1. Evaluate the following integrals. SHOW YOUR WORK (12 points)

(a) $\int \sin^3 x \, dx$

$= \int \sin x \sin^2 x \, dx = \int \sin x (1 - \cos^2 x) dx$

$= \int \sin x - \sin x \cos^2 x \, dx$

$= \int \sin x \, dx - \int \sin x \cos^2 x \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$= \int \sin x + \int u^2 du$

$= -\cos x + \frac{1}{3} u^3 + C = -\cos x + \frac{1}{3} \cos^3 x + C$

(b) $\int_1^\infty \frac{1}{\sqrt{x}} \, dx$

$= \lim_{b \to \infty} \int_1^b x^{-1/2} dx$

$= \lim_{b \to \infty} 2\sqrt{x} \Big|_1^b$

$= \lim_{b \to \infty} 2\sqrt{b} - 2\sqrt{1}$

$= \lim_{b \to \infty} 2(\sqrt{b} - 1)$

$\lim_{b \to \infty} \sqrt{b} = \infty$

Thus the integral diverges

(c) $\int_1^\infty x^{-2} \ln x \, dx$ (Hint: use parts with u = $\ln x$)

$u = \ln x \quad dv = x^{-2}$

$du = \frac{1}{x} \quad v = \frac{-1}{x}$

Thus, $\int_1^\infty x^{-2} \ln x \, dx = \frac{-\ln x}{x} \Big|_1^\infty - \int_1^\infty -x^{-2} dx$

$= \lim_{b \to \infty} \frac{-\ln x}{x} \Big|_1^b - \int_1^b -x^{-2} dx$

$= \lim_{b \to \infty} \frac{-\ln x}{x} - \frac{1}{x} \Big|_1^b$

$= \lim_{b \to \infty} -\frac{\ln x - 1}{x} \Big|_1^b = \lim_{b \to \infty} \frac{-\ln b - 1}{b} - \frac{-\ln(1) - 1}{1}$

Now, $\lim_{b \to \infty} -\frac{\ln b + 1}{b} = -\frac{\infty}{\infty}$

Thus, l'Hopital's rule applies, and so we have $\lim_{b \to \infty} -\frac{\frac{1}{b}}{1} = \lim_{b \to \infty} -\frac{1}{b} = 0$

Hence, $\lim_{b \to \infty} \frac{-\ln b - 1}{b} - \frac{-\ln(1) - 1}{1} = 0 + 1 = 1$

2. Write out the partial fraction decomposition for the function below. You should leave your answer in the form $\frac{A}{\text{stuff}} + \frac{B}{\text{stuff}} + \frac{Cx + D}{\text{stuff}} + \ldots$. You do not need to find the constants A, B, etc. but the "stuff" needs to be correct (4 points).

$$\frac{x^2 - 3x + 5}{(x-2)^3 (x+2)(x^2+4)} =$$

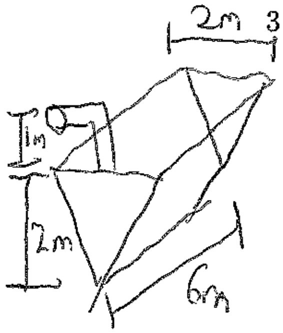
We have $(x-2)^3$, thus we need a $\frac{A}{x-2}, \frac{B}{(x-2)^2}$, and $\frac{C}{(x-2)^3}$.

We also have $(x+2)$, thus we need $\frac{D}{x+2}$.

Finally, we have (x^2+4), thus we need $\frac{Ex+F}{(x^2+4)}$.

Hence, $\frac{x^2 - 3x + 5}{(x-2)^3 (x+2)(x^2+4)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{D}{x+2} + \frac{Ex+F}{(x^2+4)}$

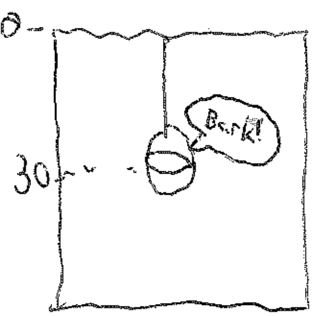
3. Set up but DO NOT EVALUATE the integral you would take to find the following. (8 points)



(a) The work required to pump the water out of the fish tank shown (note: there is no water in the spout). Clearly label where you decided x = 0 is, and which direction you took to be positive. You may use that the density of water is 1,000 kg/m^3.

A rectangular "slice" of water is Δx m. thick and lying x m. above the bottom has width x m. and volume $6x\Delta x$ m.^{3} It weighs about $(9.8\times1000)(6x\Delta x)$ N, and must be lifted $(3-x)$ m. by the pump, so the work needed is about $(9.8\times10^3)(3-x)(6x\Delta x)$ J. The total work required is $W\approx\int_0^2(9.8\times10^3)(3-x)(6x)dx = 9800\int_0^2 18x-6x^2dx$ $=9800[9x^2-2x^3]_0^2 = 9800[36-16] = 9800[20] = 196000$ J. Thus the work is approximately 196000J

(b) A rope 30 meters long and weighing 10 kg hangs over the side of a building. It has a basket of puppies tied to the end of it, weighing 5 kg total. Set up the integral you would take to find the work required to pull the rope to the top of the building. As above, clearly label where x = 0 and x = 30 are.



