 to be the field $\overline{\mathbb{Q}}$ of algebraic numbers. Then the following Λ-modules M_1, M_2 are $\mathrm{Gal}(K/K_0)$-equivalent: They have projective cover $P = (\Lambda e_1)^2$, endowed with distinguished top elements z_1, z_2, both normed by e_1. We define M_j to be P/C_j for $j = 1, 2$, where

$$C_1 = \Lambda(\beta_1\alpha - \pi^3\beta_2\alpha)z_1 \ + \ \Lambda[\beta_1\alpha z_1 + (\pi\beta_1\alpha - \sqrt[4]{\pi}\beta_2\alpha)z_2],$$

and
$$C_2 = \Lambda(\beta_1\alpha - (1/\pi)^3\beta_2\alpha)z_1 \ + \ \Lambda[\beta_1\alpha z_1 + \big((1/\pi)\beta_1\alpha + i(1/\sqrt[4]{\pi})\beta_2\alpha\big)z_2].$$

On the other hand, M_1 and M_2 fail to be isomorphic.

As pointed out, $\mathrm{Gal}(K/K_0)$-equivalent modules have the same sets of skeleta; clearly, they also have coinciding hypergraphs (see 3.9 for our conventions). In our example, the hypergraph of the equivalent modules M_1 and M_2 with respect to the skeleton $\sigma = \{z_1, \alpha z_1, \beta_1\alpha z_1, z_2, \alpha z_2, \beta_1\alpha z_2\}$ is the same as that of the module P/C described in 3.9.

Definition 4.2 of $\mathbb{S}$-, (T, d)-, and d-generic modules. Suppose that $\mathcal{C}$ is an irreducible component of $\mathfrak{Grass}(\mathbb{S})$. A module $G \in \Lambda$-mod is called $\mathbb{S}$-*generic for $\mathcal{C}$* (relative to K_0) if
- $G \cong P/C$ for some $C \in \mathcal{C}$, and
- G has all $\mathrm{Gal}(K/K_0)$-stable generic properties of the modules in $\mathcal{C}$. By this we mean that $\mathrm{Aut}_\Lambda(P).C$ has nonempty intersection with every $\mathrm{Gal}(K/K_0)$-stable open subset of $\mathcal{C}$.

If the closure of $\mathcal{C}$ in $\mathfrak{Grass}_d^T$ is an irreducible component of the latter variety, then an $\mathbb{S}$-generic module G for $\mathcal{C}$ is also called (T, d)-*generic*, or, more precisely, (T, d)-*generic for the closure of $\mathcal{C}$ in $\mathfrak{Grass}_d^T$*. Finally, let $\mathcal{C}'$ be the irreducible component of $\mathbf{Mod}(\mathbb{S})$ which corresponds to $\mathcal{C}$ via the bijection of Proposition 2.2. If the closure of $\mathcal{C}'$ in $\mathbf{Mod}_d(\Lambda)$ is an irreducible component of $\mathbf{Mod}_d(\Lambda)$, the module G is also referred to as d-*generic*, or, by an abuse of language, d-*generic for the closure of $\mathcal{C}$ in $\mathbf{Mod}_d(\Lambda)$*.

Note that the second requirement we imposed on a generic module for $\mathcal{C}$ has the following equivalent formulation: $\mathrm{Aut}_\Lambda(P).C$ has nonempty intersection with the $\mathrm{Gal}(K/K_0)$-closure of any dense open subset of $\mathcal{C}$.

For several explicit samples of $\mathbb{S}$-, (T, d)-, and d-generic modules, see Example 4.8 and Section 5. Clearly, the property of being generic on one of the indicated levels is passed on from a module G to any module isomorphic to $F_\phi(G) = G_\phi$; in other words, being $\mathbb{S}$-generic (or (T, d)-generic, or d-generic) is a $\mathrm{Gal}(K/K_0)$-stable property. In the next theorem, we establish existence of generic modules, as well as their uniqueness up to $\mathrm{Gal}(K/K_0)$-equivalence.

Theorem 4.3. *Let $\mathbb{S} = (\mathbb{S}_0, \ldots, \mathbb{S}_L)$ be any semisimple sequence with $\mathfrak{Grass}(\mathbb{S}) \neq \varnothing$.*

Given any irreducible component $\mathcal{C}$ of $\mathfrak{Grass}(\mathbb{S})$, there exists an $\mathbb{S}$-generic Λ-module $G = G(\mathbb{S}, \mathcal{C})$ for $\mathcal{C}$. It is unique up to $\mathrm{Gal}(K/K_0)$-equivalence (i.e., unique up to a shift by some auto-equivalence F_ϕ of Λ-Mod).

Supplement 1: *Every skeleton σ with $\mathfrak{Grass}(\sigma) \cap \mathcal{C} \neq \varnothing$ is a skeleton of G, and any hypergraph of G (see 3.9) is shared by all other generic modules for $\mathcal{C}$. More precisely, for each such skeleton σ, the generic module G has a minimal projective presentation as follows: Let $P = \bigoplus_{1 \leq r \leq \mathfrak{t}} \Lambda z_r$ be the distinguished projective cover of the top $T = \mathbb{S}_0$ of G. Then $G \cong P/C$, where*

$$C = \sum_{\alpha p \widehat{z}_r \ \sigma\text{-critical}} \Lambda\left(\alpha p z_r - \sum_{q\widehat{z}_s \in \sigma(\alpha p \widehat{z}_r)} c(\alpha p \widehat{z}_r, q\widehat{z}_s)\, q z_s \right)$$

for a suitable family of scalars $(c(\alpha p \widehat{z}_r, q \widehat{z}_s))$ in $\mathfrak{Grass}(\sigma) \cap \mathcal{C}$, such that the transcendence degree of the field $K_0\big(c(\alpha p \widehat{z}_r, q \widehat{z}_s) \mid \alpha p \widehat{z}_r \text{ is } \sigma\text{-critical}, \ q \widehat{z}_s \in \sigma(\alpha p \widehat{z}_r)\big)$ over K_0 equals the dimension of $\mathcal{C}$. Conversely, every module with a presentation of this type is generic for $\mathcal{C}$.

Supplement 2: *One can choose generic modules $G(\mathbb{S}, \mathcal{C})$ so as to have the following generic properties relative to one another: Given two semisimple sequences $\mathbb{S}$ and $\mathbb{S}'$, together with irreducible components $\mathcal{C}$ and $\mathcal{C}'$ of $\mathfrak{Grass}(\mathbb{S})$ and $\mathfrak{Grass}(\mathbb{S})'$, respectively, any generic $\mathrm{Gal}(K/K_0)$-stable property of pairs of modules in $\mathcal{C} \times \mathcal{C}'$ is shared by the pair $(G(\mathbb{S}, \mathcal{C}), G(\mathbb{S}', \mathcal{C}'))$.*

The gist of Supplement 1 is this: Understanding the module G boils down to understanding the geometry of $\mathfrak{Grass}(\sigma)$. It has immediate applications to algebras for which one has a full grasp of the $\mathfrak{Grass}(\sigma)$, notably truncated path algebras.

In light of Section 2, every irreducible component $\mathcal{D}$ of $\mathbf{Mod}_d(\Lambda)$ (or $\mathbf{Mod}_d^T$) is the closure in $\mathbf{Mod}_d(\Lambda)$ (resp. in $\mathbf{Mod}_d^T$)) of a unique irreducible component $\mathcal{C}$ of $\mathbf{Mod}(\mathbb{S})$. In this situation, the module $G(\mathbb{S}, \mathcal{C})$ is even d-generic for $\mathcal{D}$, or (T, d)-generic for $\mathcal{D}$, respectively. Consequently, Theorem 4.3 has two spinoffs, namely exact analogues of its assertions for d- and (T, d)-generic modules.

The lion's share of the theorem rests on a general fact concerning "generic orbits" of algebraic group actions, which allows for multiple variations in different directions. purpose, we let Γ be a connected algebraic group over K acting morphically on an irreducible quasi-projective K-variety $\mathcal{C}$, which is given in terms of an embedding into a projective K-space. We suppose that $\mathcal{C}$ is defined over K_0, so that the $\mathrm{Gal}(K/K_0)$-action on the encompassing projective space restricts to a $\mathrm{Gal}(K/K_0)$-action on $\mathcal{C}$. This action is denoted in the form $(\phi, C) \mapsto C_\phi$, while the Γ-action is given by $(g, C) \mapsto g.C$. Moreover, we suppose that $\mathcal{C}$ has a finite open affine cover (V_σ) such that each V_σ is stable under this $\mathrm{Gal}(K/K_0)$-action. In other words, we assume the V_σ to be isomorphic to closed subvarieties of some affine space $\mathbb{A}^N(K)$, cut out by suitable polynomials in $K_0[X_\nu \mid \nu \in N]$, such that the restriction to any V_σ of the $\mathrm{Gal}(K/K_0)$-action on $\mathbb{A}^N(K)$ coincides with the restricted $\mathrm{Gal}(K/K_0)$-action on $\mathcal{C}$. Finally, we assume the following compatibility condition for the two actions: If points C and D in $\mathcal{C}$ belong to the same Γ-orbit, then so do C_ϕ and D_ϕ for any ϕ, that is, the $\mathrm{Gal}(K/K_0)$-action on $\mathcal{C}$ induces an action of $\mathrm{Gal}(K/K_0)$ on the set of orbits of Γ in $\mathcal{C}$. We note that, given any open subset U of $\mathcal{C}$, the closure of U under the $\mathrm{Gal}(K/K_0)$-action is again Zariski-open.

In this situation, we call a Γ-orbit $\Gamma.C$ of $\mathcal{C}$ *generic* (relative to K_0) if $\Gamma.C$ has nonempty intersection with the $\mathrm{Gal}(K/K_0)$-closure of any dense open subset of $\mathcal{C}$.

Lemma 4.4. *Let $\mathcal{C}$ be a Γ-space as above. Then there exists a generic orbit $\Gamma.C$ in $\mathcal{C}$. It is unique in the following sense: Any two generic orbits $\Gamma.C$ and $\Gamma.C'$ are equivalent under the $\mathrm{Gal}(K/K_0)$-action on the set of Γ-orbits of $\mathcal{C}$, i.e., there exists $\phi \in \mathrm{Gal}(K/K_0)$ such that $(\Gamma.C')_\phi = \Gamma.C$.*

More strongly: If $(\mathcal{C}_k)_{k \in \mathbb{N}}$ is a sequence of irreducible Γ-spaces as specified above, then given any positive integer n, there exists a generic Γ-orbit in the Γ-space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_n$. It also has the described uniqueness property.

The existence proof is standard; yet, we include the argument for the convenience of the reader. It is the uniqueness argument which rests on the specifics of the setup.

Proof of Lemma 4.4. We start by ensuring existence of a generic orbit. To provide a setting which works for the strengthened statement concerning a sequence $(\mathcal{C}_k)$ of irreducible Γ-spaces, let B be an infinite subset of K which is algebraically independent over K_0, and $B = \bigsqcup_{i \in \mathbb{N}} B_i$ a partition of B into infinite disjoint subsets B_i. For each i, we denote by K_i the algebraic closure of $K_0(b \mid b \in B_i)$ in K. We first focus on a fixed index $i \in \mathbb{N}$, and suppose $\mathcal{C} = \mathcal{C}_i$.

Let (V_σ) be a finite open affine cover of $\mathcal{C}$ as specified ahead of the lemma; in particular, each V_σ is a closed affine subspace of some affine space $\mathbb{A}(K)^N$, defined by polynomials from $K_0[X_\nu \mid \nu \in N]$. Pick one of the V_σ – call it V – and set $V_0 = V \cap \mathbb{A}(K_0)^N$, the latter being a closed affine subvariety of $\mathbb{A}(K_0)^N$. We abbreviate $\dim V = \dim V_0 = \dim \mathcal{C}$ to v. Further, we denote by $\mathfrak{Q}(V)$ the rational function field of V and write the coordinate functions in $\mathfrak{Q}(V)$ (relative to the standard coordinatization of V in $\mathbb{A}^N$) as X_ν, $\nu \in N$. Suppose $X_{\nu_1}, \ldots, X_{\nu_v}$ form a transcendence basis for $\mathfrak{Q}(V)$ over K. The corresponding coordinate functions of V_0, again denoted by $X_{\nu_1}, \ldots, X_{\nu_v}$, then form a transcendence basis for $\mathfrak{Q}(V_0)$ over K_0. Pick arbitrary distinct elements $c_{\nu_1}, \ldots, c_{\nu_v} \in B_i \subseteq K_i$ – by construction they are algebraically independent over K_0 – and embed the subfield $K_0(X_{\nu_1}, \ldots, X_{\nu_v})$ of $\mathfrak{Q}(V_0)$ into K, by sending X_{ν_j} to c_{ν_j} and mapping K_0 identically to itself. Since K_i is algebraically closed, this map extends to a K_0-embedding $\rho : \mathfrak{Q}(V_0) \to K_i \subseteq K$. For $\nu \in N \setminus \{\nu_1, \ldots, \nu_v\}$, set $c_\nu = \rho(X_\nu)$. Then $C = \big(c_\nu\big)_{\nu \in N}$ is a point in V.

To prove that the orbit $\Gamma.C$ is generic, we will, in fact, show that C belongs to any $\mathrm{Gal}(K/K_0)$-stable dense open subset U of $\mathcal{C}$. It is harmless to assume $U \subseteq V$, since $U \cap V$ is again nonempty, open, and stable under the $\mathrm{Gal}(K/K_0)$-action. If $C \in U$, we are done. Otherwise, we consider the nonempty proper closed subset $\tilde{U} = V \setminus U$ of V, viewing the coordinate ring of $\tilde{U}$ as a suitable factor ring of that of V, and hence of the polynomial ring $K[X_\nu \mid \nu \in N]$; again, we do not make a notational distinction between the coordinate functions of $\tilde{U}$ and the variables X_ν. Since $\dim \tilde{U}$ is strictly smaller than $v = \dim \mathcal{C}$, the coordinate functions $X_{\nu_1}, \ldots, X_{\nu_v}$ of $\tilde{U}$, where $\{\nu_1, \ldots, \nu_v\}$ are as in the construction of C, are algebraically dependent over K. Let F be an intermediate field of the extension $K_0 \subseteq K$ which has finite transcendence degree over K_0 such that $\tilde{U}$ is defined over F and the coordinate functions $X_{\nu_1}, \ldots, X_{\nu_v}$ of $\tilde{U}$ are algebraically dependent over F. Next, let $(d_{\nu_1}, \ldots, d_{\nu_v})$ be any family of elements in K which are algebraically independent over F and thus, a fortiori, over K_0. Then there exists a point D in V whose ν_j-th coordinate equals d_{ν_j} for $1 \leq j \leq v$, together with a K_0-automorphism ψ of K which sends d_ν to c_ν for all $\nu \in N$; such a point is obtained as in the construction of C. This means $D_\psi = C$, showing that C is $\mathrm{Gal}(K/K_0)$-equivalent to D. On the other hand, in view of the algebraic dependence over F of the coordinate functions $(X_{\nu_1}, \ldots, X_{\nu_v})$ of $\tilde{U}$, the point D does not belong to $\tilde{U}$. Thus $D \in U$, whence $C \in U$, due to $\mathrm{Gal}(K/K_0)$-stability of U.

The extended existence assertion of the lemma concerning generic Γ-orbits of finite direct products of the $\mathcal{C}_k$ is proved analogously, given that B_k is algebraically independent over the composite of $K_1, \ldots, K_m$, whenever $k > m$.

To verify uniqueness of $\Gamma.C$, suppose that $\Gamma.C'$ is another generic orbit. We let V,

$v = \dim V$, and $X_{\nu_1}, \ldots, X_{\nu_v}$ with $\nu_1, \ldots, \nu_v \in N$ be as in the construction of C. In view of its generic status, the orbit $\Gamma.C'$ nontrivially intersects V, since V is a $\mathrm{Gal}(K/K_0)$-closed dense open subset of $\mathcal{C}$. It is therefore harmless to assume that $C' \in V$. In a first step, we show that the orbit $\Gamma.C'$ contains a point $D = (d_\nu)_{\nu \in N} \in V$ with the property that $d_{\nu_1}, \ldots, d_{\nu_v}$ are algebraically independent over K_0.

