Review Problems for Math 34A Spring 2009, Midterm II

The test will be 10 problems, with 4 word problems. A 3x5 notecard is allowed, but no calculators. The test will be cumulative, with a large focus on derivatives.

If you need additional problems, I strongly suggest looking at the previous review problems, as well as looking at the word problems in Section 8.13.

1) Solve for x in the following equations

(a) $4 \cdot 2^x = 3^{x-1}$

Solution: $x = \dfrac{\ln 12}{\ln 3 - \ln 2}$.

(b) $e^{2x} = 7^y$.

Solution: $x = \dfrac{y \ln 7}{2}$.

(c) $\frac{7}{3^x} = 4 \cdot 5^x$.

Solution: $x = \dfrac{\ln 7 - \ln 4}{\ln 15}$.

2) Evaluate the following limits

(a) $\lim_{x \to 3} \dfrac{x^2 - 3x + 2}{x + 3}$.

Solution: The limit is 1/3.

(b) $\lim_{x \to -2} \dfrac{x^2 - 5x - 14}{x^2 + 5x + 6}$.

Solution: The limit is -9.

(c) $\lim_{x \to \infty} \dfrac{3x + 8}{7 - x}$.

Solution: The limit is -3.

3) A certain radioactive isotope is found in Frank's house. Using careful measurements, scientists determine that after one day, a third of the isotope remained. What is the half-life of the isotope, in hours?

Solution: Let t be the time that has passed in hours. We know that $A(t) = A_0 \left(\frac{1}{2}\right)^{t/k}$, where k is the half-life (in hours) and A_0 is the initial amount. After one day, we know $A(24) = A_0/3$. Thus

$$\begin{aligned} \frac{A_0}{3} &= A_0 \left(\frac{1}{2}\right)^{24/k} \\ \frac{1}{3} &= \left(\frac{1}{2}\right)^{24/k} \\ -\ln 3 &= \frac{-24 \ln 2}{k} \\ k &= \frac{24 \ln 2}{\ln 3}. \end{aligned}$$

That is, the half-life is $24 \frac{\ln 2}{\ln 3}$ hours.

4) Compute the derivatives of the following functions

(a) $f(x) = (x + 7)(x - 3)$.

Solution: $f'(x) = 2x + 4$.

(b) $g(x) = \frac{x^2-3}{\sqrt{x}}$.

Solution: We can write $g(x) = x^{3/2} - 3x^{-1/2}$, and so

$$g'(x) = \frac{3}{2}x^{1/2} + \frac{3}{2}x^{-3/2}.$$

(c) $h(x) = e^{2x} - 2^x$.

Solution: We can write $h(x) = e^{2x} - e^{x \ln 2}$, and so

$$h'(x) = 2e^{2x} - (\ln 2)e^{x \ln 2} = 2e^{2x} - (ln2)2^x.$$

5) Approximate $\sqrt{5}$ using

(a) Linear interpolation between $\sqrt{4}$ and $\sqrt{9}$.

Solution: We use linear interpolation between the points $(4, \sqrt{4}) = (4, 2)$ and $(9, \sqrt{9}) = (9, 3)$. The slope of this line is $\frac{3-2}{9-4} = \frac{1}{5}$, so the equation of the line is (using point-slope form)

$$\begin{aligned} y - 2 &= \frac{1}{5}(x - 4) \\ y &= \frac{1}{5}(x - 4) + 2 \\ y &= \frac{1}{5}x + \frac{6}{5}. \end{aligned}$$

Plugging in $x = 5$, we then see that $\sqrt{5} \approx \frac{11}{5} = 2.2$.

(b) The tangent line approximation to $f(x) = \sqrt{x}$ at $x = 4$.

Solution: We know that $f'(x) = \frac{1}{2}x^{-1/2}$, so the line goes through the point $(4, \sqrt{4}) = (4, 2)$ and has slope $f'(4) = 1/4$. Using point slope form, that means the equation of the tangent line is

$$\begin{aligned} y - 2 &= \frac{1}{4}(x - 4) \\ y &= \frac{1}{4}(x - 4) + 2 \\ y &= \frac{1}{4}x + 1. \end{aligned}$$

Plugging in $x = 5$, we see that $\sqrt{5} \approx 9/4 = 2.25$.

(c) Which method do you expect to give the better answer and why?

Solution: We expect the tangent line approximation to be better, because tangent lines most closely approximate the graph of a function.

6) Consider the function $f(x) = x^3 - 6x^2 - 15x + 2$.

(a) Where is $f(x)$ increasing and decreasing?

Solution: We need to find where $f'(x)$ is positive and negative. We first calculate

$$f'(x) = 3x^2 - 12x - 15.$$

We then see that $f'(x) = 0$ when

$$\begin{aligned} 0 &= 3x^2 - 12x - 15 \\ &= 3(x^2 - 4x - 5) \\ &= 3(x-5)(x+1), \end{aligned}$$

so when $x = 5$ or $x = -1$. Thus there are three regions to check: when $x < -1$, when $-1 < x < 5$, and when $x > 5$. We will then test the sign of $f'(x)$ when $x = -2$, $x = 0$, and $x = 6$. Plugging in, we have

$$\begin{aligned} f'(-2) &= 21 > 0 \\ f'(0) &= -15 < 0 \\ f'(6) &= 21 > 0. \end{aligned}$$

That is, $f(x)$ is increasing when $x < -1$ or $x > 5$, and $f(x)$ is decreasing when $-1 < x < 5$.

