SOLUTIONS TO MIDTERM 2.2
MATH 3B - WINTER 2009

(1) Consider the region S bounded by the three curves

$$y = \arccos x, \quad x = 0 \quad \text{and} \quad y = \frac{\pi}{3}$$

Calculate the area of S

Answer: One way of writing the required integral is $\int_0^{\frac{1}{2}} \arccos x - \pi/3\, dx$. But we do not know how to do this integral.

The critical observation is that $y = \arccos x$ is the same as saying $x = \cos y$. So the area is

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos y\, dy = \sin y|_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin\frac{\pi}{3} = 1 - \frac{\sqrt{3}}{2}.$$

(2) Using the cylindrical shell method, find the resulting volume if the region between the three curves $y = \cos x$, $y = 0$ and $x = \frac{\pi}{3}$ is rotated around the y-axis.

Answer: The region described is exactly the region of problem 1, with the roles of x and y reversed.

The cylinder through the value x has height $\cos x$ so it has area

$$A(x) = 2\pi x \cos x.$$

It follows that the volume of the resulting solid is

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} A(x)\, dx = 2\pi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} x \cos x\, dx.$$

This can be integrated by parts, with $u = x$ and $v = \sin x$. Then

$$\int x\cos x\, dx = x \sin x - \int \sin x\, dx = x\sin x + \cos x.$$

So the volume is

$$2\pi(x\sin x + \cos x)|_{\frac{\pi}{3}}^{\frac{\pi}{2}} = 2\pi(\frac{\pi}{2} - \frac{\pi}{3}\frac{\sqrt{3}}{2} - \frac{1}{2}) = \pi^2(1 - \frac{1}{\sqrt{3}}) - \pi$$

(3) A reluctant burro is pulled along a path by a man who must exert a force of

$$10/(1+x)^2$$

pounds when the burro is a distance x feet from the beginning of the path. How much work does he need to do to move the burro 4 feet down the path?

Answer: The work

$$W = \int F(x)dx = 10\int_0^4 \frac{dx}{(1+x)^2}.$$

If we make the substitution $u = x+1 \Rightarrow dx = du$ we get

$$10\int_{u=1}^{u=5} \frac{du}{u^2} = -10\frac{1}{u}|_1^5 = 10(1-1/5) = 8ft-lbs.$$

(4) On the planet PsK! the standard unit of length is the gronka, abbreviated gr. Acceleration due to gravity is always $20gr/sec^2$ downwards. A ball is dropped from the top of a very tall tower.

- What will the velocity of the ball be after t seconds?
  Answer: Acceleration is the derivative of velocity, so $v = 20t + C$ downwards. When $t = 0$ the velocity is 0, so $C = 0$. Hence $v = 20t\ gr/sec$ downwards.
- How far will the ball have dropped after t seconds?
  Answer: Velocity is the derivative of the distance dropped, so $r(t) = \int 20t dt = 10t^2 + C$. When $t = 0$ the ball has not dropped at all, so $C = 0$. Hence $r = 10t^2\ gr$.
- What will the velocity be when the ball has dropped r gronkas?
  Answer: Since $r = 10t^2, t = \sqrt{r/10}$. Then $v = 20t = 20\sqrt{r/10} = 2\sqrt{10r}$
- What is the average velocity of the ball over the first 5 gronkas?
  Answer: Since $v = 2\sqrt{10r}$, the average will be $\frac{\int_0^5 2\sqrt{10r}dr}{5} = \frac{\frac{4}{3}\sqrt{10}\,r^{3/2}\Big]_0^5}{5} = \frac{4}{3}\sqrt{50} = \frac{20\sqrt{2}}{3}gr/sec.$

(5) The huge City University of Elbonia admits anyone with a Math SAT of 400 or over. Here is a graph of the number of students admitted for each SAT score between 400 and 800 (a perfect score). For example, according to the graph, about 57 students had a Math SAT of 635. Using $n = 4$ on the interval $[400, 800]$ estimate the total number of students admitted.

Answer: In this case $\Delta x = (800-400)/4 = 100, x_0 = 400, x_1 = 500, x_2 = 600, x_3 = 700, x_4 = 800$. According to the graph, $f(x_0) = 30, f(x_1) = 50, f(x_2) = 60, f(x_3) = 40, f(x_4) = 20$.

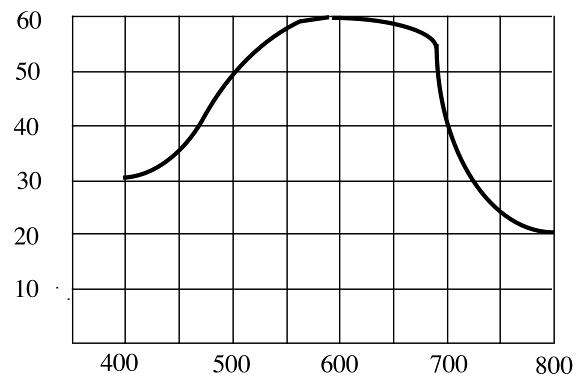

FIGURE 1. Elbonian SAT's

- For the Trapezoidal rule: $f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4) = 30+100+120+80+20 = 350$ so the estimate is $350\frac{\Delta x}{2} = 350 \cdot 50 = 17,500$.
- For Simpson's rule: $f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4) = 30+200+120+160+20 = 530$ so the estimate is $530\frac{\Delta x}{3} \sim 530 \cdot 33\frac{1}{3} = 17,667$.