Consider the intersection $U = (\Gamma.C') \cap V$, a dense open subset of the irreducible variety $\Gamma.C'$ (keep in mind that Γ is connected), let u be its dimension, and $\overline{U}$ the closure of U in V. Moreover, let $\mathfrak{Q}(U) = \mathfrak{Q}(\overline{U})$ be the function field of U, the latter again expressed as the field of fractions of a suitable factor ring of the coordinate ring of the closed subvariety $V \subseteq \mathbb{A}^N = \mathbb{A}(K)^N$. Next, we choose an algebraically closed intermediate field F of the extension $K_0 \subseteq K$ which has finite transcendence degree over K_0, such that $\overline{U}$ is defined over F, and denote by $\mathfrak{Q}(U_F)$ the function field of the restricted variety $U_F = U \cap \mathbb{A}(F)^N$. The coordinate functions in each of these fields will be labeled X_ν, $\nu \in N$, but will be further identified by the function field in which they live. For any point $A = (a_\nu) \in V$, we finally set $K_0(A) = K_0(a_{\nu_j} \mid j \leq v)$ and $F(A) = F(a_{\nu_j} \mid j \leq v)$, both subfields of K. By the choice of $\nu_1, \ldots, \nu_v$ in the existence proof, the subfield $K_0(X_\nu \mid \nu \in N)$ of $\mathfrak{Q}(V)$ is algebraic over the subfield $K_0(X_{\nu_1}, \ldots, X_{\nu_v})$ of this function field, and a fortiori, the subfield $F(X_\nu \mid \nu \in N)$ of $\mathfrak{Q}(U)$ is algebraic over the subfield $F(X_{\nu_1}, \ldots, X_{\nu_v})$ of the latter function field. Consequently, we may choose a transcendence base of $\mathfrak{Q}(U)$ over F among the coordinate functions $X_{\nu_1}, \ldots, X_{\nu_v}$. Without loss of generality, we may assume that $X_{\nu_1}, \ldots, X_{\nu_u} \in \mathfrak{Q}(U)$ constitute such a transcendence base. As we shift to the restricted setting of U_F, the coordinate functions $X_{\nu_1}, \ldots, X_{\nu_u}$ still form a transcendence base for $\mathfrak{Q}(U_F) = \mathfrak{Q}(\overline{U}_F)$ over F. Letting $d_{\nu_1}, \ldots, d_{\nu_u}$ in K be any sequence of scalars algebraically independent over F, we pick a point $D \in U$, whose coordinates labeled by $\nu_1, \ldots, \nu_u$ coincide with the given scalars $d_{\nu_1}, \ldots, d_{\nu_u}$. Say $D = (d_\nu)_{\nu \in N}$ in the coordinatization of U and V. Then the subfield $F(D)$ of K is F-isomorphic to $\mathfrak{Q}(U_F)$, via an isomorphism which maps the class of X_ν to d_ν for all $\nu \in N$. Indeed, there is an F-algebra homomorphism from the coordinate ring of $\overline{U}_F$ to $F[d_\nu \mid \nu \in N]$ sending the coordinate function X_ν of $\overline{U}_F$ to d_ν, since D satisfies the defining equations of $\overline{U}_F$. This map is, in fact, an F-algebra isomorphism (compare transcendence degrees), and therefore extends to the desired field isomorphism. In particular, it restricts to the identity on K_0.

Let w be the transcendence degree of $K_0(D)$ over K_0. Then $u \leq w \leq v$, and it is clearly harmless to assume that $d_{\nu_1}, \ldots, d_{\nu_w}$ form a transcendence base for $K_0(D)$ over K_0. If $w = v$, our intermediate claim is proved. So let us assume $w < v$. By construction, this implies that the coordinate function $X_{\nu_v} \in \mathfrak{Q}(U_F)$ is algebraically dependent over the subfield $K_0(X_{\nu_1}, \ldots, X_{\nu_w})$ of $\mathfrak{Q}(U_F)$, and any pertinent algebraic dependence relation yields such a relation in the field $\mathfrak{Q}(U)$. In other words, the coordinate function X_{ν_v} of U is algebraic over the subfield $K_0(X_{\nu_1}, \ldots, X_{\nu_w})$ of $\mathfrak{Q}(U)$. We thus obtain a nontrivial polynomial g in the polynomial ring $K_0[X_{\nu_i} \mid i \in \{1, \ldots, w\} \cup \{v\}]$ which is satisfied by all points in U. On the other hand, the point $C \in V$ which we constructed in the existence proof is not a root of g. Consequently, the set of nonroots of g in V is dense. It is clearly $\mathrm{Gal}(K/K_0)$-stable, due to the invariance of g under K_0-automorphisms of K. But, by construction, $\Gamma.C'$ has no point in common with this set, a contradiction to the generic

status of $\Gamma.C'$. Thus $w = v$.

In the following, let $D \in U$ be such that $d_{\nu_1}, \ldots, d_{\nu_v}$ are algebraically independent over K_0. Thus, we obtain a field isomorphism

$$\tau : L = K_0(C) \to R = K_0(D)$$

fixing the elements of K_0 and sending c_{ν_j} to d_{ν_j} for $1 \le j \le v$. We let $(Y_\nu)_{\nu \in N \setminus \{\nu_1, \ldots, \nu_v\}}$ be a family of independent variables over K, abbreviated to Y, and also denote by τ the induced isomorphism $L[Y] \to R[Y]$ on the level of polynomial rings. Let μ be any index in $N \setminus \{\nu_1, \ldots, \nu_v\}$ and $f \in L[Y]$ the minimal polynomial of c_μ over L. By construction of C, replacement of each c_{ν_j} by X_{ν_j} for $j \le v$ in the coefficients of f yields the minimal polynomial of X_μ over $K_0(X_j \mid 1 \le j \le v)$ in the function field of V. Therefore, the coordinate d_μ of the point $D \in V$ is a root of the irreducible polynomial $\tau(f)$, which shows $\tau(f)$ to be the minimal polynomial of d_μ over R. Thus τ extends to a field isomorphism $L(c_\mu) \to R(d_\mu)$, which sends c_μ to d_μ. An obvious induction provides us with an extension to an isomorphism $K_0(c_\nu \mid \nu \in N) \to K_0(d_\nu \mid \nu \in N)$ sending c_ν to d_ν for all $\nu \in N$. Algebraic closedness of K now guarantees a further extension to a K_0-automorphism of K, which we still label τ. We thus obtain $D = C_\tau$, whence $\Gamma.D = (\Gamma.C)_\tau$ as required.

The uniqueness argument for the expanded claim requires no adjustments, since it rests on a restriction of the focus to a finite subset of $\mathbb{N}$. □

Theorem 4.3 is readily deduced from Lemma 4.4 and its proof.

Proof of Theorem 4.3. We take $\Gamma = \mathrm{Aut}_\Lambda(P)$. Given an irreducible component $\mathcal{C}$ of $\mathfrak{Grass}(\mathbb{S})$, together with the affine open cover (V_σ) consisting of all nonempty intersections $\mathcal{C} \cap \mathfrak{Grass}(\sigma)$, the hypotheses of Lemma 4.4 are satisfied. Indeed, whenever $\mathcal{C} \cap \mathfrak{Grass}(\sigma) \ne \varnothing$, this intersection is dense open in $\mathcal{C}$ and closed in V_σ; so, in particular, it is again affine. It is defined over K_0 in the standard coordinatization of $\mathfrak{Grass}(\sigma)$, since both $\mathcal{C}$ and $\mathfrak{Grass}(\sigma)$ are. If $\mathrm{Aut}_\Lambda(P).D$ is a generic orbit as guaranteed by Lemma 4.4 – say $D \in V_\sigma$ – the module P/D is $\mathbb{S}$-generic for $\mathcal{C}$ by definition, and the proof of the lemma yields a point $C = (c_\nu)_{\nu \in N} \in \mathrm{Aut}_\Lambda(P).D$ such that the transcendence degree of $K_0(c_\nu \mid \nu \in N)$ over K_0 equals the dimension of $\mathcal{C}$. This provides us with a presentation P/C of the generic module P/D as postulated in Theorem 4.3. The remaining claims are now straightforward in light of Lemma 4.4. □

The $\mathbb{S}$-generic modules can be constructed from Q and I, provided we are handed a sufficiently large subset of K which is algebraically independent over K_0: As described in Observation 3.6, the irreducible components of $\mathfrak{Grass}(\mathbb{S})$ are determined by those of the covering $\mathfrak{Grass}(\sigma)$, and the latter are available algorithmically in terms of generators (defined over K_0) for their prime ideals. $\mathcal{C}$ is an irreducible component of some $\mathfrak{Grass}(\mathbb{S})$ and σ a skeleton with $\mathcal{C} \cap \mathfrak{Grass}(\sigma) \ne \varnothing$. Then a projective presentation of the generic module G for $\mathcal{C}$ is available via the coordinate ring of the irreducible affine variety $\mathcal{C} \cap \mathfrak{Grass}(\sigma)$. Of course, a detailed analysis of representation-theoretic features of G hinges on the understanding of that coordiante ring.

Let $\mathcal{C}$ be an irreducible component of $\mathfrak{Grass}(\mathbb{S})$, and C a point in $\mathcal{C}$, say $C = (c_\nu)_{\nu \in N} \in \mathcal{C} \cap \mathfrak{Grass}(\sigma)$, where N again stands for the set of all pairs $(\alpha p\widehat{z_r}, q\widehat{z_s})$ with a σ-critical

path $\alpha p \widehat{z}_r$ in the first slot and a path $q \widehat{z}_s \in \sigma(\alpha p \widehat{z}_r)$ in the second. If $K_0(c_\nu \mid \nu \in N)$ has transcendence degree $\dim \mathcal{C}$ over K_0, then any module $G \cong P/C$ is $\mathbb{S}$-generic for $\mathcal{C}$ by Theorem 4.3. In this situation, we call P/C a *generic presentation* of G. Conversely, Theorem 4.3 gurantees that every generic module for $\mathcal{C}$ has a generic presentation. For a simple example, consider $\Lambda = KQ$, where Q is $1 \to 2$, and $\mathbb{S} = (S_1^2, S_2)$; the only arrow of the quiver is labeled γ. Given a sequence of top elements for the projective cover $P = (\Lambda e_1)^2$ of the top S_1^2, say $\{z_1 = e_1 z_1, z_2 = e_1 z_2\}$, the only module represented by $\mathfrak{Grass}(\mathbb{S})$, up to isomorphism, is $M = \Lambda z_1 \oplus (\Lambda z_2 / J z_2)$. So M is obviously $\mathbb{S}$-generic. However, the given presentation is not generic, since the orbit dimension of M is $\dim \mathfrak{Grass}(\mathbb{S}) = 1$, while for either of the two possible skeleta σ of M (one is $\{\widehat{z}_1 = e_1 \widehat{z}_1, \gamma \widehat{z}_1, \widehat{z}_2 = e_1 \widehat{z}_2\}$, the other is symmetric), the σ-coordinates of the submodule we factored out of $\Lambda z_1 \oplus \Lambda z_2$ to obtain M belong to the subset $\{0, 1\}$ of K. A generic presentation is, for instance, $M \cong (\Lambda z_1 \oplus \Lambda z_2)/\Lambda(\gamma z_1 + k \gamma z_2)$, where $k \in K$ is transcendental over K_0.

We glean that the set of all points $C \in \mathcal{C}$, which give rise to generic presentations (of generic modules for $\mathcal{C}$) is neither open nor $\mathrm{Aut}_\Lambda(P)$-stable in general. On the other hand, as is readily seen, it *is* dense in $\mathcal{C}$. We record the positive observations for future reference.

Corollary 4.5. *Retain the hypotheses and notation of Theorem 4.3, and suppose G is $\mathbb{S}$-generic for $\mathcal{C}$. Then G has a generic presentation $G \cong P/C$, and the $\mathrm{Gal}(K/K_0)$-orbit of C is dense in $\mathcal{C}$. In other words, any nonempty open subset U of $\mathcal{C}$ contains a point D such that P/D is a generic presentation of a generic module for $\mathcal{C}$.* $\square$

In light of Sections 2 and 3, the list of all $\mathbb{S}$-generic modules (where $\mathbb{S}$ traces the d-dimensional semisimple sequences) includes the (T, d)-generic modules, and those, in turn, include the d-generic ones. In the next corollary we therefore need not separate the cases where $\mathcal{C}$ is an irreducible component of $\mathfrak{Grass}(\mathbb{S})$, $\mathfrak{Grass}_d^T$, or $\mathbf{Mod}_d(\Lambda)$.

Corollary 4.6. *Suppose that the module G is generic for some irreducible component $\mathcal{C}$ of $\mathfrak{Grass}(\mathbb{S})$, or $\mathfrak{Grass}_d^T$, or $\mathbf{Mod}_d(\Lambda)$.*

Then the syzygies of G are generic for the syzygies of the modules in $\mathcal{C}$ in the following weakened sense: If $\mathfrak{U} \subseteq \mathcal{C}$ is the subset consisting of the points D such that $\Omega^k(P/D)$ is $\mathrm{Gal}(K/K_0)$-equivalent to $\Omega^k(G)$ for all $k \geq 0$, then $\mathfrak{U}$ is dense in $\mathcal{C}$ (however, $\mathfrak{U}$ need not contain a noempty open subset of $\mathcal{C}$).

Suppose, in addition, that G is indecomposable, and let $\mathcal{A}(G)$ be that connected component of the Auslander-Reiten quiver of Λ-mod which contains G. Then $\mathcal{A}(G)$ is generic for $\mathcal{C}$ in the same sense: Namely, there exists a dense subset $\mathfrak{V}$ of $\mathcal{C}$ such that, for all $D \in \mathfrak{V}$, the module P/D is indecomposable and the connected component $\mathcal{A}(P/D)$ is $\mathrm{Gal}(K/K_0)$-equivalent to $\mathcal{A}(G)$ in the sense of Definition 4.1.

Proof. It suffices to apply Corollary 4.5, to obtain a dense subset $W \subseteq \mathcal{C}$ such that P/D is $\mathrm{Gal}(K/K_0)$-equivalent to G for all $D \in W$. This means that there exists a map $\phi \in \mathrm{Gal}(K/K_0)$ such that $P/D \cong F_\phi(G)$ (see Observation 4.1 for notation). Since F_ϕ induces a category self-equivalence of Λ-mod, both claims follow. $\square$

All of the generic properties to which we will apply Theorem 4.3 are actually stable under arbitrary autmorphisms of the field K, so that the choice of K_0 becomes irrelevant

for practical purposes. If the subfield $K_0 \subseteq K$ is chosen larger than necessary, this simply excludes certain elements in K from being eligible as scalars in generic presentations. In the second installment of consequences to Theorem 4.3 and Lemma 4.4, the direct sum of two semisimple sequences $\mathbb{S}$ and $\mathbb{S}'$ is defined in the obvious way, as resulting from componentwise summation.

Part (a.3) of the next corollary is known; see [**6**]. While most of the assertions follow from the statement of Theorem 4.3, the final assertion of part (d) is a consequence of its proof.

Corollary 4.7. *Let $\mathcal{C}$ be an irreducible component of $\mathfrak{Grass}(\mathbb{S})$ or $\mathfrak{Grass}_d^T$ or $\mathbf{Mod}_d(\Lambda)$, and let $\mathcal{C}'$ be an irreducible component of any other parametrizing variety "on the same level" (referring to the radical layering, or top combined with dimension, or else to dimension without further specification). Moreover, let G be generic for $\mathcal{C}$, and G' generic for $\mathcal{C}'$. Then:*

(a) *The number of indecomposable summands of G is the generic number for $\mathcal{C}$, and the dimension vectors of these direct summands are the generic ones. In fact, the collection of radical layerings of the indecomposable direct summands of G is the generic one for the indecomposable decompositions of the modules in $\mathcal{C}$.*

Moreover, given an indecomposable decomposition $G = M_1 \oplus \cdots \oplus M_s$, with $T_i = M_i/JM_i$ and $\dim M_i = d_i$, the following additional statements hold:

(a.1) *If G is $\mathbb{S}$-generic, the M_i are $\mathbb{S}(M_i)$-generic for suitable irreducible components of the $\mathfrak{Grass}\,\mathbb{S}(M_i)$.*

(a.2) *If G is (T,d)-generic, the M_i are (T_i, d_i)-generic for suitable irreducible components of the $\mathfrak{Grass}_{d_i}^{T_i}$.*

(a.3) *If G is d-generic, the M_i are d_i-generic for suitable irreducible components of the $\mathbf{Mod}_{d_i}(\Lambda)$.*

(b) *The dimension of the socle of G is the minimum of the socle dimensions of the modules in $\mathcal{C}$.*

(c) $\operatorname{proj\,dim} G = \min\{\operatorname{proj\,dim} M \mid M \in \mathcal{C}\}$ *and* $\operatorname{inj\,dim} G = \min\{\operatorname{inj\,dim} M \mid M \in \mathcal{C}\}$. *Moreover,*
$$\dim \operatorname{End}_\Lambda(G, G) = \min\{\dim \operatorname{End}_\Lambda(M, M) \mid M \in \mathcal{C}\}$$
and
$$\dim \operatorname{Ext}_\Lambda^i(G, G) = \min\{\dim \operatorname{Ext}_\Lambda^i(M, M) \mid M \in \mathcal{C}\},$$
for $i \geq 1$, with an analogous result holding for Tor-*spaces.*

In particular, the dimension of the $(\operatorname{Aut}_\Lambda(P)$- or GL_d-) orbit corresponding to G is maximal among the orbit dimensions in $\mathcal{C}$.

(d) *If we choose G and G' with the relative generic properties spelled out in* Supplement 2 *to Theorem 4.3, then*
$$\dim \operatorname{Ext}_\Lambda^i(G, G') = \min\{\dim \operatorname{Ext}_\Lambda^i(M, N) \mid M \in \mathcal{C}, \, N \in \mathcal{C}'\},$$

and

$$\dim \operatorname{Tor}_i^\Lambda(G, G') = \min\{\dim \operatorname{Tor}_i^\Lambda(M, N) \mid M \in \mathcal{C},\, N \in \mathcal{C}'\}$$

for $i \geq 0$, whenever $\mathcal{C} \neq \mathcal{C}'$.

If $\mathcal{C} = \mathcal{C}'$, then the analogous result holds, provided that $G' = G_\phi$, where $G = P/C$ with $C = (c_\nu)_{\nu \in N}$, and the join of the fields $K_0(c_\nu \mid \nu \in N)$ and $K_0(\phi(c_\nu) \mid \nu \in N)$ has transcendence degree $2 \cdot \dim \mathcal{C}$ over K_0.

