MATH 34B
Practice Midterm 2 Solutions

1)

1. True, since the increase in volume to first order is $V'(R)(\delta R) = 4\pi R^2 \cdot \delta R$. Given $R = 1$, $\delta R = 0.01$, so plug these numbers in.
2. True
3. True
4. True
5. False. Let $f(x) = x^3 e^x$. Then $f'(x) = 3x^2 e^x + x^3 e^x$, $f''(x) = 6xe^x + 6x^2 e^x + x^3 e^x$, and $f'''(x) = 6e^x + 12xe^x + 9x^2 e^x + x^3 e^x$ (check this). When $x = 0$, this satisfies the given conditions, but it is easy to check that near $x = 0$ the graph of $f(x)$ looks like $y = x^3$ so there is no minimum at $x = 0$.
6. True
7. False
8. True
9. False. To see this, draw the line $y = 1$, but change it so that it dips down to get minima at $x = 2$ and $x = 3$.
10. False. The linear interpolation is $f(x) = 50 - 5x$ (check this), so $f(5) = 25 \neq 15$.

2)

a) The key here is that $y(x)$ is a function of x, so regard t as a constant and just integrate. Thus

$$\begin{aligned} y''(x) &= \frac{1}{t}e^{x/t} + \frac{1}{x^3} \\ y'(x) &= e^{x/t} - \frac{2}{x^2} + C \\ y(x) &= te^{x/t} + \frac{2}{x} + Cx + D \end{aligned}$$

where C and D are constants.

b) The solution is $y(t) = 73 + Ae^{-t}$ where A is any constant. Check that this works by taking the derivative.

c) This can be rewritten as $y' = y - 73y^2 = y(1 - 73y)$, which looks like the logistic equation with $k = 1$ and $M = 1/73$. Using the formula for this solution,

$$y(t) = \frac{1}{73Ae^{-t} + 73} = \frac{1}{Be^{-t} + 73}$$

Where A and B are any constants, related so that $B = 73A$.

3)

This problem is one where it is *very* important to draw a picture! From the problem, we are asked to find the dimensions to maximize the light, so we want a pair of numbers that tell us the length and the width of the rectangle part of the Norman window (the semicircle's dimensions are defined by the width of the rectangle if you drew it as a semicircle on top of a rectangle). If we let x be the height and y the width of the rectangle, the perimeter relationship tells us that

$$2x + y + \frac{\pi}{2}y = 24$$

(refer to your drawing to get this). Also, we have the areas of the rectangle and semicircle, call them A_R and A_C, respectively, which are

$$A_R = xy$$

$$A_C = \frac{\pi}{2}r^2 = \frac{\pi}{2}\left(\frac{y}{2}\right)^2 = \frac{\pi}{8}y^2$$

If we let L denote the amount of light that passes through the window, we have

$$L = A_R + \frac{1}{2}A_C = xy + \frac{\pi}{16}y^2$$

Now, use the perimeter equation to find x in terms of y, that way we get L as a function of y, which we will need in order to take the derivative. Thus,

$$x = 12 - \left(\frac{2+\pi}{4}\right)y$$

(again, check this). Substituting into our equation for L yields

$$\begin{aligned} L = xy + \frac{\pi}{16}y^2 &= \left(12 - \left(\frac{2+\pi}{4}\right)y\right)y + \frac{\pi}{16}y^2 \\ &= 12y + \left(-\frac{2+\pi}{4} + \frac{\pi}{16}\right)y^2 \\ &= 12y - \left(\frac{8+3\pi}{16}\right)y^2 \end{aligned}$$

Now, take the derivative and set it equal to 0 to find value for y that maximizes L:

$$\begin{aligned} 0 = L' &= 12 - \left(\frac{8+3\pi}{8}\right)y \\ y &= \frac{96}{8+3\pi} \end{aligned}$$

As a quick check, look at L'' to verify that this will indeed give us a maximum. Now, we have our width, so let's find the height by substituting back into the equation for x:

$$\begin{aligned} x &= 12 - \frac{2+\pi}{4}\left(\frac{96}{8+3\pi}\right) \\ &= 12 - \frac{48+24\pi}{8+3\pi} \\ &= \frac{96+36\pi}{8+3\pi} - \frac{48+24\pi}{8+3\pi} \\ &= \frac{48+12\pi}{8+3\pi} \end{aligned}$$

Since we started with feet and there were never any other units to worry about, we know that these are the right answers. Also, one can check that the perimeter does indeed work out, and find that $x \approx 4$ and $y \approx 6$, which are reasonable dimensions.

4)

We're given that $f(x) = x^2 - 7x + 6$, which after factoring is the same as $f(x) = (x-6)(x-1)$. Thus the x-intercepts are $(1,0)$ and $(6,0)$ and the y-intercept is $(0,6)$. If we take the derivative, we get $f'(x) = 2x - 7$, so there is a critical point at $x = \frac{7}{2}$. Since $f''(x) = 2 > 0$, this critical point must be a minimum. Also, f must be concave up everywhere since f'' is always positive. So, plugging our x value that minimizes f back into the equation for $f(x)$, we get that the minimum is at $(\frac{7}{2}, -\frac{25}{4})$. Noting that this is the equation for a parabola, it is then easy to graph the function.