Rules: You have the full 70 minutes for the test. You are allowed **one** 3x5 notecard, and can use both sides. You may **not** use a calculator.

Test: The test will be 6 questions of variable difficulty. It will cover all material covered during the course, except for approximating integrals (that is, you won't have to approximate any integrals using Simpson's Rule or Riemann sums). Three of the problems will be pretty straightforward, 2 will be of moderate difficulty, and one will be difficult.

Topics: The following are the topics that I think are most important in the course, and a few ideas on how they may or may not be tested.

(1) Substitution. Know how to use substitution, which says

$$\int f'(u(x))u'(x)dx = \int f(u)du.$$

For example, know how to evaluate

$$\int xe^{x^2} dx.$$

Also know how to use substitution to evaluate definite integrals. For instance, be able to find

$$\int_1^4 \frac{\sin \ln x}{x} dx.$$

(2) Integration By Parts. Know how to use integration by parts, which says

$$\int u \; dv = uv - \int v \; du.$$

For example, know how to evaluate

$$\int xe^x dx, \qquad \int \ln x \; dx \qquad \int e^x \sin x \; dx.$$

(3) Integrating Powers of Trig Functions. Know how to evaluate the following integrals for any integers m, n:

$$\int \sin^m x \; \cos^n x \; dx \qquad \int \sec^m x \; \tan^n x \; dx \qquad \int \csc^m x \; \cot^n x \; dx.$$

These are always done by substituting u equals a trig function in the integral. The following identities are useful in coming to an answer:

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
1 + \tan^2 x &= \sec^2 x \\
1 + \cot^2 x &= \csc^2 x \\
\sin^2 x &= \frac{1 - \cos(2x)}{2} \\
\cos^2 x &= \frac{1 + \cos(2x)}{2} \\
\int \sec x \; dx &= \ln|\sec x + \tan x| + C \\
\int \csc x \; dx &= -\ln|\csc x + \cot x| + C.
\end{aligned}$$

Also remember that to evaluate $\int \sec^3 x \; dx$, $\int \sec^5 x \; dx$, and so on, you need to use integration by parts.

(4) Finding volumes. Be able to use both the method of disks and washers and the method of cylindrical shells to compute volumes of revolution. Try not to memorize formulas here, but remember the geometric method we used to derive the formula (you will no doubt receive points for a good picture, and may lose points for not having a picture!) The basic formulas to remember here are

$$\begin{aligned}
\text{Volume of a Washer} &= \pi \times ((\text{outer radius})^2 - (\text{inner radius})^2) \times \text{thickness} \\
\text{Volume of a Shell} &= 2\pi \times \text{radius} \times \text{height} \times \text{thickness}.
\end{aligned}$$

(5) Improper Integrals. Remember that we can only use the Fundamental Theorem of Calculus to integrate $\int_a^b f(x)dx$ if f is continuous on the closed interval $[a, b]$. If f is discontinuous at a, remember that we define

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx,$$

and a similar statement holds if f is discontinuous at b. If f is discontinuous at some point $d \in (a, b)$, then we need to write

$$\int_a^b f(x)dx = \int_a^d f(x)dx + \int_d^b f(x)dx$$

and use the above method.

Also remember that we can evaluate integrals with infinite limits of integration in a similar manner. Specifically, we defined

$$\int_a^\infty f(x)dx = \lim_{n \to \infty} \int_a^n f(x)dx,$$

and similarly for $\int_{-\infty}^a f(x)dx$. If we wanted to integrate $\int_{-\infty}^\infty f(x)dx$, we would have to split

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

and use the above method. Remember that BOTH integrals must converge (exist) for the full integral to exist in this case!

(6) Arclength. Remember that if a curve is given with coordinates $(f(t), g(t))$ for $a \leq t \leq b$, then the length of the curve is

$$\int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}dt.$$

Thus, for example, the curve $y = f(x)$, $a \leq x \leq b$ is the curve $(x, f(x)$, and so has length

$$\int_a^b \sqrt{1 + (f'(x))^2}dx.$$

(7) Surface Areas of Revolution. Remember that if we rotate the curve $y = f(x)$, $a \leq x \leq b$, around the x-axis, then the surface area of the resulting shape is given by

$$2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}dx.$$

(8) Work and Hydrostatic Pressure. Remember that the amount of work required to move an object with constant force F a constant distance d is $W = Fd$. If either is not constant, we need to break up the task into smaller tasks for which force and distance are constant, which leads us to the general equation

$$\text{Work} = \int \text{Force} \times \text{Distance},$$

where either the force or the distance will have to include a dx because of our approximations (compare this to our derivation of volumes).

Similarly, the hydrostatic force on a thin plate of area A in a liquid of density ρ and at depth d is $F = \rho dA$. To find the hydrostatic force on a general object, we then approximate it by thin plates and compute

$$\text{Force} = \int \text{density} \times \text{depth} \times \text{Area}.$$

(9) Center of Mass. Recall that if $\rho(x)$ is the density of a long, thin rod given by $a \leq x \leq b$, then its mass is

$$m = \int_a^b \rho(x)dx$$

and its first moment is

$$M = \int_a^b x\rho(x)dx.$$

The center of mass is then given by M/m. We can use this to compute the center of mass of 2 and 3-dimensional objects by breaking those shapes up into rectangles or rectangular prisms, and using that the center of mass of a rectangle with constant density is just the center of the rectangle.

With regards to the applications to physics and the volumes and surface areas of revolution, I am more interested in whether or not you understand how to set up the integral than that you can get the correct answer. As such, you should make it your priority to demonstrate you know how to set up the integral, and not just

messily write down an integral expression which "magically" gives the right answer. Most points on those problems will come from demonstrating you can set up the integral (and as such, will likely be easy integrals, but don't count on this).
I will, however, definitely have a question that really tests your ability to put all the techniques together to evaluate an integral. We've done integrals where we had to use trig substitution, partial fractions, and integration by parts in the same integral. This may happen on the test — do not expect that one integral you are given is "the trig substitution problem" and one is "the partial fractions" problem.

There are a lot of problems on WebWork in the optional homework set that have all of these topics represented. You may also find it useful to go back through your old homework problems and redo them. Certain topics in particular only have so many good, computable examples, after all...