(e) *The set of skeleta of G equals the generic set of skeleta of the modules represented by $\mathcal{C}$. If $\mathcal{C} \subseteq \mathfrak{Grass}(\mathbb{S})$, this set consists of all σ with $\mathcal{C} \cap \mathfrak{Grass}(\sigma) \neq \varnothing$.*

Proof. (a) In [**15**], Kac showed that every irreducible component $\mathcal{C}$ of $\mathbf{Mod}_d(\Lambda)$ contains a nonempty open subset V such that all modules $\tilde{M}$ in V have the same number of indecomposable summands $\tilde{M}_i$ and the dimension vectors of the latter are invariant on V. Combined with the arguments of Section 2, this yields a nonempty open $U \subseteq V$ such that even the radical layerings $\mathbb{S}(\tilde{M}_i)$ of the indecomposable summands $\tilde{M}_i$ of the $\tilde{M}$ in U are constant. Further, we deduce that the same statements hold for irreducible components of $\mathbf{Mod}_d^T$ (resp. $\mathfrak{Grass}_d^T$) and $\mathbf{Mod}(\mathbb{S})$ (resp. $\mathfrak{Grass}(\mathbb{S})$). Clearly, all these quantities are $\operatorname{Gal}(K/K_0)$-stable, whence the first two assertions follow from Theorem 4.3.

For (a.1), we set $\mathbb{S}_i = \mathbb{S}(M_i)$. Moreover, we choose a point $C \in \mathcal{C}$ which represents G, next to an open subset $U \subseteq \mathcal{C}$ as specified above. In particular, each module $\tilde{M}$ in U can be written in the form $\tilde{M} = \tilde{M}_1 \oplus \cdots \oplus \tilde{M}_s$ such that $\tilde{M}_i$ is indecomposable with top T_i and dimension d_i. Without loss of generality, $C = \bigoplus_{1 \leq i \leq s} C_i$ with $C_i \subseteq JP_i$, where $P = \bigoplus_{1 \leq i \leq s} P_i$ is a suitable decomposition of P. We take P_i as the distinguished projective cover of T_i in the realization of $\mathfrak{Grass}_{d_i}^{T_i}$ and consider the morphism Ψ from the projective variety $\prod_{1 \leq i \leq s} \mathfrak{Grass}_{d_i}^{T_i}$ to $\mathfrak{Grass}_d^T$, defined by $\Psi(D_1, \ldots, D_s) = \bigoplus_i D_i$ whenever the D_i are submodules of the JP_i of codimension d_i in P_i, respectively. Clearly, Ψ induces an isomorphism between the direct product and a suitable closed subvariety W of $\mathfrak{Grass}_d^T$, which restricts to an isomorphism between $\prod_{1 \leq i \leq s} \mathfrak{Grass}\, \mathbb{S}_i$ and $W \cap \mathfrak{Grass}(\mathbb{S})$. Since U is an irreducible open subset of $\mathfrak{Grass}(\mathbb{S})$, the intersection $U \cap W$ is irreducible and open in W. Moreover, our choice of U guarantess that every $\operatorname{Aut}_\Lambda(P)$-orbit Z in $\mathfrak{Grass}(\mathbb{S})$ which has nonempty intersection with U intersects nontrivially with $U \cap W$, whence the closure of $Z \cap W$ in $\mathfrak{Grass}(\mathbb{S})$ coincides with the closure of Z. We infer the existence of irreducible components $\mathcal{C}_i$ of $\mathfrak{Grass}\, \mathbb{S}_i$ such that $U \cap W \subseteq \Psi\big(\prod_{1 \leq i \leq s} \mathcal{C}_i\big) \subseteq \mathcal{C}$ and conclude that the closure of $\Psi\big(\prod_{1 \leq i \leq s} \mathcal{C}_i\big)$ in $\mathfrak{Grass}(\mathbb{S})$ equals $\overline{U \cap W} = \overline{U} = \mathcal{C}$. For each i, we now pick an $\mathbb{S}_i$-generic module $G(\mathbb{S}_i, \mathcal{C}_i)$ for $\mathcal{C}_i$, such that these generic objects even enjoy the relative generic properties described in the final statement of Theorem 4.3. By construction, the direct sum

$$\bigoplus_{1 \leq i \leq s} G(\mathbb{S}_i, \mathcal{C}_i)$$

is $\mathbb{S}$-generic for $\mathcal{C}$. In light of the uniqueness part of Theorem 4.3, we deduce that this direct sum is $\operatorname{Gal}(K/K_0)$-equivalent to G, say isomorphic to $G_\tau \cong \bigoplus_{1 \leq i \leq s} (M_i)_\tau$ for some $\tau \in \operatorname{Gal}(K/K_0)$. Using the Krull-Schmidt theorem, we conclude that $G(\mathbb{S}_i, \mathcal{C}_i)$ is isomorphic to $(M_{\pi(i)})_\tau$ for some permutation π of $\{1, \ldots, s\}$, with the property that

$\mathbb{S}\big(M_{\pi(j)}\big) = \mathbb{S}(M_j)$ for all j. This shows the $(M_i)_\tau$, and therefore also the M_i, to be $\mathbb{S}_i$-generic for the components $\mathcal{C}_{\pi(i)}$.

Parts (a.2) and (a.3) are proved similarly.

Part (b), as well as the statements regarding Hom-dimensions under (c) and (d), follow from upper semicontinuity of the function $\dim \mathrm{Hom}(-,-)$ on $\mathcal{C} \times \mathcal{C}'$. For part (b) note, moreover, that $\dim \mathrm{Hom}_\Lambda(S_i, M)$ is the dimension of the S_i-homogeneous component of M, the automorphisms of K leave the S_i invariant, up to isomorphism (cf. Observation 4.1), and $\dim \mathrm{Hom}_\Lambda(N, M) = \dim \mathrm{Hom}_\Lambda\big(N_\phi, M_\phi\big)$ for all $N \in \Lambda\text{-mod}$ and $\phi \in \mathrm{Gal}(K/K_0)$.

For the remaining statements under (c) and (d), we use upper semicontinutiy of the functions $\mathrm{proj\,dim}(-)$ and $\mathrm{inj\,dim}(-)$ to ascertain that the indicated minimal values are the generic ones (see [**17**, Corollary 5.4]), together with the fact that both $\mathrm{proj\,dim}\,M$ and $\mathrm{inj\,dim}\,M$ are $\mathrm{Gal}(K/K_0)$-invariant attributes of a Λ-module M, an analogous equality holding for injective dimensions. Moreover, in light of upper semicontinuity of $\mathrm{Hom}_\Lambda(-,-)$, Propositions 5.3 and 5.6 in [**17**] (attributed to Bongartz), yield upper semicontinuity of $\mathrm{Ext}\,\Lambda^i(-,-)$ and $Tor_i^\Lambda(-,-)$ (see also [CBS, Lemma 4.3] for Ext). Given that application of an automorphism ϕ to both arguments leaves the dimensions of the resulting Ext or Tor-spaces untouched, this completes the proof of (c) and (d).

For (e), it suffices to observe that the collection of skeleta of the modules in $\mathcal{C}$ is generic and again invariant under application of automorphisms $\phi \in \mathrm{Gal}(K/K_0)$, by Observation 4.1. $\square$

We illustrate the theory with a first example.

Example 4.8. Let $\Lambda = KQ/I$, where Q is the quiver.
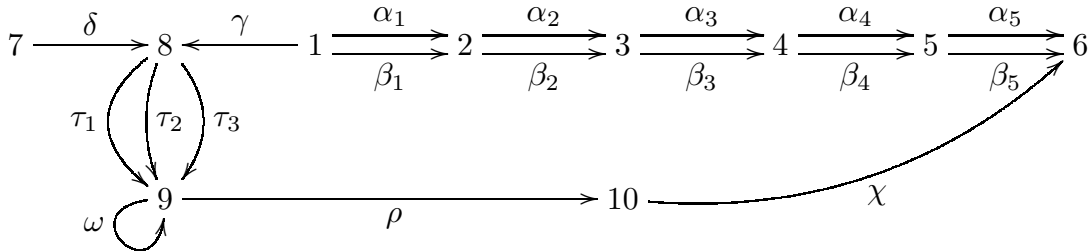
and $I \subseteq KQ$ is the ideal generated by the following relations: $\alpha_5\alpha_4\alpha_3\beta_2\beta_1 - \beta_5\beta_4\beta_3\alpha_2\alpha_1$, $\alpha_5\alpha_4\alpha_3\alpha_2\beta_1 - \alpha_5\alpha_4\alpha_3\beta_2\alpha_1$, $\alpha_5\alpha_4\beta_3\alpha_2\alpha_1 - \alpha_5\beta_4\alpha_3\alpha_2\alpha_1$, $\alpha_5\beta_4\alpha_3\alpha_2\alpha_1 - \beta_5\alpha_4\alpha_3\alpha_2\alpha_1$, $\tau_i\gamma$ for $i = 1, 2, 3$, $\sum_{1\le i<3}\tau_i\delta$, and ω^3. Then $L = 6$, and we may take K_0 to be the algebraic closure of the prime field of K. First choose

$$\mathbb{S} = \big(S_1 \oplus S_7, S_2 \oplus S_8, S_3 \oplus S_9^2, S_4 \oplus S_9, S_5 \oplus S_9, S_6\big).$$

Then $\mathfrak{Grass}(\mathbb{S})$ is irreducible and coincides with $\mathfrak{Grass}(\sigma)$, where σ is the skeleton $\sigma^{(1)} \sqcup \sigma^{(2)}$ such that $\sigma^{(1)}$ is the set of all initial subpaths of $\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1\widehat{z}_1$ and $\sigma^{(2)}$ the set of all initial subpaths of $\delta\tau_1\omega_2\widehat{z}_2$ and $\delta\tau_2\omega\widehat{z}_2$. One computes $\mathfrak{Grass}(\sigma) \cong V(Y^2 - X^3) \times \mathbb{A}^2$. We give a generic presentation of "the" generic module G – we know it to be unique up to $\mathrm{Gal}(K/K_0)$-equivalence – relative to σ, and follow with the graph of G relative to the same skeleton.
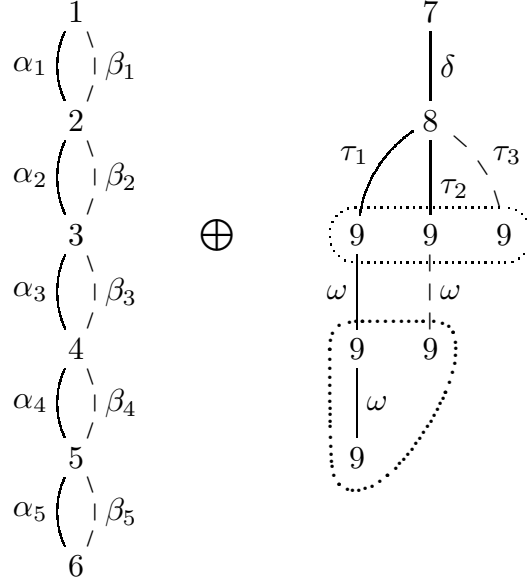
Given scalars $c_1, c_2, c_3 \in K$ which are algebraically independent over K_0 and a projective cover $P = \Lambda z_1 \oplus \Lambda z_2$ of the top $S_1 \oplus S_7$ of $\mathbb{S}$ with $z_1 = e_1 z_1$ and $z_2 = e_7 z_2$, we obtain: $G \cong P/(C_1 \oplus C_2)$, where C_1 is the submodule of Λz_1 generated by

$$\beta_1 z_1 - (\sqrt{c_1})^3 \alpha_1 z_1, \ \beta_2 \alpha_1 z_1 - (\sqrt{c_1})^3 \alpha_2 \alpha_1 z_1, \ \beta_3 \alpha_2 \alpha_1 z_1 - c_1 \alpha_3 \alpha_2 \alpha_1 z_1,$$

$$\beta_4 \alpha_3 \alpha_2 \alpha_1 z_1 - c_1 \alpha_4 \alpha_3 \alpha_2 \alpha_1 z_1, \ \ \beta_5 \alpha_4 \alpha_3 \alpha_2 \alpha_1 z_1 - c_1 \alpha_5 \alpha_4 \alpha_3 \alpha_2 \alpha_1 z_1;$$

here $\sqrt{c_1}$ is a square root of c_1 in K; the submodule C_2 of Λz_2 is generated by $\sum_{1 \leq i \leq 3} \tau_i \delta z_2$ and $\omega \tau_2 z_2 - c_2 \omega \tau_1 \delta z_2 - c_3 \omega^2 \tau_1 \delta z_2$. The corresponding hypergraph is displayed at the end of the discussion.

Clearly, the generic module G is decomposable into two local direct summands $\Lambda z_1/C_1$ and $\Lambda z_2/C_2$, these being $\mathbb{S}(\Lambda z_1/C_1)$- and $\mathbb{S}(\Lambda z_2/C_2)$-generic, respectively, by Corollary 4.7. In fact, this example shows that not even for the generic module does every σ-critical path $\alpha p \widehat{z}_r$ with $\sigma(\alpha p \widehat{z}_r) \neq \varnothing$ lead to a hyperedge in the corresponding hypergraph. For instance, the σ-critical path $\gamma(e_1 \widehat{z}_1)$ leads to $\sigma(\gamma \widehat{z}_1) = \{\delta \widehat{z}_7\}$, but the corresponding coefficient disappears in every module with skeleton σ, due to the relations $\tau_i \gamma = 0$ in Λ; hence, these latter relations are responsible for the decomposability of G. The socle of G is $S_6 \oplus S_9^2$, whence $S_6 \oplus S_9^2$ is contained in the socles of all modules in $\mathfrak{Grass}(\mathbb{S})$. Moreover, generically, the modules with radical layering $\mathbb{S}$ have endomorphism rings of dimension at least 2. Further, we observe that G is also (T, d)-generic, where $T = S_1 \oplus S_7$ and $d = 12$; in fact, G is even 12-generic. While $\mathfrak{Grass}_d^T$ is still irreducible, making G the only (T, d)-generic module, up to equivalence – it obviously has class 0 in the terminology of Observation 2.1 – the full variety $\mathbf{Mod}_d(\Lambda)$ has many additional irreducible components; the one represented by G has class 1. Note that the set of skeleta of $\Lambda z_1/C_1$ has cardinality 2^5, while $\Lambda z_2/C_2$ has 6 different skeleta. The singularities of $\mathfrak{Grass}(\sigma)$, here identified with $V(Y^2 - X^3) \times \mathbb{A}^2$, are precisely the points that parametrize the modules with fewer than $2^5 \cdot 6$ skeleta; in fact, each of the latter modules has precisely 6 skeleta. Moreover, by Theorem 4.3, the syzygies $\Omega^i(G)$ are the generic syzygies for the irreducible component $\mathcal{C}$ of $\mathbf{Mod}_d(\Lambda)$ obtained by closing $\mathbf{Mod}(\mathbb{S})$ in $\mathbf{Mod}_d(\Lambda)$; they are obtainable combinatorially from the graph of G, showing that $\operatorname{proj} \dim G = \infty$. By Corollary 4.7, we conclude further that all modules in $\mathcal{C}$ have infinite projective dimension.

$$
\begin{array}{ccc}
1 & & 7 \\
\alpha_1\ \Big(\ \big|\ \beta_1 & & \big|\ \delta \\
2 & & 8 \\
\alpha_2\ \Big(\ \big|\ \beta_2 & & \tau_1\quad \tau_2\quad \tau_3 \\
3 & \oplus & 9\quad 9\quad 9 \\
\alpha_3\ \Big(\ \big|\ \beta_3 & & \omega\quad \big|\ \omega \\
4 & & 9\quad 9 \\
\alpha_4\ \Big(\ \big|\ \beta_4 & & \big|\ \omega \\
5 & & 9 \\
\alpha_5\ \Big(\ \big|\ \beta_5 & & \\
6 & &
\end{array}
$$

Finally, we remark: If we replace $\mathbb{S}$ by the sequence $\mathbb{S}'$, which differs from $\mathbb{S}$ only in the penultimate slot, taking $\mathbb{S}'_4 = \mathbb{S}_5 \oplus S_{10}$ versus $\mathbb{S}_4 = S_5 \oplus S_9$, then the $\mathbb{S}'$-generic module is indecomposable and has socle $S_6 \oplus S_9$. $\square$

5. Irreducible components and generic modules over truncated path algebras

From now on, we assume that Λ is a truncated path algebra, i.e., $\Lambda = KQ/I$, where $I = I(L) \subseteq KQ$ is the ideal generated by all paths of length $L + 1$ for some L. We will keep L fixed in our discussion.

As we will see in Theorem 5.3 below, in this situation, the $\mathfrak{Grass}(\mathbb{S})$ are irreducible smooth rational varieties. To describe their structure in greater detail, the fact that Λ is graded by path lengths will be pivotal. In fact, it will be advantageous to simultaneoulsy explore, from the start, varieties of graded modules with radical layering $\mathbb{S}$, next to the full class of modules parametrized by $\mathbb{S}$. We recall from [1] the tools required for studying the homogeneous points in $\mathfrak{Grass}(\mathbb{S})$.

As before, let $\mathbb{S}$ be a d-dimensional semisimple sequence with top $T = \mathbb{S}_0$ and P a projective cover of T, endowed with its distinguished sequence $z_1, \ldots, z_{\mathfrak{t}}$ of top elements. Moreover, let $P = \bigoplus_{0 \leq l \leq L} P_l = \bigoplus_{1 \leq r \leq \mathfrak{t}} \left(\bigoplus_{0 \leq l \leq L} \Lambda_l z_r \right)$ be the natural grading of P, where Λ_l denotes the K-subspace of Λ generated by the paths of length l and $P_l = \bigoplus_{1 \leq r \leq \mathfrak{t}} \Lambda_l z_r$. The *homogeneous points* of $\mathfrak{Grass}_d^T$ are those of the form $C = \bigoplus_{1 \leq l \leq L} C_l$, where $C_l = C \cap P_l$. For proofs of the following observations, see [1].

Definitions and Observations 5.1.