First, we notice that the total work equals the sum of the work to raise the rope and the work to raise the basket. The force needed to raise the rope is $\frac{10}{30}\cdot9.8\cdot\Delta x = \frac{1}{3}\cdot9.8\cdot\Delta x = \frac{49}{15}\Delta x$. The distance is denoted by x_i. Hence, $W_R\approx\int_0^{30}\frac{49}{15}x\,dx = \frac{49}{15}\int_0^{30}x\,dx = \frac{49}{15}\left[\frac{1}{2}x^2\right]_0^{30}$ $=1470$J. Now, the force on the basket is constant. Hence, $W_B\approx Fd=(ma)d=5(9.8)(30)=1470$J. Thus, the total work $W=W_R+W_B=1470+1470=2940$J

Congratulations! You are now covered in puppies.

4. Evaluate the following integrals. (12 points)

(a) $\displaystyle\int \frac{1}{x^2+2x+3}\,dx$ (Hint: You cannot factor the denominator)

$$\int \frac{1}{x^2+2x+3}\,dx$$

$$x^2+2x+3 = x^2+2x+1+2 = (x+1)^2+2$$

Thus, $\displaystyle\int \frac{1}{x^2+2x+3}\,dx = \int \frac{1}{(x+1)^2+2}\,dx$

$$u = x+1$$
$$du = dx$$

$= \displaystyle\int \frac{1}{u^2+2}\,du$, The integral of $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$.

Thus, $\displaystyle\int \frac{1}{u^2+2}\,du = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+C$

(b) $\displaystyle\int e^{\sqrt{x}}\,dx$

$u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$

$x = u^2$

$dx = 2u\,du$

Hence, $\displaystyle\int e^{\sqrt{x}}\,dx = \int 2ue^u\,du$

$v = u \quad dw = e^u$

$dv = 1 \quad w = e^u$

Hence, $\displaystyle\int 2ue^u\,du = 2ue^u - 2\int e^u\,du$

$= 2ue^u - 2e^u + C$

$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$

5. Use your wits to solve this problem! (8 points)

(a) Find $f'(x)$ where $f(x) = \int_{-1}^{x} \sqrt{e^{-2t} - 1}\, dt$

By the Fundamental Theorem of Calculus, part 1, $f'(x) = \sqrt{e^{-2x} - 1}$

(b) With $f(x)$ as above, find the arclength of $y = f(x)$ from $x = 0$ to $x = 4$

$$\text{Arc length} = \int_0^4 \sqrt{1 + [f'(x)]^2}\, dx$$

$$= \int_0^4 \sqrt{1 + (\sqrt{e^{-2x} - 1})^2}\, dx$$

$$= \int_0^4 \sqrt{1 + e^{-2x} - 1}\, dx$$

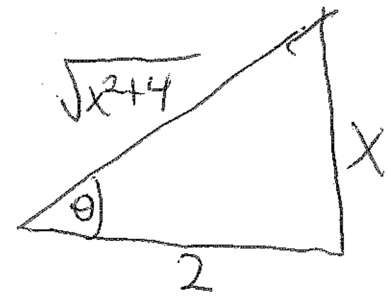
$$= \int_0^4 \sqrt{e^{-2x}}\, dx$$

$$= \int_0^4 (e^{-2x})^{1/2}\, dx$$

$$= \int_0^4 e^{\frac{-2x}{2}}\, dx = \int_0^4 e^{-x}\, dx$$

$$= -e^{-x}\Big|_0^4 = -e^{-4} + e^{-0} = -\frac{1}{e^4} + 1$$

BONUS 1: Evaluate $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$



$\tan\theta = \frac{x}{2} \Rightarrow x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$\frac{\sqrt{x^2+4}}{2} = \sec\theta \Rightarrow \sqrt{x^2+4} = 2\sec\theta$

Thus, we have the substitution

$$\int \frac{\cancel{2}\sec^2\theta\, d\theta}{(2\tan\theta)^2 \cdot \cancel{2\sec\theta}} = \int \frac{\sec\theta}{4\tan^2\theta}\, d\theta$$

$$= \int \frac{\sec\theta}{4\left(\frac{\sin^2\theta}{\cos^2\theta}\right)}\, d\theta = \int \frac{\sec\theta\cos^2\theta}{4\sin^2\theta}\, d\theta = \int \frac{\frac{1}{\cancel{\cos\theta}}\cdot\cos^2\theta}{4\sin^2\theta}\, d\theta$$

$$= \int \frac{\cos\theta}{4\sin^2\theta}\, d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$\int \frac{1}{4} u^{-2}\, du = \frac{-1}{4} u^{-1} + C = \frac{-1}{4\sin\theta} + C$$

Going back to the triangle, we see $\sin\theta = \frac{x}{\sqrt{x^2+4}}$

Hence,

$$\int \frac{1}{x^2\sqrt{x^2+4}}\, dx = \frac{-1}{4\left(\frac{x}{\sqrt{x^2+4}}\right)} + C = -\frac{1}{4}\,\frac{\sqrt{x^2+4}}{x} + C$$

(more bonus on next page. It's easier and funner)