(b) Where is $f(x)$ concave up and concave down?

Solution: We need to find when $f''(x)$ is positive and negative. First, we know that

$$f''(x) = 6x - 12.$$

Thus $f''(x) = 0$ when $6x - 12 = 0$, or $x = 2$, and so we need to check the regions $x < 2$ and $x > 2$. We will use the values $x = 0$ and $x = 3$.

$$\begin{aligned} f''(0) &= -12 < 0 \\ f''(3) &= 6 > 0, \end{aligned}$$

so $f(x)$ is concave up when $x > 2$ and $f(x)$ is concave down when $x < 2$.

7) What is the maximum value of the function $f(x) = -2x^2 + 3x + 7$?

Solution: The maximum must occur when $f'(x) = 0$, so we have

$$0 = f'(x) = -4x + 3.$$

Thus the maximum must occur when $x = 3/4$. We check though that $f''(x) = -4 < 0$, which means that a maximum indeed occurs when $x = 3/4$. Thus the maximum is

$$f(3/4) = 8.125.$$

8) Albert hits a baseball upwards with an initial velocity of 45 meters per second. Using that acceleration is 10 meters per second per second,

(a) How long is the ball in the air?

Solution: We know that the height of the ball is given by $h(t) = -5t^2 = 45t$. The ball hits the ground when $h(t) = 0$, so when

$$\begin{aligned} 0 &= -5t^2 + 45t \\ &= -5t(t-9), \end{aligned}$$

so when $t = 0$ or $t = 9$. The ball was initially on the ground, so it comes back down when $t = 9$.

(b) What is the maximum height of the ball?

Solution: The ball is at the maximum height when $h'(x) = 0$. But $h'(x) = -10x + 45$, so $h'(x) = 0$ when $x = 4.5$. We check that $h''(x) = -10 < 0$, so the maximum indeed occurs when $x = 4.5$, making the maximum height

$$h(4.5) = -5(4.5)^2 + 45(4.5) = 101.25.$$

(c) What does the initial velocity need to be for the maximum height to be 200 meters? (Hint: what is the maximum height when the initial velocity is v_0?)

Solution: Now we have $h(t) = -5t^2 + v_0 t$. We know the maximum occurs when

$$0 = h'(t) = -10t + v_0,$$

so when $t = v_0/10$. The maximum height is then

$$h\left(\frac{v_0}{10}\right) = -5\left(\frac{v_0}{10}\right)^2 + v_0 \cdot \frac{v_0}{10} = \frac{5v_0^2}{100}.$$

We then need to solve the equation $200 = \frac{v_0^2}{20}$, which gives $v_0 = \sqrt{4000}$.

9) A rectangular field will have one side made of a brike wall and the other three sides made of wooden fence. Brick wall costs \$20 per meter. Wooden fence costs \$40 for 4 meters. The area of the field is to be 2400 m^2. What length should the brick wall be to give the lowest total cost of wall plus fence?

Solution: First we need to write a formula for the cost of the wall and fence in terms of the length of the wall. If L is the length of the wall and W is the length of the fence, then we know the total cost is

$$C = 20L + 10(L + 2W) = 30L + 20W.$$

We also know that the area is $2400 = LW$, so $W = 2400L^{-1}$. That means that the cost in terms of L is

$$C(L) = 30L + 20 \cdot 2400L^{-1} = 30L + 48000L^{-1}.$$

Next we need to maximize, so we need to find when $C'(L) = 0$. We know that

$$C'(L) = 30 - 48000L^{-2},$$

so setting $C'(L) = 0$ gives

$$\begin{aligned} 0 &= 30 - 48000L^{-2} \\ 48000L^{-2} &= 30 \\ 48000 &= 30L^2 \\ L^2 &= 1600 \\ L &= \pm 40. \end{aligned}$$

Only the value $L = 40$ makes sense, so we know that the cost will be minimized when the length of the brick wall is 40 m. Alternatively, we can check that

$$C''(L) = 96000L^{-3},$$

and so $C''(40) = 3 > 0$, and so $L = 40$ is indeed a minimizer.

10) Consider the line $f(x) = x - 1$.

(a) Draw a graph of $f(x)$, letting x lie between -4 and 4.

Solution: The line goes through the points $(-4, -5)$ and $(4, 3)$.

(b) Calculate $\int_0^3 f(x)\,dx$.

Solution: We need to calculate the signed area between $x = 0$ and $x = 3$ of the line. The endpoints are then $(0, -1)$ and $(3, 2)$, and the line goes through the x-axis at the point $(1, 0)$. This means the line is below the x-axis between the points $(0, -1)$ and $(1, 0)$ and above the x-axis between the points $(1, 0)$ and $(3, 2)$. Thus the area below the graph is

$$A_- = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

and the area above the graph is

$$A_+ = \frac{1}{2} \cdot 2 \cdot 2 = 2.$$

Thus the integral is

$$\int_0^3 (x-1)dx = A_+ - A_- = 2 - \frac{1}{2} = \frac{3}{2}.$$