(1) The set Gr-$\mathfrak{Grass}_d^T$ of all homogeneous points in $\mathfrak{Grass}_d^T$ is a closed subvariety. In particular, Gr-$\mathfrak{Grass}_d^T$ is in turn projective.

(2) The set Gr-$\mathfrak{Grass}(\mathbb{S})$ of all homogeneous points in $\mathfrak{Grass}(\mathbb{S})$ is a *projective* subvariety of $\mathfrak{Grass}(\mathbb{S})$; in particular, Gr-$\mathfrak{Grass}(\mathbb{S})$ is closed in $\mathfrak{Grass}(\mathbb{S})$. (Note that, by contrast, $\mathfrak{Grass}(\mathbb{S})$ fails to be projective, in general.)

(3) For each abstract skeleton σ compatible with $\mathbb{S}$, the set Gr-$\mathfrak{Grass}(\sigma)$ of all homogeneous points in $\mathfrak{Grass}(\sigma)$ is an affine open subvariety of Gr-$\mathfrak{Grass}(\mathbb{S})$, and the family of these subvarieties, as σ traces all eligible skeleta, covers Gr-$\mathfrak{Grass}(\mathbb{S})$.

(4) The subgroup Gr-$\mathrm{Aut}_\Lambda(P)$ of $\mathrm{Aut}_\Lambda(P)$, consisting of the homogeneous automorphisms of P, acts morphically on Gr-$\mathfrak{Grass}_d^T$ and Gr-$\mathfrak{Grass}(\mathbb{S})$. The orbits of Gr-$\mathfrak{Grass}_d^T$ under this action are in one-to-one correspondence with the graded-isomorphism classes of those graded d-dimensional modules with top T which are generated in degree zero; analogously, the Gr-$\mathrm{Aut}_\Lambda(P)$-orbits of Gr-$\mathfrak{Grass}(\mathbb{S})$ are in one-to-one correspondence with the graded-isomorphism classes of those graded modules which have radical layering $\mathbb{S}$ and are generated in degree zero.

For truncated path algebras Λ, it is exceptionally easy to recognize the semisimple sequences $\mathbb{S}$ for which $\mathfrak{Grass}(\mathbb{S}) \neq \varnothing$ or – as it turns out, equivalently – Gr-$\mathfrak{Grass}(\mathbb{S}) \neq \varnothing$. Indeed, the final two of the equivalent conditions below are immediately checkable via the (tree) graphs of the indecomposable projective modules Λe_i, and these graphs can in turn be simply read off the quiver.

Observation 5.2. *Suppose $\Lambda = KQ/I$ is a truncated path algebra, and $\mathbb{S} = (\mathbb{S}_0, \ldots, \mathbb{S}_L)$ a semisimple sequence. Then the following conditions are equivalent:*

(a) *$\mathfrak{Grass}(\mathbb{S})$ is nonempty.*
(b) *Gr-$\mathfrak{Grass}(\mathbb{S})$ is nonempty.*
(c) *There exists an abstract skeleton compatible with $\mathbb{S}$.*
(d) *For each positive integer $l < L$, the following holds:*

$$\sum_{1 \leq i \leq n} \bigl(\text{the number of arrows in } Q \text{ from } e_i \text{ to } e_j\bigr) \cdot \dim e_i \mathbb{S}_l \quad \geq \quad \dim e_j \mathbb{S}_{l+1},$$

for all $j = 1, \ldots, n$.

Moreover, any abstract skeleton σ consisting of paths of length $\leq L$ in Q is the skeleton of a (graded) Λ-module. In other words, Gr-$\mathfrak{Grass}(\sigma) \neq \varnothing$, and a fortiori, $\mathfrak{Grass}(\sigma) \neq \varnothing$, for every abstract skeleton σ. $\square$

Condition (d) is always necessary for $\mathfrak{Grass}(\mathbb{S})$ to be nonempty, but need not be sufficient in the non-truncated case. For example, consider the algebra KQ/I, based on the quiver
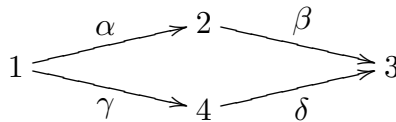
where I is generated by $\beta\alpha - \delta\gamma$. Then the sequence (S_1, S_2, S_3) satisfies the two final conditions under 5.2, but fails to arise as the sequence of radical layers of a Λ-module.

As an obvious consequence of the following theorem, we obtain that the set of skeleta of any $\mathbb{S}$-generic module over a truncated path algebra is the full set of skeleta compatible with $\mathbb{S}$.

For the definition of $N(\mathbb{S})$ and $N_0(\mathbb{S})$, see Observation 3.10.

Theorem 5.3. (*Irreducibility of* $\mathfrak{Grass}(\mathbb{S})$ *and* Gr-$\mathfrak{Grass}(\mathbb{S})$, *and first structure results.*) *Suppose that Λ is a truncated path algebra and $\mathbb{S}$ any semisimple sequence such that $\mathfrak{Grass}(\mathbb{S}) \neq \varnothing$.*

Then both $\mathfrak{Grass}(\mathbb{S})$ and Gr-$\mathfrak{Grass}(\mathbb{S})$ *are irreducible varieties endowed with finite open covers consisting of copies of an affine space, namely, of $\mathbb{A}^{N(\mathbb{S})}$ in case of $\mathfrak{Grass}(\mathbb{S})$, and $\mathbb{A}^{N_0(\mathbb{S})}$ in case of* Gr-$\mathfrak{Grass}(\mathbb{S})$. *In particular, both $\mathfrak{Grass}(\mathbb{S})$ and* Gr-$\mathfrak{Grass}(\mathbb{S})$ *are rational and smooth.*

More precisely: If σ is any skeleton compatible with $\mathbb{S}$, then $\mathfrak{Grass}(\sigma) \cong \mathbb{A}^{N(\mathbb{S})}$ is dense and open in $\mathfrak{Grass}(\mathbb{S})$, and Gr-$\mathfrak{Grass}(\sigma) \cong \mathbb{A}^{N_0(\mathbb{S})}$ *is dense and open in* Gr-$\mathfrak{Grass}(\mathbb{S})$.

Proof. We only address the claims for ungraded modules. The proofs of the twin statements for Gr-$\mathfrak{Grass}(\mathbb{S})$ are proved analogously. However, we will use some observations about the graded case to deal with the ungraded one.

As above, we let $\Lambda = \bigoplus_{0 \le l \le L} \Lambda_l$ and $P = \bigoplus_{l \le L} P_l$ be the decompositions of Λ and P into homogeneous subspaces. Note that, whenever σ is an abstract skeleton and $C \in$ Gr-$\mathfrak{Grass}_d^T$, the requirement that σ be a skeleton of P/C is equivalent to the formally weaker condition that σ yields a basis for P/C.

Let σ be compatible with $\mathbb{S}$. Whenever we do not explicitly insist on identification of σ with a subset of P, we go back to the original definition of skeleta in Definition 3.1, viewing the elements of σ as paths in the KQ-module $\widehat{P} = \bigoplus_{1 \le r \le \mathfrak{t}} (KQ)\widehat{z}_r$. By $\sigma^{(r)}$ we denote the set of paths in σ that start in $\widehat{z}_r$, and by $\sigma_{li}^{(r)}$ the subset of paths of length l in $\sigma^{(r)}$ which end in the vertex e_i.

That $\mathfrak{Grass}(\sigma) \cong \mathbb{A}^{N(\mathbb{S})}$ is obvious. Since the affine patches corresponding to the skeleta form an open cover of $\mathfrak{Grass}(\mathbb{S})$, it thus suffices to show that, given any other skeleton compatible with $\mathbb{S}$ – call it $\tilde{\sigma}$ – the intersection $\mathfrak{Grass}(\sigma) \cap \mathfrak{Grass}(\tilde{\sigma})$ is non-empty; indeed, $\mathfrak{Grass}(\sigma) \cap \mathfrak{Grass}(\tilde{\sigma})$ is then a dense open subset of both $\mathfrak{Grass}(\sigma)$ and $\mathfrak{Grass}(\tilde{\sigma})$. Clearly, $\sigma \cap \tilde{\sigma}$ is the disjoint union of the intersections $\sigma_{li}^{(r)} \cap \tilde{\sigma}_{li}^{(r)}$; the union $\sigma \cup \tilde{\sigma}$ breaks up similarly.

By induction on $L \ge 0$, we will construct a point $C \in$ Gr-$\mathfrak{Grass}(\mathbb{S})$, that is, a homogeneous submodule $\bigoplus_{l \le L} C_l$ of P, together with a bijection $f : \tilde{\sigma} \to \sigma$ whose restriction to $\sigma \cap \tilde{\sigma}$ equals the identity on $\sigma \cap \tilde{\sigma}$, such that the following additional conditions are satisfied:

- $C \in \mathfrak{Grass}(\sigma) \cap \mathfrak{Grass}(\tilde{\sigma})$;
- whenever $\tilde{p} \in \tilde{\sigma}_{li}^{(r)}$, the path $f(\tilde{p})$ belongs to $\sigma_{li}^{(s)}$ for some s, and the difference $\tilde{p} - f(\tilde{p})$, now viewed as an element of P, belongs to the homogenous subspace $C_{\text{length}(\tilde{p})}$ of C.
- $pz_r = 0$ in P, for any path $p\widehat{z}_r \in \widehat{P}$ which does not belong to $\sigma \cup \tilde{\sigma}$.

Note that, by the first condition, the families of residue classes $(\tilde{p} + C)$ and $(f(\tilde{p}) + C)$ in P/C form bases for P/C; here we again identify σ and $\tilde{\sigma}$ with subsets of P under the

conventions of Definition 3.1. In particular, none of the elements $\tilde{p}$ or $f(\tilde{p})$ of P belongs to C. Moreover, the second requirement forces f to induce a bijection $\tilde{\sigma}_{li} \to \sigma_{li}$, for each $0 \leq l \leq L$ and $1 \leq i \leq n$.

The proposed construction is trivial for $L = 0$. So, given L, we assume that our requirements can be met in any setup of the described ilk whenever $L - 1$ is an upper bound on the lengths of nonvanishing paths. Suppose $L' = L - 1$. Let Λ' be the vector space $\bigoplus_{0 \leq l \leq L'} \Lambda_l$ identified with the algebra Λ/J^L, and let $P' = \bigoplus_{0 \leq l \leq L'} P_l$ be the corresponding truncation of P, clearly a projective Λ'-module. (Up to isomorphism, P' is the same as $P/J^L P$ as a Λ'-module; but as a vector space, P' is contained in P by definition, so that, for a subset $U \subseteq P'$, it makes sense to talk about the Λ-submodule ΛU of P generated by U.) Moreover, we set $\mathbb{S}' = (\mathbb{S}_0, \ldots, \mathbb{S}_{L'})$, and let σ', $\tilde{\sigma}'$ be the skeleta resulting from σ and $\tilde{\sigma}$ through deletion of the paths of length L. Observe that σ' and $\tilde{\sigma}'$ are both compatible with $\mathbb{S}'$. Therefore, our induction hypothesis yields a homogeneous submodule $C' \subseteq P'$ and a bijection $f' : \tilde{\sigma}' \to \sigma'$ satisfying the conditions listed above. Compatibility of σ and $\tilde{\sigma}$ with $\mathbb{S}$ allows us to extend f' to a bijection $f : \tilde{\sigma} \to \sigma$ that coincides with the identity on $\sigma \cap \tilde{\sigma}$ and induces bijections $\tilde{\sigma}_{Li} \to \sigma_{Li}$ for $1 \leq i \leq n$. Our choice of C' and f then guarantees the following: Suppose q is a path of length $L - 1$, α an arrow and $r \leq \mathfrak{t}$. In case $q\widehat{z}_r \notin \sigma \cup \tilde{\sigma}$, we have $\Lambda \alpha q z_r = K \alpha q z_r \subseteq \Lambda C'$.

To construct $C \in \mathfrak{Grass}_d^T$ with the required properties, we will add to $\Lambda C'$ suitable cyclic submodules of P generated by linear combinations of paths $q z_r$ in P, where $r \leq \mathfrak{t}$ and q is a path of length L in KQ. In light of the preceding comment, our interest is focused on paths of the form $\alpha p \widehat{z}_r \in \widehat{P}$, where α is an arrow and $p\widehat{z}_r$ belongs to $\sigma_{L-1} \cup \tilde{\sigma}_{L-1}$. We will now, for each path of this ilk, construct a term $C(\alpha p \widehat{z}_r) \subseteq P$ to be added to $\Lambda C'$ so as to yield C as desired.

First suppose $p\widehat{z}_r \in \sigma_{L-1}$, and let α be an arrow such that $\alpha p \widehat{z}_r$ is σ-critical (that is, $\alpha p \widehat{z}_r \notin \sigma$) and $\alpha f^{-1}(p\widehat{z}_r) \notin \tilde{\sigma}$. Then also $\alpha p \widehat{z}_r \notin \tilde{\sigma}$. Indeed, suppose $\alpha p \widehat{z}_r \in \tilde{\sigma}$. Then $p\widehat{z}_r \in \sigma \cap \tilde{\sigma}$, whence $f^{-1}(p\widehat{z}_r) = p\widehat{z}_r$, and consequently also $\alpha f^{-1}(p\widehat{z}_r) = \alpha p \widehat{z}_r$ belongs to $\tilde{\sigma}$, contrary to our hypthesis. In particular, this means that we want the element $\alpha p z_r$ of P to belong to C in order to meet the third of the conditions we are targeting. Accordingly, we let $C(\alpha p \widehat{z}_r) \subseteq P$ be the Λ-submodule $K \alpha p z_r$ in that case.

Now suppose $p\widehat{z}_r \in \sigma_{L-1}$ such that $\alpha p \widehat{z}_r$ is σ-critical, while $\alpha f^{-1}(p\widehat{z}_r) \in \tilde{\sigma}$. We define the correction term as $C(\alpha p \widehat{z}_r) = K\big(\alpha p z_r - f(\alpha f^{-1}(p\widehat{z}_r))\big)$. In the latter difference, $f(\alpha f^{-1}(p\widehat{z}_r))$ is identified with an element in P; that is, $f(\alpha f^{-1}(p\widehat{z}_r))$ stands for $\alpha q z_s$ if $f^{-1}(p\widehat{z}_r) = q\widehat{z}_s$. Thirdly, in case $\alpha p \widehat{z}_r \in \sigma$ but $\alpha f^{-1}(p\widehat{z}_r) \notin \tilde{\sigma}$, we set $C(\alpha p \widehat{z}_r) = K\big(\alpha p z_r - f^{-1}(\alpha p \widehat{z}_r)\big)$; in this difference, we again view $f^{-1}(\alpha p \widehat{z}_r)$ as an element in P. Finally, if $\alpha p \widehat{z}_r \in \sigma$ and $\alpha f^{-1}(p\widehat{z}_r) \in \tilde{\sigma}$, we take $C(\alpha p \widehat{z}_r)$ to be zero; in that case, our induction hypothesis guarantees that $p z_r - f^{-1}(p\widehat{z}_r) \in C'$, whence $\alpha p z_r - \alpha f^{-1}(p\widehat{z}_r) \in \Lambda C'$; once more, $f^{-1}(p\widehat{z}_r)$ is viewed as an element in P.

Then

$$C = \Lambda C' + \sum_{\alpha \text{ arrow}, \, p\widehat{z}_r \in \sigma_{L-1} \cup \tilde{\sigma}_{L-1}} C(\alpha p \widehat{z}_r)$$

is a Λ-submodule of P satisfying all requirements imposed by our induction. Indeed, since C is homogeneous by construction, our initial comments show that, to prove $C \in$

$\mathfrak{Grass}(\mathbb{S}) \cap \mathfrak{Grass}(\mathbb{S}')$, one only needs to observe that each of the two families $\big(pz_r\big)_{p\widehat{z_r} \in \sigma}$ and $\big(pz_r\big)_{p\widehat{z_r} \in \widetilde{\sigma}}$ of elements of P induces a basis for P/C, whence $C \in \mathfrak{Grass}(\mathbb{S}) \cap \mathfrak{Grass}(\mathbb{S}')$. This completes the argument showing that $\mathfrak{Grass}(\sigma)$ is dense in $\mathfrak{Grass}(\mathbb{S})$.

Smoothness of $\mathfrak{Grass}(\mathbb{S})$ now follows from the fact that the $\mathfrak{Grass}(\sigma)$, where σ traces the skeleta compatible with $\mathbb{S}$, form an open cover of $\mathfrak{Grass}(\mathbb{S})$. Rationality of $\mathfrak{Grass}(\mathbb{S})$ is obvious. $\quad\square$

To contrast Theorem 5.3 with the non-truncated situation: In general *any* affine variety, not necessarily irreducible, arises as a variety $\mathfrak{Grass}(\sigma)$, up to isomorphism, for suitable Λ and σ; see [**10**, Theorem G]. Moreover, arbitrary affine or projective varieties arise in the form $\mathfrak{Grass}(\mathbb{S})$; see [**1**, Section 5] and [**8**, Example]. With regard to the following corollary, keep in mind that there are only finitely many semisimple sequences of any given dimension d.

Corollary 5.4. Rationality of the components of $\mathfrak{Grass}_d^T$. *Suppose Λ is a truncated path algebra, and consider the set*

$$\{\overline{\mathfrak{Grass}(\mathbb{S})} \mid \mathbb{S} \text{ is a semisimple sequence with } \mathbb{S}_0 = T \text{ and } \underline{\dim}\mathbb{S} = d\}$$

of closures of the varieties $\mathfrak{Grass}(\mathbb{S})$ in $\mathfrak{Grass}_d^T$. The maximal elements in this set are precisely the irreducible components of $\mathfrak{Grass}_d^T$. They are determined by skeleta as follows:

$$\overline{\mathfrak{Grass}(\mathbb{S})} \; = \; \overline{\mathfrak{Grass}(\sigma)},$$

whenever σ is a skeleton compatible with $\mathbb{S}$.

Analogously, the irreducible components of $\mathrm{Gr}\text{-}\mathfrak{Grass}_d^T$ are the maximal ones among the closures of the varieties $\mathrm{Gr}\text{-}\mathfrak{Grass}(\mathbb{S})$ in $\mathrm{Gr}\text{-}\mathfrak{Grass}_d^T$.

In particular, all irreducible components of $\mathfrak{Grass}_d^T$ and $\mathrm{Gr}\text{-}\mathfrak{Grass}_d^T$ are rational varieties. $\quad\square$

Theorem 5.3 and Corollary 5.4 can be transposed to the classical affine setting via Proposition 2.2: Namely, each variety $\mathbf{Mod}(\mathbb{S})$ is irreducible and smooth, and the irreducible components of the varieties $\mathbf{Mod}_d^T$ and $\mathbf{Mod}_d(\Lambda)$ are the maximal candidates among the closures of the $\mathbf{Mod}(\mathbb{S})$ in the larger varieties. While we expect the $\mathbf{Mod}(\mathbb{S})$ to also be rational, in tandem with the $\mathfrak{Grass}(\mathbb{S})$, it is only in the following slightly weakened form that we could carry over rationality to the classical parametrizing varieties. Fortunately, this weaker result still suffices to yield Corollary 5.6.

For the representation-theoretic audience, we recall that an irreducible variety V over K is *unirational* if its function field embeds into a purely transcendental extension of K.

Corollary 5.5. Unirationality of the components of $\mathbf{Mod}_d^T$ and $\mathbf{Mod}_d(\Lambda)$. *For any positive integer d, the irreducible components of the affine variety $\mathbf{Mod}_d(\Lambda)$ are unirational, as are the irreducible components of the quasi-affine varieties $\mathbf{Mod}_d^T$.*

Moreover: Suppose $\mathcal{C}$ is any irreducible component of $\mathbf{Mod}_d(\Lambda)$. If $\mathbb{S}$ is the generic radical layering of the modules in $\mathcal{C}$, and σ any skeleton compatible with $\mathbb{S}$, then $\mathcal{C}$ contains a dense open subset consisting of modules with skeleton σ.

Proof. Let $\mathcal{C}$ be an irreducible component of $\mathbf{Mod}_d(\Lambda)$ or some $\mathbf{Mod}_d^T$, and let $\mathbb{S}$ be the generic radical layering of the modules in $\mathcal{C}$. Since $\mathbf{Mod}(\mathbb{S})$ is irreducible, $\mathcal{C}$ contains $\mathbf{Mod}(\mathbb{S})$ as a dense locally closed subvariety. Let σ be any skeleton compatible with $\mathbb{S}$.

We construct a (Zariski-closed) subset, labeled $\mathbf{Mod}(\sigma)$, of $\mathbf{Mod}(\mathbb{S})$ as follows: First, we index a basis for K^d by the elements of σ, say $(b_{\widehat{p}})_{\widehat{p}\in\sigma}$, and identify any map in $\mathrm{End}_K(K^d)$ with its matrix relative to the sequence $(b_{\widehat{p}})$, indexing rows and columns by the elements of σ. Next, we consider the K-algebra generators of $\Lambda = KQ/I$ given by (the I-residues) of the elements in $Q^* = Q_0 \cup Q_1$, and define $\mathbf{Mod}(\sigma)$ as the set of those points $(x_u) \in \prod_{u\in Q^*} \mathrm{End}_K(K^d) = \prod_{u\in Q^*} M_d(K)$ which satisfy the following conditions: For $u = e_i$, the matrix x_u carries 1 in position $(\widehat{p}, \widehat{p})$, whenever the path $\widehat{p}$ ends in the vertex e_i, and carries 0 in all other positions; for $u = \alpha \in Q_1$, the $\widehat{p}$-th column of the matrix $x_u = x_\alpha$ has the following form: in case $\alpha\widehat{p} \in \sigma$, it carries 1 in position $(\alpha\widehat{p}, \widehat{p})$ and zeros in the other positions, while, in case $\alpha\widehat{p} \notin \sigma$ (meaning that $\alpha\widehat{p}$ is σ-critical), the $\widehat{p}$-th column is required to have zeros in all positions $(\widehat{q}, \widehat{p})$ having row index $\widehat{q}$ outside $\sigma(\alpha\widehat{p})$, with no conditions imposed on the remaining entries. It is readily seen that $\mathbf{Mod}(\sigma) \subseteq \mathbf{Mod}(\mathbb{S})$; indeed, by construction, the sequences of matrices in $\mathbf{Mod}(\sigma)$ describe multiplication by the elements $u \in Q^*$ on the factor modules P/C for $C \in \mathfrak{Grass}(\sigma)$, relative to the basis $(pz_r + C)_{\widehat{p}=p\widehat{z}_r\in\sigma}$ of P/C, respectively. Thus, Theorem 5.3 ensures that all those positions in the matrix sequences above, which are not specified to be 0 or 1 in the definition of $\mathbf{Mod}(\sigma)$, will independently take on arbitrary values in K. In other words, $\mathbf{Mod}(\sigma) \cong \mathbb{A}^{N(\mathbb{S})}$.

Let V be the closure of $\mathfrak{Grass}(\sigma)$ in $\mathfrak{Grass}(\mathbb{S})$ under the $\mathrm{Aut}_\Lambda(P)$-action. Then

$$V = \bigcup_{f\in\mathrm{Aut}_\Lambda(P),\, C\in\mathfrak{Grass}(\sigma)} f.C$$

is open and dense in $\mathfrak{Grass}(\mathbb{S})$, since by Theorem 5.3 we know $\mathfrak{Grass}(\sigma)$ to be open and dense. Proposition 2.2 therefore shows the corresponding GL_d-stable subvariety W of $\mathbf{Mod}(\mathbb{S})$ to be open and dense in $\mathbf{Mod}(\mathbb{S})$. On the other hand, by construction, W is the closure of $\mathbf{Mod}(\sigma)$ under the GL_d-action. This provides us with a dominant morphism

$$\mathrm{GL}_d \times \mathbf{Mod}(\sigma) \to \mathbf{Mod}(\mathbb{S}), \quad (g, (x_u)_u) \mapsto (g^{-1}x_u g)_u.$$

Therefore $\mathbf{Mod}(\mathbb{S})$ is unirational, and so is $\mathcal{C}$. To conclude the argument, we observe that the GL_d-orbits in W are in one-to-one correspondence with the isomorphism classes of modules having skeleton σ. $\square$

Let us return to the hierarchy of irreducible components of $\mathfrak{Grass}_d^T$ introduced in Corollary 2.7. Over truncated path algebras, the irreducible components of class 0 are just the closures in $\mathfrak{Grass}_d^T$ of the irreducible varieties $\mathfrak{Grass}\,\mathbb{S}^{(0,i)}$, where $\mathbb{S}^{(0,1)}, \ldots, \mathbb{S}^{(0,n_0)}$ are the distinct semisimple sequences which are minimal with respect to $\mathfrak{Grass}\,\mathbb{S}^{(0,i)} \neq \varnothing$. Hence, the irreducible components of class 0 of $\mathfrak{Grass}_d^T$ are in one-to-one correspondence with the sequences $\mathbb{S}$ that are minimal with respect to satisfying the equivalent conditions of Observation 5.2. They can be picked out on sight. Next, let $\mathbb{S}^{(1,1)}, \ldots, \mathbb{S}^{(1,n_1)}$ be the distinct semisimple sequences which are minimal among the sequences $\mathbb{S}'$ with the property

that $\mathfrak{Grass}(\mathbb{S})'$ is not contained in the closure of $\bigcup_{1 \le i \le n_0} \mathfrak{Grass}\, \mathbb{S}^{(0,i)}$ in $\mathfrak{Grass}_d^T$. These sequences are much harder to recognize in general, a task which will be addressed in a sequel to this paper, with the aid of Corollary 5.6 below. The (necessarily distinct) closures of the $\mathfrak{Grass}\, \mathbb{S}^{(1,i)}$ in $\mathfrak{Grass}_d^T$ are precisely the irreducible components of class 1 in the bigger variety, and so forth. Irreducible components of $\mathfrak{Grass}_d^T$ (alias $\mathbf{Mod}_d^T$) of arbitrarily high class numbers are realizable over truncated path algebras; see Examples 5.7(2) below.

An analogous sifting process produces the irreducible components of class 0 of $\mathbf{Mod}_d(\Lambda)$ as closures of certain varieties $\mathbf{Mod}(\mathbb{S}^{(0,n_0)})$ in $\mathbf{Mod}_d(\Lambda)$, the 0-th generation being again easy to detect. Recognizing non-embeddedness in identifying the sequences that lead to irreducible components of higher class numbers again requires additional theory regarding closures. The main tool in accessing such closures will result from Corollaries 5.4 and 5.5 above, combined with the following fact, due to J. Kollár (for a proof, see [**13**, Proposition 3.6]): Whenever V is a unirational projective variety of positive dimension and $x, y \in V$, there exists a curve $\psi : \mathbb{P}^1 \to V$, the image of which connects x and y. Applications as indicated of the next Corollary will follow in a sequel to this article.

Corollary 5.6.

• *Let T be semisimple, $\mathcal{C}$ an irreducible component of the projective variety $\mathfrak{Grass}_d^T$, and $\mathbb{S}$ the generic radical layering of $\mathcal{C}$. Moreover, suppose that σ is any skeleton compatible with $\mathbb{S}$. Then any point $C \in \mathcal{C}$ belongs to the image of a curve $\psi : \mathbb{P}^1 \to \mathcal{C}$ mapping a dense open subset of $\mathbb{P}^1$ to $\mathfrak{Grass}(\sigma)$.*

• *Let $\mathcal{C}$ be an irreducible component of the affine variety $\mathbf{Mod}_d(\Lambda)$, $\mathbb{S}$ again its generic radical layering, and σ a skeleton compatible with $\mathbb{S}$. Then any point $x \in \mathcal{C}$ belongs to the image of a rational map $\psi : \mathbb{A}^1 \to \mathcal{C}$ which maps a dense open subset of $\mathbb{A}^1$ to the locally closed subvariety of $\mathcal{C}$ consisting of the modules with skeleton σ.* $\square$

Provided that the closure of some $\mathbf{Mod}(\mathbb{S}^{(1,i)})$ in $\mathbf{Mod}_d(\Lambda)$ is maximal irreducible, its class among the irreducible components of $\mathbf{Mod}_d(\Lambda)$ may change in either direction when compared with the class of its closure in the pertinent $\mathbf{Mod}_d^T$; see Examples 5.7, (2) and (3). The movement in the class numbers of irreducible components, as one progresses to closures on the next level in the hierarchy of parametrizing varieties, in fact, encodes a substantial amount of information about Λ-mod. It warrants a separate study of these invariants, to link them more directly to the quiver Q and the Loewy length of Λ.

Examples 5.7.

(1) Let Q be the quiver $\quad\circlearrowleft 1 \longrightarrow 2$, and take $L = 2$, that is, $\Lambda = KQ/I$, where I is generated by the paths of length 3. Moreover, let $d = 3$, $T = S_1$, and $\mathbb{S} = (S_1, S_1, S_2)$. Then $\mathfrak{Grass}(\mathbb{S}) \cong \mathbb{A}^1$, and the closure of $\mathfrak{Grass}(\mathbb{S})$ in $\mathfrak{Grass}_d^T$ equals $\mathfrak{Grass}_d^T$, the latter variety being isomorphic to $\mathbb{P}^1$. Thus, the variety $\mathbf{Mod}_d^T$ is also irreducible. Its closure in $\mathbf{Mod}_d(\Lambda)$ is an irreducible component of class 0. On the other hand, the closure of the irreducible variety $\mathfrak{Grass}\,(S_1, S_1 \oplus S_2, 0)$ in $\mathfrak{Grass}_d^T$ (a singleton) is not an irreducible component of $\mathfrak{Grass}_d^T$, as it is properly contained in the closure of $\mathfrak{Grass}(\mathbb{S})$.

(2) For every nonnegative integer h, there exist a truncated path algebra $\Lambda = KQ/I$ and a semisimple sequence $\mathbb{S}$, with top T and dimension d say, such that the closure of $\mathfrak{Grass}(\mathbb{S})$

in $\mathfrak{Grass}_d^T$ is an irreducible component of class h. We give examples for $h = 1, 2$, which make it clear how to move on to higher values of h.

Let Q be the quiver
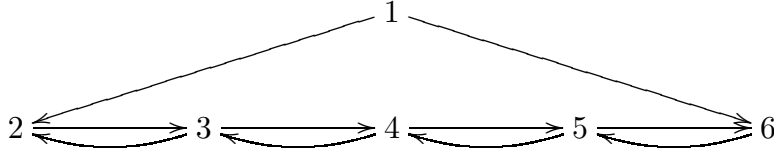
and Λ the truncated path algebra with quiver Q and Loewy length $L + 1 = 6$. Moreover, take $T = S_1^2$ and $d = 7$. Then the closure of

$$\mathfrak{Grass}(\mathbb{S}) := \mathfrak{Grass}\left(S_1^2, S_2, S_3, S_4, S_5, S_6\right)$$

in $\mathfrak{Grass}_d^T$ is an irreducible component of class 0 of $\mathfrak{Grass}_d^T$, clearly the only one. The closure of

$$\mathfrak{Grass}\left(\mathbb{S}'\right) := \mathfrak{Grass}\left(S_1^2, S_2 \oplus S_6, S_3, S_4, S_5, 0\right)$$

in $\mathfrak{Grass}_d^T$ is an irreducible of component of class 1: Indeed, given that $\mathbb{S} < \mathbb{S}'$, we only need to show that $\mathfrak{Grass}\left(\mathbb{S}'\right)$ is not contained in the closure of $\mathfrak{Grass}(\mathbb{S})$ in $\mathfrak{Grass}_d^T$; but this follows from the fact that the arrow $6 \to 5$ annihilates every module in $\mathfrak{Grass}(\mathbb{S})$, and hence annihilates all modules in the closure, whereas $\mathfrak{Grass}\left(\mathbb{S}'\right)$ contains an indecomposable module not annihilated by this arrow. Similarly one shows that the closure of

$$\mathfrak{Grass}\left(\mathbb{S}''\right) := \mathfrak{Grass}\left(S_1^2, S_2 \oplus S_6, S_3 \oplus S_5, S_4, 0, 0\right)$$

in $\mathfrak{Grass}_d^T$ is an irreducible component of class 2 in $\mathfrak{Grass}_d^T$. Indeed, one notes that $\mathbb{S}' < \mathbb{S}''$ and that the closure of $\mathfrak{Grass}\left(\mathbb{S}'\right)$ in $\mathfrak{Grass}_d^T$ is the only component of class 1; then one checks that $\mathfrak{Grass}\left(\mathbb{S}''\right)$ is not contained in the closure of $\mathfrak{Grass}\left(\mathbb{S}'\right)$. By contrast: If we move on to the closures of the corresponding irreducible subsets of $\mathbf{Mod}_d^T$ in $\mathbf{Mod}_d(\Lambda)$, we obtain three distinct irreducible components of class 0 of $\mathbf{Mod}_d(\Lambda)$, since $(S_1)^2$ is minimal among the tops of the modules with dimension vector $(2, 1, 1, 1, 1, 1)$.

(3) For every nonnegative integer h, there exists a truncated path algebra Λ of Loewy length $L + 1 = 3$, a dimension vector $\underline{d} = (d_1, d_2, \ldots, d_n)$ of total dimension $d = \sum_i d_i$, next to semisimple modules $T(0) < T(1) < \cdots < T(h)$, such that the varieties $\mathfrak{Grass}_{\underline{d}}^{T(i)}$ are irreducible, and hence, are irreducible components of class zero of the varieties $\mathfrak{Grass}_d^{T(i)}$ (for the connected components $\mathfrak{Grass}_{\underline{d}}^{T(i)}$, see Observation 2.3). On the other hand, one can arrange for the closures of the corresponding $\mathbf{Mod}_{\underline{d}}^{T(i)}$ in $\mathbf{Mod}_d(\Lambda)$ to be irreducible components of class i in $\mathbf{Mod}_d(\Lambda)$, respectively.

To give a specific example, let Λ have quiver Q as follows:
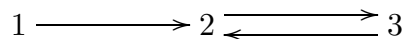
and Loewy length $L + 1 = 3$. Set $T(i) = (S_1)^h \oplus (S_3)^i$ for $0 \leq i \leq h$. Moreover, take $d = 3h$ and $\underline{d} = (h, h, h)$. Observe that $\mathfrak{Grass}_{\underline{d}}^{T(i)}$ equals $\mathfrak{Grass}\left(T(i), (S_2)^h, (S_3)^{h-i}\right)$ and is therefore irreducible by Theorem 5.3. The closure of the corresponding irreducible subset $\mathbf{Mod}_{\underline{d}}^{T(i)}$ of $\mathbf{Mod}_d(\Lambda)$ is an irreducible component of $\mathbf{Mod}_d(\Lambda)$ of class i: Clearly, the closure of $\mathbf{Mod}_{\underline{d}}^{T(0)}$ is an irreducible component of class 0 of $\mathbf{Mod}_d(\Lambda)$, the only one in $\mathbf{Mod}_{\underline{d}}$ in fact. To see that the closure of $\mathbf{Mod}_{\underline{d}}^{T(1)}$ in $\mathbf{Mod}_d(\Lambda)$ is an irreducible component of class 1, it suffices to check that $\mathbf{Mod}_{\underline{d}}^{T(1)}$ is not contained in the closure of $\mathbf{Mod}_{\underline{d}}^{T(0)}$. To see this, let M be the unique indecomposable module with radical layering $(S_1 \oplus S_3, S_2)$ and note that $(\Lambda e_1)^{h-1} \oplus M$ belongs to $\mathbf{Mod}_{\underline{d}}^{T(1)}$, but not to the closure of $\mathbf{Mod}_{\underline{d}}^{T(0)}$ in $\mathbf{Mod}_d(\Lambda)$; indeed, the arrow $3 \to 2$ annihilates each module in $\mathbf{Mod}_{\underline{d}}^{T(0)}$, and consequently annihilates each module in the closure. Continue inductively. $\square$

As we will show next, for a truncated path algebra, the irreducible varieties $\mathfrak{Grass}(\mathbb{S})$ are bundles with affine fibres over a projective base space. The projective portion, $\text{Gr-}\mathfrak{Grass}(\mathbb{S})$, is recognized as a close kin to a flag variety.

Suppose that V is an algebraic variety. By a *Grassmann bundle over V* we mean a fibre bundle over V with fibre F, where F is a direct product of classical Grassmannians $\mathfrak{Gr}(m_i, K^{n_i})$ and all of the pertinent maps are morphisms of varieties; in particular, this means that the transition maps corresponding to a suitable trivialization are automorphisms of the fibres. Moreover, we call a bundle Δ over V an *iterated Grassmann bundle* in case there are bundles $\Delta_1, \ldots, \Delta_r = \Delta$ such that Δ_1 is a Grassmann bundle over V and each Δ_{i+1} is a Grassmann bundle over Δ_i. In particular, the flag variety of any finite dimensional vector space W is an iterated Grassmann bundle over $\mathfrak{Gr}(\dim W - 1, W)$, and iterated Grassmann bundles may be viewed as generalized flag varieties. In a similar vein, a fibre bundle over V is referred to as an *affine bundle with fibre F* if F is an affine variety and, once again, all of the corresponding maps are morphisms, resp., automorphisms of varieties.

Theorem 5.8. Structure of the $\mathfrak{Grass}(\mathbb{S})$. *Suppose that Λ is a truncated path algebra and $\mathbb{S} = (\mathbb{S}_0, \ldots, \mathbb{S}_L)$ a semisimple sequence with $\mathfrak{Grass}(\mathbb{S}) \neq \varnothing$.*

Then $\mathfrak{Grass}(\mathbb{S})$ is an affine bundle over the projective variety $\text{Gr-}\mathfrak{Grass}(\mathbb{S})$ with fibre $\mathbb{A}^{N_1}$, where $N_1 = N_1(\mathbb{S})$ is the invariant of $\mathbb{S}$ introduced in Observation 3.10. The base space of this bundle, $\text{Gr-}\mathfrak{Grass}(\mathbb{S})$, is an iterated Grassmann bundle over a direct product of classical Grassmannian varieties. More precisely, the base space of $\text{Gr-}\mathfrak{Grass}(\mathbb{S})$ is isomorphic to $\mathfrak{Grass}(\mathbb{S}_0, \mathbb{S}_1, 0, \ldots, 0)$.

Proof. As before, we denote by T the top $\mathbb{S}_0$ of $\mathbb{S}$, and by $P = \bigoplus_{l \geq 0} P_l$ a projective cover of T, endowed with the natural grading and its distinguished sequence $z_1, \ldots, z_t$ of top elements. To any point $C \in \mathfrak{Grass}(\mathbb{S})$ we assign the following point in $\text{Gr-}\mathfrak{Grass}(\mathbb{S})$:

$$C_{\text{hmg}} = \bigoplus_{l \geq 1} C_l \quad \text{with} \quad C_l = P_l \cap (C + \bigoplus_{m > l} P_m).$$

Suppose σ is a skeleton compatible with $\mathbb{S}$. Throughout this argument, we will write the paths in our skeleton simply in the form $p \in \sigma$; in other words, p stands for $p = p'\widehat{z}_r$ in $\widehat{P}$ and also for $p'z_r$ in P whenever we work in the projective Λ-module $P = \widehat{P}/I\widehat{P}$.

Let $C \in \mathfrak{Grass}(\sigma)$. Then C_{hmg} takes on the following form in the standard affine coordinates for $\mathfrak{Grass}(\sigma)$:

$$C_{\mathrm{hmg}} = \big(c(\alpha p, q)\big)_{\alpha p \ \sigma\text{-critical}, \ q \in \sigma_0(\alpha p)}$$

where $\sigma_0(\alpha p)$ is the set of all paths in $\sigma(\alpha p)$ which have the same length as αp. Analogously, we denote by $\sigma_1(\alpha p)$ the set of all paths in $\sigma(\alpha p)$ which are strictly longer than αp. Recall that N_1 is the cardinality of the disjoint union of the sets $\{\alpha p\} \times \sigma_1(\alpha p)$.

It is readily checked that the assignment

$$\pi : \mathfrak{Grass}(\mathbb{S}) \to \mathrm{Gr}\text{-}\mathfrak{Grass}(\mathbb{S}), \quad C \mapsto C_{\mathrm{hmg}}$$

is a morphism of varieties and that the inclusion $\mathrm{Gr}\text{-}\mathfrak{Grass}(\mathbb{S}) \hookrightarrow \mathfrak{Grass}(\mathbb{S})$ is a section of π. We will show that π makes $\mathfrak{Grass}(\mathbb{S})$ an affine bundle with fibre $\mathbb{A}^{N_1}$ over $\mathrm{Gr}\text{-}\mathfrak{Grass}(\mathbb{S})$. To do so, we specify trivializations over the open affine subvarieties $\mathrm{Gr}\text{-}\mathfrak{Grass}(\sigma)$ covering $\mathrm{Gr}\text{-}\mathfrak{Grass}(\mathbb{S})$, where σ traces the skeleta compatible with $\mathbb{S}$. Fixing such a skeleton, we define a morphism

$$\mathrm{Gr}\text{-}\mathfrak{Grass}(\sigma) \times \mathbb{A}_1^N \to \pi^{-1}(\mathrm{Gr}\text{-}\mathfrak{Grass}(\sigma))$$

as follows: Given a homogeneous point $D \in \mathrm{Gr}\text{-}\mathfrak{Grass}(\sigma)$ with affine coordinates

$$\big(d(\alpha p, q)\big)_{\alpha p \ \sigma\text{-critical}, \ q \in \sigma_0(\alpha p)}$$

we send the pair

$$\left(D, \big(c(\alpha p, q)\big)_{\alpha p \ \sigma\text{-critical}, \ q \in \sigma_1(\alpha p)} \right)$$

to the following submodule of P:

$$\sum_{\alpha p \ \sigma\text{-critical}} \Lambda \bigg(\alpha p - \sum_{q \in \sigma_1(\alpha p)} c(\alpha p, q) \, q - \sum_{q \in \sigma_0(\alpha p)} d(\alpha p, q) \, q \bigg).$$

By Theorem 5.3, arbitrary choices of coefficients $c(\alpha p, q)$ in K^{N_1} lead to such points in $\pi^{-1}(\mathrm{Gr}\text{-}\mathfrak{Grass}(\sigma))$, independently of the given homogeneous point D. This makes the above assignment a well-defined isomorphism of varieties.

To verify that the transition maps are automorphisms of affine N_1-space, suppose $D \in \mathrm{Gr}\text{-}\mathfrak{Grass}(\sigma) \cap \mathrm{Gr}\text{-}\mathfrak{Grass}(\tilde{\sigma})$ with affine coordinates

$$\big(d(\alpha p, q)\big)_{\alpha p \ \sigma\text{-critical}, \ q \in \sigma_0(\alpha p)} \quad \text{and} \quad \big(\tilde{d}(\alpha p, q)\big)_{\alpha p \ \sigma\text{-critical}, \ q \in \sigma_0(\alpha p)}$$

relative to σ and $\tilde{\sigma}$, respectively. Moreover, let $C \in \mathfrak{Grass}(\mathbb{S})$ be a point in the fibre above D. Then $C \in \mathfrak{Grass}(\sigma) \cap \mathfrak{Grass}(\tilde{\sigma})$, since any homogeneous point D has the same set of

skeleta as the points in $\pi^{-1}(D)$ (by an abuse of language, we refer to skeleta of P/D also as skeleta of D). Let

$$\big(c(\alpha p, q)\big)_{\alpha p \ \sigma\text{-critical, } q \in \sigma_1(\alpha p)} \quad \text{and} \quad \big(\tilde{c}(\alpha p, q)\big)_{\alpha p \ \sigma\text{-critical, } q \in \sigma_1(\alpha p)}$$

be the supplementary nonhomogeneous coordinates relative to σ and $\tilde{\sigma}$, respectively. For reasons of symmetry, it suffices to show that the $\tilde{c}(\alpha p, q)$ are polynomials in the $c(\alpha p, q)$ with coefficients in $K\big(d(\alpha p, q)\big)$, if we treat the $c(*)$'s, $\tilde{c}(*)$'s and $d(*)$'s as independent variables over K.

Let l be any integer between 1 and L and $\sigma_l = \{p_1, \ldots, p_s\}$; the set $\tilde{\sigma}$ has the same cardinality, say $\tilde{\sigma}_l = \{\tilde{p}_1, \ldots, \tilde{p}_s\}$. Then each element $\tilde{p}_i$ in P can be expanded, modulo C, in the format

$$\text{(CONG)} \qquad\qquad \tilde{p}_i \equiv \sum_{1 \le j \le s} a_{ij} p_j + A_i,$$

where the a_{ij} form an invertible $s \times s$-matrix over $K[d(*)]$, and each A_i is a linear combination of paths in σ of lengths exceeding l, with coefficients in $K[c(*), d(*)]$; congruence means congruence modulo C, and again we identify σ and $\tilde{\sigma}$ with subsets of P. Solving for the elements $p_i \in P$ yields the latter as linear combinations

$$\text{(CONG-}l\text{)} \qquad\qquad p_i \equiv \sum_{1 \le j \le s} b_{ij} \tilde{p}_j + B_i,$$

modulo C, where the b_{ij} are coefficients in $K\big(d(*)\big)$ and B_i is a linear combination of paths in σ which are longer than l, with coefficients in $K\big(d(*)\big)[c(*)]$. Now suppose that $\tilde{\alpha}\tilde{p}$ is a $\tilde{\sigma}$-critical path with $\tilde{p} \in \tilde{\sigma}_l$; say $\tilde{p} = \tilde{p}_1$. On multiplying the first of the s congruences labeled (CONG) from the left by $\tilde{\alpha}$, we expand, modulo C, the element $\tilde{\alpha}\tilde{p}$ of P in terms of σ. Then we successively insert the congruences $\big(\text{CONG-}(l+1)\big)$, CONG-$(l+2)$, ... into the expansion. This process terminates, because paths longer than L vanish. It thus displays $\tilde{\alpha}\tilde{p}$, modulo C, as a linear combination of terms $\tilde{q}$ from $\tilde{\sigma}$ with coefficients in $K\big(d(*)\big)[c(*)]$. Comparison of coefficients shows the $\tilde{c}_{\tilde{\alpha}\tilde{p}}$ to have the required form. This completes the proof of the first assertion of the theorem.

To prove the statement concerning Gr-$\mathfrak{Grass}(\mathbb{S})$, we start by noting that nonemptiness of $\mathfrak{Grass}(\mathbb{S})$ implies that

$$\text{Gr-}\mathfrak{Grass}(\mathbb{S}_0, \mathbb{S}_1, \ldots, \mathbb{S}_l, 0, \ldots, 0) \neq \varnothing$$

for all $l \le L$. In a first step, we will show that Gr-$\mathfrak{Grass}(\mathbb{S}_0, \mathbb{S}_1, 0, \ldots, 0)$ is a direct product of classical Grassmannians. Indeed, this variety consists of all homogeneous submodules $C = \bigoplus_{1 \le l \le L} C_l$ of JP which are of the form $C_1 \oplus J^2 P = C_1 \oplus \bigoplus_{l \ge 2} P_l$ such that $\dim e_i C_1 = \dim e_i P_1 - \dim e_i \mathbb{S}_1$. Moreover, we observe that, for any K-subspace U of P_1,

the space $U \oplus \bigoplus_{l \geq 2} P_l$ is a Λ-submodule of JP. Set $m_i = \dim e_i \mathbb{S}_1$ and $n_i = \dim e_i P_1$, that is, $\mathbb{S}_1 = \bigoplus_{1 \leq i \leq n} S_i^{m_i}$ and $JP/J^2P = \bigoplus_{1 \leq i \leq n} S_i^{n_i}$. Clearly, the map

$$\psi : \prod_{1 \leq i \leq n} \mathfrak{Gr}(n_i - m_i, P_1) \to \text{Gr-}\mathfrak{Grass}(\mathbb{S}_0, \mathbb{S}_1, 0, \ldots, 0)$$

which sends $(U_i)_{i \leq n}$ to $\left(\bigoplus_{1 \leq i \leq n} U_i \right) \oplus \left(\bigoplus_{l \geq 2} P_l \right)$ is an isomorphism of varieties, and the initial claim is established.

Finally, we prove that, for any integer $l \geq 2$ which is smaller than L, the following map ψ_l endows Gr-$\mathfrak{Grass}(\mathbb{S}_0, \ldots, \mathbb{S}_{l+1}, 0, \ldots, 0)$ with the structure of a Grassmann bundle over Gr-$\mathfrak{Grass}(\mathbb{S}_0, \ldots, \mathbb{S}_l, 0, \ldots, 0)$:

$$\psi_l : \text{Gr-}\mathfrak{Grass}(\mathbb{S}_0, \ldots, \mathbb{S}_{l+1}, 0, \ldots, 0) \to \text{Gr-}\mathfrak{Grass}(\mathbb{S}_0, \ldots, \mathbb{S}_l, 0, \ldots, 0)$$

sends $C' = \bigoplus_{1 \leq j \leq L} C'_j = \left(\bigoplus_{1 \leq j \leq l+1} C'_j \right) \oplus \left(\bigoplus_{l+2 \leq j \leq L} P_j \right)$ to

$$C' + J^{l+1}P = \left(\bigoplus_{1 \leq j \leq l} C'_j \right) \oplus \left(\bigoplus_{l+1 \leq j \leq L} P_j \right).$$

Next we provide a trivialization over each patch of the open affine cover $(\text{Gr-}\mathfrak{Grass}(\sigma))_\sigma$ of Gr-$\mathfrak{Grass}(\mathbb{S}_0, \ldots, \mathbb{S}_l, 0, \ldots, 0)$, where σ traces the skeleta compatible with the semisimple sequence $(\mathbb{S}_0, \ldots, \mathbb{S}_l, 0, \ldots, 0)$. Let $\sigma = \bigcup_{1 \leq r \leq \mathsf{t}} \sigma^{(r)}$ be such a skeleton, suppose $\mathbb{S}_{l+1} = \bigoplus_{1 \leq i \leq n} S_i^{u_i}$, and consider the subspaces

$$V_i = \sum_{1 \leq r \leq \mathsf{t}} \sum_{p \in \sigma_l^{(r)}} K\langle e_i Q_1 p \rangle$$

of P, where Q_1 is the set of arrows in the quiver Q, and $K\langle A \rangle$ the subspace generated by a subset A of P (again we identify σ with a subset of P whenever called for). If $v_i = \dim V_i$, then $u_i \leq v_i$ in view of the fact that Gr-$\mathfrak{Grass}(\mathbb{S}_0, \ldots, \mathbb{S}_{l+1}, 0, \ldots, 0) \neq \varnothing$. As we will see, the fibre of ψ_l over any point $C \in \text{Gr-}\mathfrak{Grass}(\sigma)$ is isomorphic to

$$F := \prod_{1 \leq i \leq n} \mathfrak{Gr}(v_i - u_i, V_i).$$

Note that $\sum_{1 \leq i \leq n} V_i = \bigoplus_{1 \leq i \leq n} V_i \subseteq P_{l+1}$. This setup permits us to describe a trivialization of ψ_l over Gr-$\mathfrak{Grass}(\sigma)$ as follows:

$$\tau_\sigma : F \times \text{Gr-}\mathfrak{Grass}(\sigma) \longrightarrow \psi_l^{-1}\left(\text{Gr-}\mathfrak{Grass}(\sigma) \right)$$

with

$$\left((U_i)_{1 \leq i \leq n}, C \right) \mapsto C \oplus K\langle Q_1 C_l \rangle \oplus \left(\bigoplus_{1 \leq i \leq n} U_i \right) \oplus \left(\bigoplus_{j \geq l+2} P_j \right),$$

where $K\langle Q_1 C_l\rangle$ is the subspace of P generated by all elements αc, where α is an arrow and $c \in C_l$. To ascertain well-definedness, start by noting that the K-subspaces which are attained as images under τ_σ are actually Λ-submodules of JP. To see that each of these submodules belongs to $\mathfrak{Grass}\,(\mathbb{S}_0, \ldots, \mathbb{S}_{l+1}, 0, \ldots, 0)$, observe that

$$e_i P_{l+1} = V_i \oplus K\langle e_i Q_1 C_l\rangle,$$

since the layered graph of P relative to $z_1, \ldots, z_r$ is a forest with the same tree sitting underneath each vertex labeled e_i in the l-th layer of this graph. Therefore the codimension of $U_i \oplus K\langle e_i Q_1 C_l\rangle$ in $e_i P_{l+1}$ is u_i as desired. That application of the map ψ_l to any point in the image of τ_σ yields a point $D \in \text{Gr-}\mathfrak{Grass}(\mathbb{S}_0, \ldots, \mathbb{S}_l, 0, \ldots, 0)$ with the property that P/D has skeleton σ, is obvious from our construction. More strongly, $\psi_l \circ \tau_\sigma\big((U_i)_{i \leq n}, C\big) = C$. Thus τ_σ is an isomorphism with inverse $C' \mapsto \big((e_i C_{l+1})_i, \psi_l(C')\big)$.

Concerning compatibility of these trivializations: Given two skeleta σ and $\tilde{\sigma}$ compatible with $(\mathbb{S}_0, \ldots, \mathbb{S}_l, 0, \ldots, 0)$, it is routine to check that the corresponding transition map $\text{pr}_F \circ \tau_\sigma^{-1} \circ \tau_{\tilde{\sigma}}(-, C)$ for $C \in \text{Gr-}\mathfrak{Grass}(\sigma) \cap \text{Gr-}\mathfrak{Grass}(\tilde{\sigma})$ is an isomorphism of F, where $\text{pr}_F : F \times \text{Gr-}\mathfrak{Grass}(\sigma) \to F$ denotes the projection onto the fibre. $\square$

In general, the bundles of Theorem 5.8 are nontrivial. A small example exhibiting nontriviality is as follows.

Example 5.9. Let $\Lambda = KQ/I$, where Q is the quiver
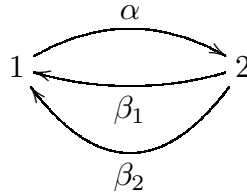
and I is generated by the paths of length 3. Let $T = S_1 \oplus S_2$, $d = 4$, and $\mathbb{S} = (T, S_2, S_1)$. Then it is readily checked that $\mathfrak{Grass}(\mathbb{S})$ is a nontrivial $\mathbb{A}^2$-bundle over $\text{Gr-}\mathfrak{Grass}(\mathbb{S}) \cong \mathbb{P}^1$. On the other hand, if $\mathbb{S}' = (T, S_1, S_2)$, then $\mathfrak{Grass}\,(\mathbb{S}')$ is isomorphic to the trivial $\mathbb{A}^1$-bundle over $\mathbb{P}^1$. This example will be revisited after Theorem 5.11.

We follow with properties of the $\mathbb{S}$-generic module for any sequence $\mathbb{S}$. As in Section 4, we assume that K has infinite transcendence degree over its prime field. Moreover, we let $K_0 \subseteq K$ be the algebraic closure of the prime field. Clearly, any truncated path algebra is defined over K_0 so that the prerequisites of Section 4 are in place.

For easier readability, we will in the sequel identify the elements of a skeleton σ, a priori living in the projective KQ-module $\widehat{P}$, with their canonical images in the projective Λ-module P. This is harmless in the truncated situation, since path lengths in Λ, and hence in P, are well-defined, which makes passage to the corresponding KQ-module superfluous.

The following lemma shows in particular that all syzygies of Λ-modules are direct sums of cyclic modules, each isomorphic to a left ideal of Λ generated by a path. (That this holds for second syzygies already follows from [**9**, Theorem A].)

Lemma 5.10. *Suppose Λ is a truncated path algebra and $C \in \mathfrak{Grass}_d^T$. Then the syzygy C of P/C is a direct sum of cyclic modules, each of which is isomorphic to a left ideal of Λ. More precisely, if $C \in \mathfrak{Grass}(\sigma)$, the elements*

$$\omega(\alpha p z_r) := \alpha p z_r - \sum_{q z_s \in \sigma(\alpha p z_r)} c(\alpha p z_r, q z_s) \, q z_s \; \in \; P,$$

where $\alpha p z_r$ traces the σ-critical paths and the $c(\alpha p z_r, q z_s)$ are the affine coordinates of C in $\mathfrak{Grass}(\sigma)$ (see Section 3.C), generate nonzero independent cyclic submodules of P such that

$$C = \bigoplus_{\alpha p z_r \; \sigma\text{-critical}} \Lambda \omega(\alpha p z_r) \qquad and \qquad \Lambda \omega(\alpha p z_r) \cong \Lambda \alpha p \qquad for \ each \ \sigma\text{-critical path } \alpha p z_r.$$

Proof. Suppose $C \in \mathfrak{Grass}(\sigma)$. We already know that C is generated by the $\omega(\alpha p z_r)$; see the remarks preceding Theorem 3.8. By definition, all of the $\omega(\alpha p z_r)$ are nonzero in P, as the paths involved in the pertinent expansions have lengths $\leq L$. Assume, to the contrary of our claim, that 0 is a sum of certain nontrivial Λ-multiples of $\omega(\alpha p z_r)$'s in P. Let

$$(\dagger) \qquad\qquad\qquad \sum_{1 \leq i \leq \nu} \lambda_i \, \omega(\alpha_i p_i z_{r_i}) = 0$$

be such a nontrivial dependence relation in P, such that ν is minimal and the λ_i are K-linear combinations of paths in Λ, the lengths of which are bounded by the differences $L - \mathrm{length}(\alpha_i p_i)$, respectively; the latter assumption is legitimate as, by construction, each path in $\sigma(\alpha_i p_i z_{r_i})$ is at least as long as $\alpha_i p_i z_{r_i}$. To reach a contradiction, we use the K-linear independence of the formally distinct elements of the form $p z_r$ in P, where p is a path of length at most L in Λ which starts in the vertex $e(r)$ that norms the top element z_r. Let l_0 be the minimum of the lengths of the $\alpha_i p_i$, and assume that $\alpha_1 p_1$ has length l_0. Next, observe the following: If $\lambda_1 = \sum_j k_j q_j$, where the k_j are nonzero elements of K and the q_j are distinct paths with $0 \leq \mathrm{length}(q_j) \leq L - l_0$, then the occurrence of the path $q_1 \alpha_1 p_1 z_{r_1}$ in the term $\lambda_1 \alpha_1 p_1 z_{r_1}$ is the only one in $(\dagger)$. Indeed, as we already remarked, the paths $p z_r$ in the union of the $\sigma(\alpha_i p_i z_{r_i})$ are at least as long as $\alpha_1 p_1 z_{r_1}$, and they all belong to σ; since $\alpha_1 p_1 z_{r_1}$ does not belong to σ, we infer that this path does not arise in the K-linear path expansion in P of any Λ-multiple of a path $q z_s \in \bigcup_{1 \leq i \leq \nu} \sigma(\alpha_i p_i z_{r_i})$. That the path expansions of the Λ-multiples of the $\alpha_i p_i z_{r_i}$ with $2 \leq i \leq \nu$ in P do not contain nontrivial K-multiples of $q_1 \alpha_1 p_1 z_{r_1}$ either is clear, because the minimal initial subpath not belonging to σ of any $q \alpha_i p_i z_{r_i}$ is $\alpha_i p_i z_{r_i}$ and thus different from $\alpha_1 p_1 z_{r_1}$ for $i > 1$ (due to minimality of ν). This yields the required contradiction and thus proves independence of the cyclic modules $\Lambda \omega(\alpha p z_r)$.

To see that each $\Lambda \omega(\alpha p z_r) \subseteq P$ is isomorphic to some left ideal of Λ, let e_s be the endpoint of α. Then one readily checks that $\Lambda \omega(\alpha p z_r) \cong \Lambda \alpha p$, in view of the fact that all paths $q z_s \in \sigma(\alpha p z_r)$ are at least as long as $\alpha p z_r$ and end in the vertex e_s as well. $\quad\square$

Remark. Lemma 5.10 clearly extends to syzygies of nonfinitely generated modules, the same argument being applicable. For that purpose, one generalizes the concept of a skeleton to the infinite dimensional case and adapts the definitions of σ-critical paths and the scalars $c(\alpha p z_r, q z_s)$ in the obvious way. This strengthening is immaterial here, but is used in [**7**].

Most of the statements of the following theorem are immediate consequences of Theorems 4.3 and 5.3 and their proofs; for the sharper result concerning generic syzygies under (1), which is key to easy applicability, we use Lemma 5.10. Note in particular that, for our present choice of Λ, the $\mathbb{S}$-generic modules can be read off the quiver, without requiring any computation, since the auxiliary varieties $\mathfrak{Grass}(\sigma)$ are known to be affine spaces in this case. The uniqueness assertion under (1) also improves on the corresponding one in Theorem 4.3; the considerably stronger version holding in the truncated situation is apparent from the high level of symmetry of the hypergraph of $G(\mathbb{S})$, again a consequence of the projective presentation of $G(\mathbb{S})$ given below. Concerning part (2) of the upcoming theorem: The results of Theorem 4.3 and Corollary 4.7 for $\mathbb{S}$-generic modules carry over to $\mathbb{S}$-generic graded modules, if we replace "isomorphism" by "graded-isomorphism". Mutatis mutandis, the final assertion of part (1) is true for $\mathbb{S}$-generic graded modules as well.

Theorem 5.11. (The $\mathbb{S}$-generic modules and the $\mathbb{S}$-generic graded modules over a truncated path algebra.) *Suppose Λ is a truncated path algebra, and let $\mathbb{S}$ be any semisimple sequence with $\mathfrak{Grass}(\mathbb{S}) \neq \varnothing$. As before, P is the distinguished projective cover of the top $T = \mathbb{S}_0$.*

(1) *The $\mathbb{S}$-generic module $G(\mathbb{S})$ for the irreducible variety $\mathfrak{Grass}(\mathbb{S})$ – equivalently, for the classical variety $\mathbf{Mod}(\mathbb{S})$ – has a projective presentation as follows: Let σ be any skeleton compatible with $\mathbb{S}$ and $\big(x(\alpha p z_r, q z_s)\big)$ any family of scalars which is algebraically independent over K_0. Then $G(\mathbb{S}) = P/C(\mathbb{S})$, up to equivalence, where $C(\mathbb{S})$ is the submodule of P generated by the differences $\alpha p z_r - \sum_{q z_s \in \sigma(\alpha p z_r)} x(\alpha p z_r, q z_s) q z_s$, with $\alpha p z_r$ tracing the σ-critical paths. (By definition, $\sigma(\alpha p z_r)$ consists of those paths in σ which have lengths between length(αp) and L and end in the same vertex as αp.) In particular, the syzygy*

$$
C(\mathbb{S}) = \bigoplus_{\alpha p z_r\ \sigma\text{-critical}} \Lambda\left(\alpha p z_r - \sum_{q z_s \in \sigma(\alpha p z_r)} x(\alpha p z_r, q z_s)\, q z_s\right) \quad \cong \quad \bigoplus_{\alpha p z_r\ \sigma\text{-critical}} \Lambda \alpha p
$$

of $G(\mathbb{S})$ is a direct sum of cyclic modules, each of which is isomorphic to a left ideal of Λ generated by a path. It is completely determined by $\mathbb{S}$, up to isomorphism (not only up to $\mathrm{Gal}(K/K_0)$-equivalence).

Moreover, any submodule H of $G(\mathbb{S})$, which – modulo $C(\mathbb{S})$ – is generated by homogeneous elements of a fixed degree in P, is the $\mathbb{S}(H)$-generic module, up to equivalence.

(2) *The $\mathbb{S}$-generic graded module $\mathrm{Gr}\text{-}G(\mathbb{S})$ generated in degree 0 has a projective presentation as follows. Denote by N_0 the disjoint union of the sets $\{\alpha p z_r\} \times \sigma_0(\alpha p z_r)$, where $\sigma_0(\alpha p z_r)$ consists of those paths in $\sigma(\alpha p z_r)$ which have the same length as αp. Let σ be any skeleton compatible with $\mathbb{S}$ and $\big(x(\alpha p z_r, q z_s)\big)$ a K_0-algebraically independent family*

of scalars indexed by N_0. Then $\mathrm{Gr}\text{-}G(\mathbb{S})$ *is equivalent to* $P/(\mathrm{Gr}\text{-}C(\mathbb{S}))$, *where*

$$\mathrm{Gr}\text{-}C(\mathbb{S}) = \bigoplus_{\alpha p z_r \, \sigma\text{-cricital}} \Lambda\left(\alpha p z_r - \sum_{q z_s \in \sigma_0(\alpha p z_r)} x(\alpha p z_r, q z_s)\, q z_s \right) \;\cong\; C(\mathbb{S}). \qquad \square$$

In light of Corollary 4.7, Theorem 5.11 thus reduces structural problems regarding the $\mathbb{S}$-generic modules to combinatorial tasks. In particular, this is true for the questions regarding decomposability of the $\mathbb{S}$-generic module, the structure of its indecomposable summands, generic socles, higher syzygies, etc. For example, the module $G(\mathbb{S})$ is indecomposable if and only if all of its (finitely many) hypergraphs relative to full sequences of top elements are connected.

Example 5.9 revisited. Let $d = 4$. For the dimension vector $\underline{d} = (2, 2)$, we list all irreducible components of $\mathbf{Mod}_d(\Lambda)$ which are contained in the connected component $\mathbf{Mod}_{\underline{d}}$ and display their generic modules. There are precisely four, and the generic modules have top dimension ≤ 2. We will start by displaying the $\mathbb{S}$-generic modules for each of the six nonempty varieties $\mathfrak{Grass}(\mathbb{S})$, where $\mathbb{S}$ is a semisimple sequence of dimension vector $\underline{d}$ with $\dim \mathbb{S}_0 \leq 2$; these are: $\mathbb{S}^{(1)} = (S_1^2, S_2^2, 0)$, $\mathbb{S}^{(2)} = (S_2^2, S_1^2, 0)$, $\mathbb{S}^{(3)} = (S_1 \oplus S_2, S_1 \oplus S_2, 0)$, $\mathbb{S}^{(4)} = (S_2, S_1^2, S_2)$, $\mathbb{S}^{(5)} = (S_1 \oplus S_2, S_2, S_1)$, and $\mathbb{S}^{(6)} = (S_1 \oplus S_2, S_1, S_2)$. Instead of formally presenting the generic modules $G_i = G(\mathbb{S}^{(i)})$ (which is straightforward in view of Theorem 5.11), we provide hypergraphs relative to suitable top elements and skeleta, which is more informative at a glance.
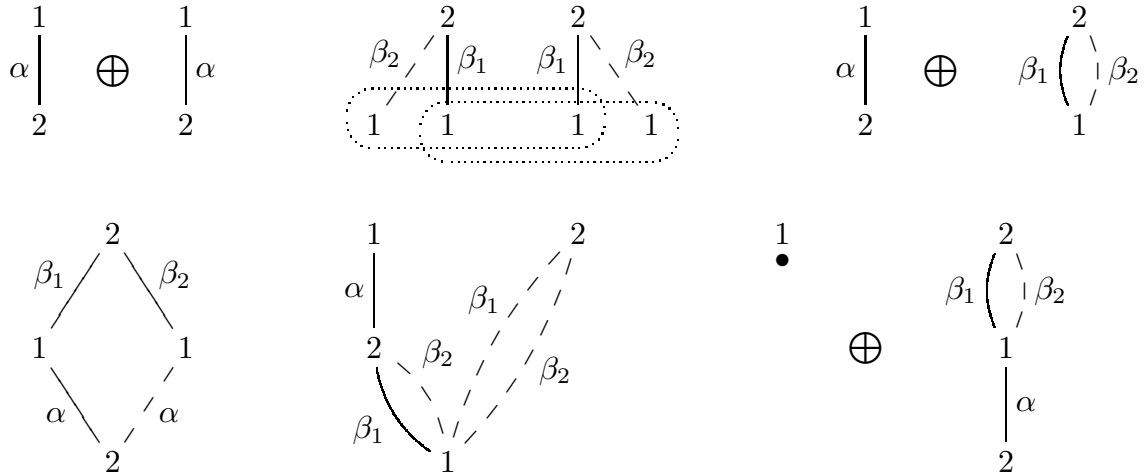

Note that G_5 has socle S_1; indeed, due to the fact that $\beta_2 \alpha z_1$ and $\beta_1 z_2$, $\beta_2 z_2$ are "attached to" $\beta_1 \alpha z_1$ by means of scalars which are algebracially independent over K_0, the simple S_2 does not occur in $\mathrm{Soc}\, G_5$. Concerning G_6: The displayed hypergraph does not correspond to a generic presentation of G_6 in the sense of the paragraph preceding Corollary 4.5; the given non-generic presentation more clearly exhibits decomposability.

Let $\mathcal{C}_i$ be the closure of $\mathbf{Mod}(\mathbb{S}^{(i)})$ in $\mathbf{Mod}_d(\Lambda)$ for $i = 1, \ldots, 6$. Aided by Corollary 4.7, we will sift out the $\mathcal{C}_i$ which are maximal irreducible in $\mathbf{Mod}_d(\Lambda)$. Clearly, $\mathcal{C}_3$ is contained in each of $\mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6$. Next, it is straightforward to construct a curve $\psi : \mathbb{A}^1 \to \mathbf{Mod}_d(\Lambda)$ with $\psi(t) \in \mathbf{Mod}(\mathbb{S}^{(4)})$ for $t \neq 0$, such that $\psi(0)$ represents G_6; hence $\mathcal{C}_6 \subseteq \mathcal{C}_4$. Since the dimension vector of top G_i for $i = 1, 4$ is minimal among the dimension vectors of the tops of the modules in $\mathcal{C}_1, \ldots, \mathcal{C}_6$ (see Observation 2.1), $\mathcal{C}_1$ and $\mathcal{C}_4$ are irreducible components of $\mathbf{Mod}_d(\Lambda)$. Comparing tops, we further note that the only $\mathcal{C}_j$ which potentially contains $\mathcal{C}_2$ is $\mathcal{C}_4$; but, in light of Corollary 4.7, the containment $\mathcal{C}_2 \subseteq \mathcal{C}_4$ is ruled out by the fact that S_2 is evidently a summand of $\mathrm{Soc}\, G_4$, but not a summand of $\mathrm{Soc}\, G_2$. Again comparing socles of generic modules and invoking Corollary 4.7, we conclude that $\mathcal{C}_5$ is not contained in any of $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_4$. Thus the $\mathcal{C}_i$ for $i = 1, 2, 4, 5$ constitute a full irredundant list of those irreducible components of $\mathbf{Mod}_d(\Lambda)$ which are contained in the connected component representing the modules with dimension vector $\underline{d}$. Consequently, the G_i for $i = 1, 2, 4, 5$ are even 4-generic. In the top-order, $\mathcal{C}_1$ and $\mathcal{C}_4$ are irreducible components of $\mathbf{Mod}_d(\Lambda)$ of class 0, while $\mathcal{C}_2$ and $\mathcal{C}_5$ are of class 1. $\square$

We illustrate Theorem 5.11 with another, somewhat more complex, example. We also display the generic graded module $\mathrm{Gr}\text{-}G(\mathbb{S})$ for the considered semisimple sequence $\mathbb{S}$.
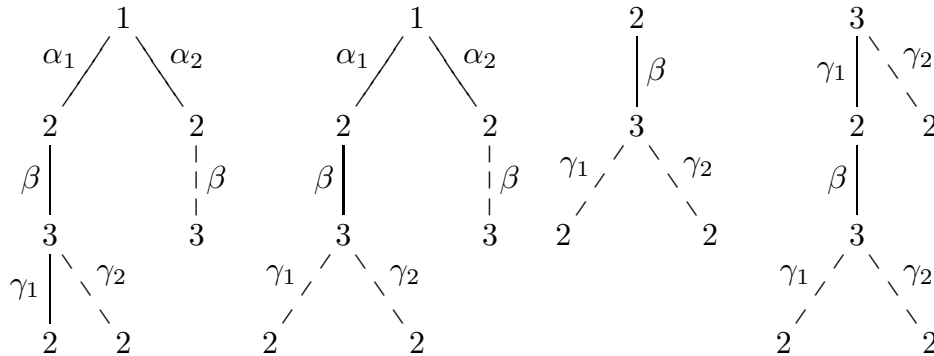
Example 5.12. Let Q be the quiver
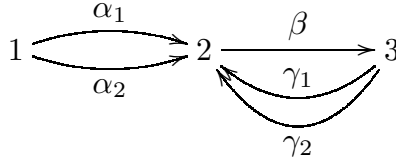
and Λ the truncated path algebra KQ/I, where I is generated by the paths of length 4; i.e., $L = 3$. We will consider the 14-dimensional semisimple sequence
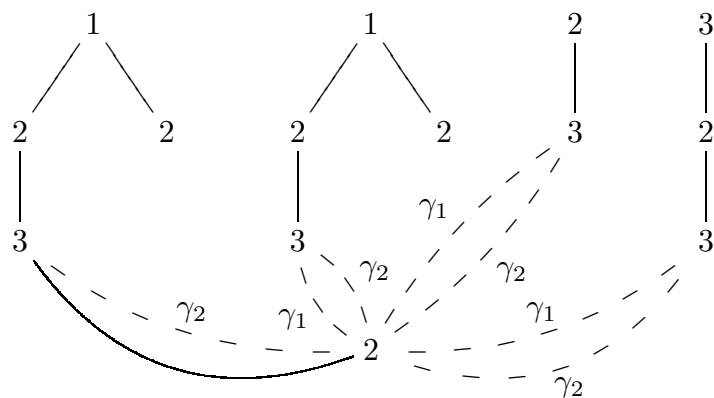
$$\mathbb{S} = (S_1^2 \oplus S_2 \oplus S_3, S_2^5 \oplus S_3, S_3^3, S_2).$$

We give the hypergraph of the corresponding $\mathbb{S}$-generic module $G(\mathbb{S})$, relative to the skeleton σ shown below; the broken edges again indicate the σ-critical paths.
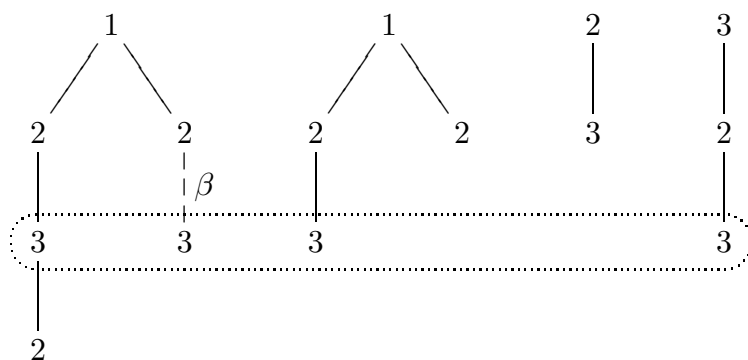

The hypergraph of $G(\mathbb{S})$ results from superposition of the following diagrams (1)–(4):
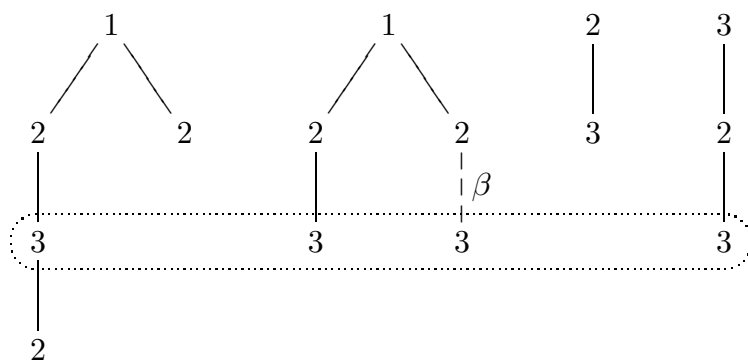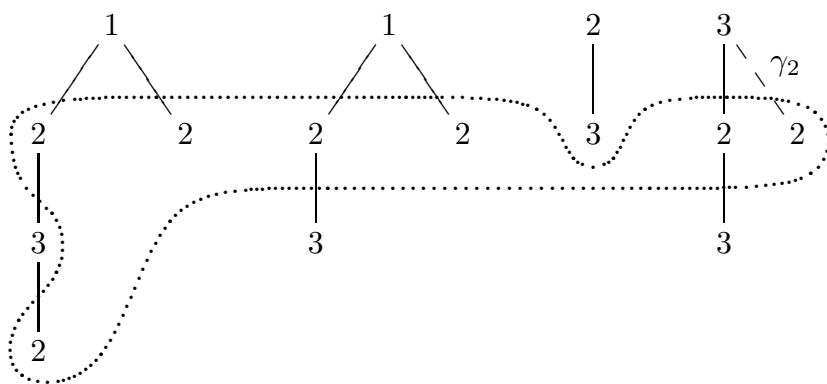
(1)

(2)

(3)

(4)

The module $G = G(\mathbb{S})$ is indecomposable, as can already be gleaned from Diagram (1); indeed, the full collection of scalars involved in the relations, which are indicated by dotted edges, is algebraically independent over K_0. The generic socle is S_2^2; one copy of S_2 in the socle is obvious, the other can be read off Diagram (3). The closure of $\mathfrak{Grass}(\mathbb{S})$ is not an irreducible component of $\mathfrak{Grass}_d^T$, where $T = S_1^2 \oplus S_2 \oplus S_3$ and $d = 14$; in fact, G arises as a top-stable degeneration of a generic module for the semisimple sequence $\mathbb{S}' = (S_1^2 \oplus S_2 \oplus S_3, S_2^5, S_3^4, S_2)$. Moreover, a purely combinatorial process yields the first syzygy of G to be $\Omega^1(G) \cong S_1^5 \oplus (\Lambda e_2/J^2 e_2)^3 \oplus (\Lambda e_2/J^3 e_2) \oplus (\Lambda e_3/J^2 e_3)^2$, which makes the higher generic syzygies for the modules with radical layering $\mathbb{S}$ readily available. In particular, $\operatorname{proj\,dim} G = \infty$, which, in view of Corollary 4.7, shows that all modules with the given radical layering have infinite projective dimension.

The $\mathbb{S}$-generic graded module Gr-$G(\mathbb{S})$ with top generated in a fixed degree has the following modified hypergraph relative to σ: It is the superposition of the diagram below with Diagrams (2) and (3) above.
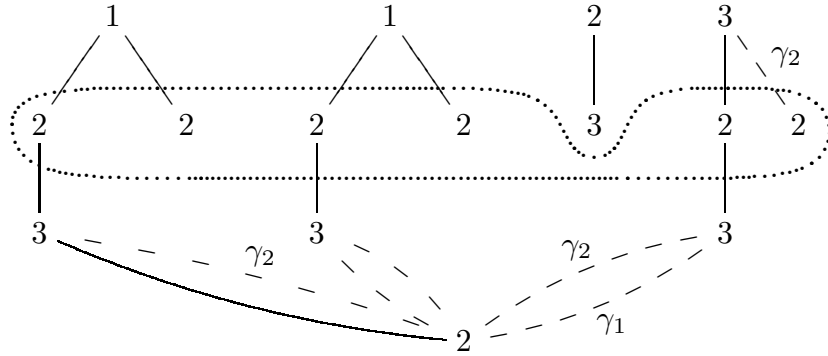
Generically, the graded modules with radical layering $\mathbb{S}$ have two indecomposable summands, with dimension vectors $(0, 1, 1)$ and $(2, 6, 4)$. The generic socle may shrink as one passes from the graded to the ungraded situation; indeed, $\operatorname{Soc} \text{Gr-}G(\mathbb{S}) \cong \operatorname{Soc} G(\mathbb{S}) \oplus S_3$ in our example. $\square$

In general, it is cumbersome to explicitly state combinatorial equivalent conditions for indecomposability of $G(\mathbb{S})$ or Gr-$G(\mathbb{S})$ in terms of $\mathbb{S}$, Q, and L. We content ourselves with presenting a straightforward necessary condition for the graded situation. The following auxiliary graph depends on a choice of skeleton, but the vertex sets of its connected components do not: The vertex set is the set $Z = \{z_1, \ldots, z_t\}$ of distinguished top elements in the projective cover P of $\mathbb{S}_0$. Given a skeleton σ compatible with $\mathbb{S}$, there is an edge connecting z_r and z_s if and only if either there exists a σ-critical path $\alpha p z_r$ and a path $q z_s \in \sigma$ with $\operatorname{length}(\alpha p) = \operatorname{length}(q)$ and $\operatorname{endpt}(\alpha p) = \operatorname{endpt}(q)$, or else this condition holds with the roles of r and s reversed.

Corollary 5.13. *Let $\mathbb{S}$ be a semisimple sequence over a truncated path algebra Λ such that $\mathfrak{Grass}(\mathbb{S}) \neq \varnothing$. Moreover, let $Z_1, \ldots, Z_\mu \subseteq Z$ be the vertex sets of the connected components of any of the auxiliary graphs introduced above, and let U_i, for $1 \leq i \leq \mu$, be the submodule of $\text{Gr-}G(\mathbb{S}) = P/\text{Gr-}C(\mathbb{S})$ generated by the residue classes $z + \text{Gr-}C(\mathbb{S})$, $z \in Z_i$.*

Then Gr-$G(\mathbb{S}) = \bigoplus_{1 \le i \le \mu} U_i$. *In particular, indecomposability of* Gr-$G(\mathbb{S})$ *implies connectedness of the auxiliary graphs.*

If the top of $\mathbb{S}$ *is squarefree, the* U_i *are the indecomposable direct summands of* Gr-$G(\mathbb{S})$, *and* Gr-$G(\mathbb{S})$ *is indecomposable if and only if any of the auxiliary graphs is connected.* $\square$

We leave the easy proof to the reader. There are obvious analogues of the two statements of Corollary 5.13 for the ungraded situation. The first is always true, while the second is not. The following general connection between the graded and ungraded situations is helpful.

Corollary 5.14. *Let* $\mathbb{S}$ *be a semisimple sequence over a truncated path algebra* Λ *such that* $\mathfrak{Grass}(\mathbb{S}) \ne \varnothing$. *If the generic graded module* Gr-$G(\mathbb{S})$ *with radical layering* $\mathbb{S}$ *is indecomposable, then so is the generic ungraded module* $G(\mathbb{S})$.

Proof. Note that the module Gr-$G(\mathbb{S})$ (resp., $G(\mathbb{S})$) is indecomposable precisely when every hypergraph of Gr-$G(\mathbb{S})$ (resp., $G(\mathbb{S})$) is connected. Since the set of skeleta of Gr-$G(\mathbb{S})$ coincides with that of $G(\mathbb{S})$, our claim is easily deduced from Theorem 5.11. $\square$

The converse of Corollary 5.14 fails in general. Indeed, let $\Lambda = KQ$, where Q is the quiver

$$1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4$$

and $\mathbb{S}$ the sequence $(S_1 \oplus S_4, S_2, S_3)$. Then the $\mathbb{S}$-generic module is indecomposable, while the $\mathbb{S}$-generic graded module decomposes.

If "structural symmetry" of a module is measured by the dimension of its endomorphism ring, the generic module $G(\mathbb{S})$ (or Gr-$G(\mathbb{S})$) has minimal structural symmetry among the modules represented by $\mathbb{S}$; see Corollary 4.7. Yet, note that the endomorphism ring of an indecomposable $\mathbb{S}$-generic module $G(\mathbb{S})$ need not be trivial: Let Q be the quiver of Examples 5.7(1), and take $L = 2$ and $\mathbb{S} = (S_1, S_1, S_2)$. Then $\operatorname{End}_\Lambda G(\mathbb{S})$ has dimension 2. By contrast, if indecomposable, the generic graded module Gr-$G(\mathbb{S})$ always has trivial homogeneous endomorphism ring.

On the other hand, in terms of its submodule structure, the module $G(\mathbb{S})$ (resp. Gr-$G(\mathbb{S})$) displays maximal symmetry among the (graded) modules with radical layering $\mathbb{S}$, in the following sense: Let σ be any skeleton compatible with $\mathbb{S}$, and $G(\mathbb{S}) = P/C(\mathbb{S})$ the corresponding generic presentation described in Theorem 5.11. Then the theorem shows in particular that, given any two paths $pz_r, qz_s \in \sigma$ with coinciding length and endpoint, the canonical images of Λpz_r and Λqz_s in $G(\mathbb{S})$ are $\operatorname{Gal}(K/K_0)$-equivalent and thus have the same hypergraphs.

References

1. E. Babson, B. Huisgen-Zimmermann, and R. Thomas, *Moduli spaces of graded representations of finite dimensional algebras*, in Algebra and its Applications (Athens, Ohio, 2005), D. V. Huynh, S. K. Jain, and S. R. López-Permouth, Eds., Contemp. Math. **419** (2006), 7-27.

2. E. Babson, B. Huisgen-Zimmermann, and R. Thomas, *Maple codes for computing* $\mathfrak{Grass}(\sigma)$*'s*, posted at http://www.math.washington.edu/ thomas/programs/programs.html.

3. K. Bongartz and B. Huisgen-Zimmermann, *The geometry of uniserial representations of algebras II. Alternate viewpoints and uniqueness*, J. Pure Appl. Algebra **157** (2001), 23-32.

4. _____, *Varieties of uniserial representations IV. Kinship to geometric quotients*, Trans. Amer. Math. Soc. **353** (2001), 2091-2113.

5. W. Crawley-Boevey, *Tame algebras and generic modules*, Proc. London Math. Soc. **63** (1991), 241-265.

6. W. Crawley-Boevey and J. Schröer, *Irreducible components of varieties of modules*, J. reine angew. Math. **553** (2002), 201-220.

7. A. Dugas, B. Huisgen-Zimmermann and J. Learned, *Truncated path algebras are homologically transparent*, in Contributions to Module Theory (Volume in Honor of A.L.S. Corner) (R. Göbel and B. Goldsmith, eds.), de Gruyter, Berlin (to appear).

8. L. Hille, *Tilting line bundles and moduli of thin sencere representations of quivers*, An. St. Univ. Ovidius Constantza **4** (1996), 76-82.

9. B. Huisgen-Zimmermann, *Homological domino effects and the first finitistic dimension conjecture*, Invent. Math. **108** (1992), 369-383.

10. _____, *The geometry of uniserial representations of finite dimensional algebras I*, J. Pure Appl. Algebra **127** (1998), 39-72.

11. _____, *The phantom menace in representation theory*, Algebra and its applications (Athens, Ohio 1999) (D. Van Huynh, S.K. Jain, and S.R. López-Permouth, eds.), Contemp. Math. Vol. 259, Amer. Math. Soc., Providence, 2000, pp. 247-278.

12. _____, *Classifying representations by way of Grassmannians*, Trans. Amer. Math. Soc. **359** (2007), 2687-2719.

13. _____, *Top-stable degenerations of finite dimensional representations I*, Proc. London Math. Soc. **96** (2008), 163-198.

14. _____, *A hierarchy of parametrizing varieties for representations*, Contemp. Math. (to appear).

15. V. Kac, *Infinite root systems, representations of graphs and invariant theory*, J. Algebra **78** (1982), 141-162.

16. A. Schofield, *General representations of quivers*, Proc. London Math. Soc. (3) **65** (1992), 46-64.

17. S. O. Smalø, *Lectures on Algebras*, Mar del Plata, Argentina, March 2006, Revista Unión Mat. Argentina **48** (2007), 21-45.

Department of Mathematics, University of California, Davis, CA 95616
E-mail address: babson@math.ucdavis.edu

Department of Mathematics, University of California, Santa Barbara, CA 93106-3080
E-mail address: birge@math.ucsb.edu

Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195-4350
E-mail address: thomas@math.washington.edu